Synopsis of
De fractionibus continuis Wallisii

(On the continued fractions of Wallis)

Thomas J Osler, Christopher Tippie and Kristin Masters

Introduction to E745

In the year 1656 John Wallis published his "Arithmetica Infinitorum", [6], in which he displayed many ideas that were to lead to the integral calculus of Newton. In this work we find the celebrated infinite product for π,

(1) $$\frac{2}{\pi}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots$$

which we now call the "Wallis product".

Using modern notation, we can say that Wallis knew the integration formula

(2) $$\int_0^c x^p dx=\frac{c^{p+1}}{p+1},$$

and could use it for values of p that were both integers and fractions. Wallis wanted to find some convenient expression for the area bound by the unit circle "in terms of integers", and (again in modern notation), he wanted to evaluate the integral

(3) $$\frac{\pi}{4}=\int_0^1 \sqrt{1-x^2}\,dx.$$

The binomial theorem for fractional exponents had not yet been discovered, and so knowing only (2), Wallis had no direct way of evaluating the integral (3). Instead, Wallis used an ingenious method of interpolation. He reasoned that the value of the integral (3) was between the two integrals $\int_0^1 \left(1-x^2\right)^0 dx$ and $\int_0^1 \left(1-x^2\right)^1 dx$, and, of course, he could evaluate both of these. To achieve this interpolation, he created a table of values the reciprocal integrals $1/\int_0^1 \left(1-x^{1/Q}\right)^P dx$ for values of *P* and *Q* that he could evaluate. (The reason for the reciprocal was probably to obtain more integer values in the table.) A very careful and ingenious study of this table lead Wallis to tease out his product (1).

Wallis's Table of the Reciprocal Integral $1/\int_0^1 \left(1-x^{1/Q}\right)^P dx$

	$P=0$	$P=1/2$	$P=1$	$P=3/2$	$P=2$	$P=5/2$	$P=3$
$Q=0$	1	$1\cdot\frac{1}{1}$	$1\cdot\frac{2}{2}$	$1\cdot\frac{1}{1}\cdot\frac{3}{3}$	$1\cdot\frac{2}{2}\cdot\frac{4}{4}$	$1\cdot\frac{1}{1}\cdot\frac{3}{3}\cdot\frac{5}{5}$	$1\cdot\frac{2}{2}\cdot\frac{4}{4}\cdot\frac{6}{6}$
$Q=1/2$	1	$\frac{2}{\pi}\cdot\frac{2}{1}$	$1\cdot\frac{3}{2}$	$\frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{4}{3}$	$1\cdot\frac{3}{2}\cdot\frac{5}{4}$	$\frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}$	$1\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}$
$Q=1$	1	$\frac{1}{2}\cdot\frac{3}{1}$	$1\cdot\frac{4}{2}$	$\frac{1}{2}\cdot\frac{3}{1}\cdot\frac{5}{3}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}$	$\frac{1}{2}\cdot\frac{3}{1}\cdot\frac{5}{3}\cdot\frac{7}{5}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}\cdot\frac{8}{6}$
$Q=3/2$	1	$\frac{4}{3\pi}\cdot\frac{4}{1}$	$1\cdot\frac{5}{2}$	$\frac{4}{3\pi}\cdot\frac{4}{1}\cdot\frac{6}{3}$	$1\cdot\frac{5}{2}\cdot\frac{7}{4}$	$\frac{4}{3\pi}\cdot\frac{4}{1}\cdot\frac{6}{3}\cdot\frac{8}{5}$	$1\cdot\frac{5}{2}\cdot\frac{7}{4}\cdot\frac{9}{6}$
$Q=2$	1	$\frac{3}{8}\cdot\frac{5}{1}$	$1\cdot\frac{6}{2}$	$\frac{3}{8}\cdot\frac{5}{1}\cdot\frac{7}{3}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}$	$\frac{3}{8}\cdot\frac{5}{1}\cdot\frac{7}{3}\cdot\frac{9}{5}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}\cdot\frac{10}{6}$
$Q=5/2$	1	$\frac{16}{15\pi}\cdot\frac{6}{1}$	$1\cdot\frac{7}{2}$	$\frac{16}{15\pi}\cdot\frac{6}{1}\cdot\frac{8}{3}$	$1\cdot\frac{7}{2}\cdot\frac{9}{4}$	$\frac{16}{15\pi}\cdot\frac{6}{1}\cdot\frac{8}{3}\cdot\frac{10}{5}$	$1\cdot\frac{7}{2}\cdot\frac{9}{4}\cdot\frac{11}{6}$
$Q=3$	1	$\frac{5}{8}\cdot\frac{7}{1}$	$1\cdot\frac{8}{2}$	$\frac{5}{8}\cdot\frac{7}{1}\cdot\frac{9}{3}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}$	$\frac{5}{8}\cdot\frac{7}{1}\cdot\frac{9}{3}\cdot\frac{11}{5}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}\cdot\frac{12}{6}$

This table proved to be seminal for further research into the gamma function, beta integral and continued fractions, especially the second row in which $Q=1/2$. First Lord Brouncker, a colleague of Wallis, used it to obtain a sequence of continued fractions:

$$2A = 1 + \frac{1^2}{2 +} \frac{3^2}{2 +} \frac{5^2}{2 +} \cdots = \frac{4}{\pi},$$

$$2B = 3 + \frac{1^2}{6 +} \frac{3^2}{6 +} \frac{5^2}{6 +} \cdots = \pi,$$

$$2C = 5 + \frac{1^2}{10 +} \frac{3^2}{10 +} \frac{5^2}{10 +} \cdots = \frac{16}{\pi},$$

$$2E = 7 + \frac{1^2}{14 +} \frac{3^2}{14 +} \frac{5^2}{14 +} \cdots = \frac{9\pi}{4},$$

$$2F = 9 + \frac{1^2}{18 +} \frac{3^2}{18 +} \frac{5^2}{18 +} \cdots = \frac{256}{9\pi}.$$

Unfortunately, Brouncker never published his method of determining this sequence. He did give a partial explanation of his method of discovery to Wallis, and Wallis did present this incomplete description in his “Arithmetica Infinitorum”.

It is here that Euler begins E745. Euler is inspired by both Wallis’s table as well as the incomplete explanation of the continued fractions. Using both heavily, Euler then gives us his own derivation of Brouncker’s sequence as well as a generalization of both Brouncker’s continued fractions and Wallis’s product.

Section 1.

Euler refers to the book by John Wallis “Arithmetica Infinitorum” in which we find a sequence of continued fractions due to Lord Brouncker. In Wallis, these are given without a complete derivation and Euler wishes to provide this. Euler incorrectly states that Brouncker discovered only the first fraction in this sequence, and Wallis found the remaining. Modern investigations [5] give Brouncker credit for conjecturing the entire sequence.

Section 2.

Inspired by Wallis's table, Euler examines the list of integrals (all definite integrals from 0 to 1).

$$\int \frac{x\partial x}{\sqrt{1-xx}} = 1 = 1$$

$$\int \frac{x^3\partial x}{\sqrt{1-xx}} = \frac{2}{3} = \frac{2\cdot 2}{2\cdot 3}$$

$$\int \frac{x^5\partial x}{\sqrt{1-xx}} = \frac{2\cdot 4}{3\cdot 5} = \frac{2\cdot 2\cdot 4\cdot 4}{2\cdot 3\cdot 4\cdot 5}$$

$$\int \frac{x^7\partial x}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$$

$$\int \frac{x^9\partial x}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}, \text{ etc}\ldots$$

Sections 3 and 4.

Euler now wishes to find expressions, A, B, C, D, E so that

$AB = 1^2$, $BC = 2^2$, $CD = 3^2$, $DE = 4^2$, etc. Then he can write the above list of integrals as

$$\int \frac{x\partial x}{\sqrt{1-xx}} = 1 = \frac{1}{A}\cdot\frac{A}{1},$$

$$\int \frac{x^3\partial x}{\sqrt{1-xx}} = \frac{BC}{2\cdot 3} = \frac{1}{A}\cdot\frac{ABC}{1\cdot 2\cdot 3},$$

$$\int \frac{x^5 \partial x}{\sqrt{1-xx}} = \frac{BCDE}{2\cdot 3\cdot 4\cdot 5} = \frac{1}{A}\cdot\frac{ABCDE}{1\cdot 2\cdot 3\cdot 4\cdot 5},$$

$$\int \frac{x^7 \partial x}{\sqrt{1-xx}} = \frac{BCDEFG}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = \frac{1}{A}\cdot\frac{ABCDEFG}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}, \text{ etc...}$$

The above integrals have odd powers in the numerators. Euler now wishes to find expressions for integrals with even powers. For this purpose, he studies the third column of expressions above and "interpolates" to get

$$\int \frac{\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot 1,$$

$$\int \frac{xx\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot\frac{AB}{1\cdot 2},$$

$$\int \frac{x^4 \partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot\frac{ABCD}{1\cdot 2\cdot 3\cdot 4},$$

$$\int \frac{x^6 \partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot\frac{ABCDEF}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}, \text{ etc...}$$

This is a conjecture totally in the spirit of Wallis. For convenience, we also use the modern notation $L(0) = A$, $L(1) = B$, $L(2) = C$, etc.

<u>Section 5.</u>

Euler knows that $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$. He calls this value q. Euler lists these values and there numerical equivalents in a table.

Difference

$$A=\frac{1}{q}=0{,}636620$$

0,934176

$$B=q=1{,}570796$$

0,975683

$$C=\frac{4}{q}=2{,}546479$$

0,987813

$$D=\frac{9q}{4}=3{,}534292$$

0,992782

$$E=\frac{4\cdot 16}{9q}=4{,}527074$$

0,995257

$$F=\frac{9\cdot 25}{4\cdot 16}q=5{,}522331$$

Section 6.

Euler now hunts for *A, B, C, D, E, etc.* He states his problem in a more general form:

Problem: Given numbers *f* and *a* we wish to find *A*, *B*, *C*, *D*, *E*, *etc*. such that $AB=f^2$, $BC=(f+a)^2$, $CD=(f+2a)^2$, etc.

Section 7.

Euler begins by noticing that $\left(f-\frac{a}{2}\right)\left(f+\frac{a}{2}\right)=f^2-\frac{a^2}{4}$. Euler asks how he might increase these two factors on the LHS so that the product is exactly f^2. He starts by trying

(7.1) $A=f-\frac{a}{2}+\frac{s/2}{A'}$, $B=f+\frac{a}{2}+\frac{s/2}{B'}$,

where s, *A'* and *B'* are to be determined. After a series of manipulations he obtains

(7.2) $\left(2f - a + \frac{s}{A}\right)\left(2f + a + \frac{s}{B}\right) = 4f^2$,

(7.3) $(A' - 2f - a)(B' - 2f + a) = 4f^2$,

and

(7.4) $2A = 2f - a + \frac{a^2}{A'}$, and $2B = 2f + a + \frac{a^2}{B'}$.

Section 8.

Euler now seeks appropriate expressions for *A'* and *B'*. He reasons that a good start is

(8.1) $A' = 4f - 2a + \frac{s'}{A''}$, and $B' = 4f + 2a + \frac{s'}{B''}$,

where *s'* , *A''*, and *B''* are to be found. He finds that

(8.2) $\left(2f - 3a + \frac{s'}{A''}\right)\left(2f + 3a + \frac{s'}{B''}\right) = 4f^2$,

(8.3) $(A'' - 2f - 3a)(B'' - 2f + 3a) = 4f^2$,

and

(8.4) $A' = 4f - 2a + \frac{3^2 a^2}{A''}$, and $B' = 4f + 2a + \frac{3^2 a^2}{B''}$.

Section 9 and 10.

Now Euler seeks expressions for *A''* and *B''*. As he reasoned above to get (8.1) he tries

(9.1) $A'' = 4f - 2a + \frac{s''}{A'''}$ and $B'' = 4f + 2a + \frac{s''}{B'''}$,

and obtains

(9.2) $\left(2f-5a+\frac{s''}{A'''}\right)\left(2f+5a+\frac{s''}{B'''}\right)=4f^{2}$,

and

(9.3) $A''=4f-2a+\frac{5^2a^2}{A'''}$ and $B''=4f+2a+\frac{5^2a^2}{B'''}$.

Section 11.

The general pattern is now emerging and he can write the continued fractions as

$$2A=2f-a+\frac{a^2}{4f-2a}\,{}_{+}\,\frac{3^2a^2}{4f-2a}\,{}_{+}\,\frac{5^2a^2}{4f-2a}\,{}_{+}\cdots,$$

$$2B=2f+a+\frac{a^2}{4f+2a}\,{}_{+}\,\frac{3^2a^2}{4f+2a}\,{}_{+}\,\frac{5^2a^2}{4f+2a}\,{}_{+}\cdots.$$

$$2C=2f+3a+\frac{a^2}{4f+6a}\,{}_{+}\,\frac{3^2a^2}{4f+6a}\,{}_{+}\,\frac{5^2a^2}{4f+6a}\,{}_{+}\cdots,$$

$$2D=2f+5a+\frac{a^2}{4f+10a}\,{}_{+}\,\frac{3^2a^2}{4f+10a}\,{}_{+}\,\frac{5^2a^2}{4f+10a}\,{}_{+}\cdots,$$

$$2E=2f+7a+\frac{a^2}{4f+14a}\,{}_{+}\,\frac{3^2a^2}{4f+14a}\,{}_{+}\,\frac{5^2a^2}{4f+14a}\,{}_{+}\cdots$$

and so forth.

Section 12.

The continued fractions of Brouncker, listed in the introduction, are obtained as special cases from the above by writing $f=a=1$.

Section 13.

Euler reminds us how he previously derived the continued fraction

$2A=1+\frac{1^2}{2}\,{}_{+}\,\frac{3^2}{2}\,{}_{+}\,\frac{5^2}{2}\,{}_{+}\cdots=\frac{4}{\pi}$ from the Gregory Leibniz series $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$.

Section 14.

Euler now poses the a second problem:

Problem: Find representations for *A, B, C, …,* as infinite products and as definite integrals.

Section 15.

Since $A=\frac{f^2}{B}$, $B=\frac{(f+a)^2}{C}$, $C=\frac{(f+2a)^2}{D}$, etc., Euler gets

$$A=\frac{f^2}{B}=\frac{f^2}{\frac{(f+a)^2}{C}}=\frac{f^2C}{(f+a)^2}=\frac{f^2}{(f+a)^2}\frac{(f+2a)^2}{D}.$$

Continuing in this way he finds the infinite product

$$A=f\cdot\frac{f(f+a)}{(f+a)(f+a)}\cdot\frac{(f+2a)(f+4a)}{(f+3a)(f+3a)}\cdot\frac{(f+4a)(f+6a)}{(f+5a)(f+5a)}\cdot etc...$$

Section 16.

Euler has previously derived the following lemma:

Integrating from 0 to 1 we have

$$\int\frac{x^{m-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-k}}}=\frac{m+k}{m}\cdot\frac{m+k+n}{m+n}\cdot\frac{m+k+2n}{m+2n}\cdot\frac{m+k+3n}{m+3n}\cdot$$

$$\frac{m+k+4n}{m+4n}\ldots\int\frac{x^{\infty}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-k}}}.$$

(There seems to be a problem here. On the RHS, the factors in the infinite product approach one as they should, however the product itself diverges to infinity. The integral approaches zero because of x^{∞} in the numerator.)

Next Euler lets $n = 2a$, $m = f$ and $k = a$ in (16.2) to get

$$\int \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}} = \frac{f+a}{f}\cdot\frac{f+3a}{f+2a}\cdot\frac{f+5a}{f+4a}\ldots\int\frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}}$$

Next Euler replaces f by $f+a$ in the last expression to get

$$\int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}} = \frac{f+2a}{f+a}\cdot\frac{f+4a}{f+3a}\cdot\frac{f+6a}{f+5a}\ldots\int\frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}}.$$

(Here we see eighteenth century mathematics. In all three of the above expressions, the infinite products on the RHSs all diverge to infinity while the integrals tend to zero. No doubt Euler is aware of this, and the results he will obtain in the end are valid. In the notes to this paper we present a modern explanation.)

Section 17.

By simply dividing the infinite products obtained above, Euler gets the ratios of integrals

$$A = f\int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}}.$$

$$B = (f+a)\int \frac{x^{f+2a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}}$$

$$C = (f + 2a)\int \frac{x^{f+3a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f+2a-1}\partial x}{\sqrt{1-x^{2a}}} \quad \text{etc...}$$

(We note that the factors f, $f + a$, and $f + 2a$ are incorrectly omitted by Euler in his paper; a rare typographical omission. Later in section 27 when he generalizes this result, he has it corrected.)

Euler ends this section by stating his third problem:

Generalized Problem: Find expressions *A, B, C, D,* etc., such that $AB = f^2 + c$; $BC = (f+a)^2 + c$; $CD = (f+2a)^2 + c$**;** etc.

Section 18.

The solution is an alteration on the previous work. The following sections 19 to 23 are a very minor variation on sections 7 to 11.

Section 19.

Euler starts with

(19.1) $AB = f^2 + c$,

and assume that the desired increase is given by

$$A = f - \frac{a}{2} + \frac{s/2}{A'}, \qquad B = f + \frac{a}{2} + \frac{s/2}{B'},$$

where s, *A'* and *B'* are to be determined. Euler obains after some manipulation

$$\left(2f - a + \frac{s}{A}\right)\left(2f + a + \frac{s}{B}\right) = 4f^2 + 4c,$$

$$(A' - 2f - a)(B' - 2f + a) = 4f^2 + 4c,$$

and

$$2A = 2f - a + \frac{a^2 + 4c}{A'}, \text{ and} \qquad 2B = 2f + a + \frac{a^2 + 4c}{B'}.$$

Section 20.

Euler now wishes to determine appropriate expressions for *A'* and *B'* and starts with

$$A' = 4f - 2a + \frac{s'}{A''}, \text{ and } \quad B' = 4f + 2a + \frac{s'}{B''},$$

where *s'* , *A''*, and *B''* are to be found. He obtains

$$\left(2f - 3a + \frac{s'}{A''}\right)\left(2f + 3a + \frac{s'}{B''}\right) = 4f^2 + 4c,$$

$$(A'' - 2f - 3a)(B'' - 2f + 3a) = 4f^2 + 4c,$$

and

$$A' = 4f - 2a + \frac{3^2 a^2 + 4c}{A''}, \text{ and } \quad B' = 4f + 2a + \frac{3^2 a^2 + 4c}{B''}.$$

<u>Section 21 and 22.</u>

Next Euler seeks appropriate expressions for *A''* and *B''*. He tries

$$A'' = 4f - 2a + \frac{s''}{A'''} \text{ and } B'' = 4f + 2a + \frac{s''}{B'''}.$$

and obtains

$$\left(2f - 5a + \frac{s''}{A'''}\right)\left(2f + 5a + \frac{s''}{B'''}\right) = 4f^2 + 4c,$$

and by a similar reasoning

$$A'' = 4f - 2a + \frac{5^2 a^2 + 4c}{A'''} \text{ and } B'' = 4f + 2a + \frac{5^2 a^2 + 4c}{B'''}.$$

Summarizing he has

$$2A = 2f - a + \frac{a^2 + 4c}{A'}, \text{ and } \quad 2B = 2f + a + \frac{a^2 + 4c}{B'},$$

$$A' = 4f - 2a + \frac{3^2 a^2 + 4c}{A''}, \text{ and } \quad B' = 4f + 2a + \frac{3^2 a^2 + 4c}{B''},$$

$$A'' = 4f - 2a + \frac{5^2 a^2 + 4c}{A'''} \quad \text{and} \quad B'' = 4f + 2a + \frac{5^2 a^2 + 4c}{B'''}.$$

Section 23.

The general pattern is now emerging and he can easily combine the above to get

$$2A = 2f - a + \frac{a^2 + 4c}{4f - 2a +} \frac{3^2 a^2 + 4c}{4f - 2a +} \frac{5^2 a^2 + 4c}{4f - 2a +} \cdots,$$

$$2B = 2f + a + \frac{a^2 + 4c}{4f + 2a +} \frac{3^2 a^2 + 4c}{4f + 2a +} \frac{5^2 a^2 + 4c}{4f + 2a +} \cdots.$$

$$2C = 2f + 3a + \frac{a^2 + 4c}{4f + 6a +} \frac{3^2 a^2 + 4c}{4f + 6a +} \frac{5^2 a^2 + 4c}{4f + 6a +} \cdots,$$

$$2D = 2f + 5a + \frac{a^2 + 4c}{4f + 10a +} \frac{3^2 a^2 + 4c}{4f + 10a +} \frac{5^2 a^2 + 4c}{4f + 10a +} \cdots,$$

$$2E = 2f + 7a + \frac{a^2 + 4c}{4f + 14a +} \frac{3^2 a^2 + 4c}{4f + 14a +} \frac{5^2 a^2 + 4c}{4f + 14a +} \cdots$$

and so forth.

Section 24.

With $AB + f^2 + c$, $BC = (f + a)^2 + c$ $CD = (f + 2a)^2 + c$, $DE = (f + 3a)^2 + c$

Euler has

$$A = \frac{f^2 + c}{B},\ B = \frac{(f + a)^2 + c}{C},\ C = \frac{(f + 2a)^2 + c}{D}, \text{ etc., so}$$

$$A = \frac{f^2 + c}{B} = \frac{f^2 + c}{\frac{(f + a)^2 + c}{C}} = \frac{(f^2 + c)C}{(f + a)^2 + c} = \frac{(f^2 + c)}{\left((f + a)^2 + c\right)} \frac{(f + 2a)^2 + c}{D}.$$

Section 25.

The product emerging is

$$A=\frac{(f^2+c)}{\left((f+a)^2+c\right)}\cdot\frac{(f+2a)^2+c}{(f+3a)^2+c}\cdot\frac{(f+4a)^2+c}{(f+5a)^2+c}\cdot\frac{(f+6a)^2+c}{(f+7a)^2+c}\cdots.$$

This product will not converge as written. Squaring we can write

$$A^2=(ff+c)\cdot\frac{(ff+c)((f+2a)^2+c)}{((f+a)^2+c)((f+a)^2+c)}\cdot\frac{((f+2a)^2+c)((f+4a)^2+c)}{((f+3a)^2+c)((f+3a)^2+c)}\cdot etc...$$

which does converge.

Case 1 in which c = - bb

Section 26.

Replacing c by $-b^2$ in the previous general continued fraction Euler gets

$$2A=2f-a+\frac{a^2-4b^2}{4f-2a}\;+\;\frac{3^2a^2-4b^2}{4f-2a}\;+\;\frac{5^2a^2-4b^2}{4f-2a}\;+\cdots$$

which can also be written as

$$2A=2f-a+\cfrac{(a+2b)(a-2b)}{4f-2a+\cfrac{(3a+2b)(3a-2b)}{4f-2a+\cfrac{(5a+2b)(5a-2b)}{4f-2a+\cfrac{(7a+2b)(7a-2b)}{4f-2a+etc...}}}}$$

In a similar way the infinite product becomes

$$A=(f-b)\cdot\frac{(f+b)(f+2a-b)}{(f+a+b)(f+a-b)}\cdot\frac{(f+2a-b)(f+4a-b)}{(f+3a+b)(f+3a-b)}\cdot etc...$$

There is a misprint in the last factor. It should be $\dfrac{(f+2a+b)(f+4a-b)}{(f+3a+b)(f+3a-b)}$.

Section 27.

This section is a slight variation on section 16. Euler obtains

$$\frac{f+a+b}{f+b}\cdot\frac{f+3a+b}{f+2a+b}\cdot\frac{f+5a+b}{f+4a+b}\cdot\cdots\cdot\int\frac{x^{\infty}\partial x}{\sqrt{(1-x^{2a})}}=\int\frac{x^{f+b-1}\partial x}{\sqrt{1-x^{2a}}}.$$

And

$$\frac{f+2a-b}{f+a-b}\cdot\frac{f+4a-b}{f+3a-b}\cdot\frac{f+6a-b}{f+5a-b}\cdots\int\frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}}=\int\frac{x^{f+a-b-1}\partial x}{\sqrt{1-x^{2a}}}.$$

Dividing these he gets

$$A=(f-b)\cdot\int\frac{x^{f+a-b-1}\partial x}{\sqrt{1-x^{2a}}}:\int\frac{x^{f+b-1}\partial x}{\sqrt{1-x^{2a}}}.$$

Note that this time he has the correct factor $(f-b)$ which he neglected in section 17 when $b=0$.

<u>Section 28.</u>

Euler exhibits the special case of

$$2A=2f-a+\frac{a^2-4b^2}{4f-2a+}\frac{3^2a^2-4b^2}{4f-2a+}\frac{5^2a^2-4b^2}{4f-2a+}\cdots$$

From section 26 with $f=2$, $a=b=1$, he gets

$$2A=3-\frac{3}{6+}\frac{5}{6+}\frac{21}{6+}\frac{45}{6+}\frac{77}{6+}\cdots.$$

He can also write the infinite product

$$A=1\cdot\frac{3\cdot3}{2\cdot4}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{9\cdot9}{8\cdot10}\cdots.$$

He also writes A as a ratio of integrals.

(The remaining sections of this paper may have been included to show the usefulness of complex variables.)

Case 2 in which c = + bb

<u>Section 29.</u>

Now Euler sets $c=b^2$ in

$$2A = 2f - a + \frac{a^2+4c}{4f-2a+}\,\frac{3^2a^2+4c}{4f-2a+}\,\frac{5^2a^2+4c}{4f-2a+}\cdots,$$

to get

$$(29.1)\quad 2A = 2f - a + \frac{a^2+4b^2}{4f-2a+}\,\frac{3^2a^2+4b^2}{4f-2a+}\,\frac{5^2a^2+4b^2}{4f-2a+}\cdots.$$

In the infinite product from section 26

$$A = (f-b)\cdot\frac{(f+b)(f+2a-b)}{(f+a+b)(f+a-b)}\cdot\frac{(f+2a-b)(f+4a-b)}{(f+3a+b)(f+3a-b)}\cdot etc...$$

he can replace b by either $+bi$ or $-bi$ to get

(29.2)

$$A = (f-b\sqrt{-1})\cdot\frac{(f+b\sqrt{-1})(f+2a-b\sqrt{-1})}{(f+a+b\sqrt{-1})(f+a-b\sqrt{-1})}\cdot\frac{(f+2a+b\sqrt{-1})(f+4a-b\sqrt{-1})}{(f+3a+b\sqrt{-1})(f+3a-b\sqrt{-1})}\cdot etc...$$

and

(29.3)

$$A = (f+b\sqrt{-1})\cdot\frac{(f-b\sqrt{-1})(f+2a+b\sqrt{-1})}{(f+a-b\sqrt{-1})(f+a+b\sqrt{-1})}\cdot\frac{(f+2a-b\sqrt{-1})(f+4a+b\sqrt{-1})}{(f+3a-b\sqrt{-1})(f+3a+b\sqrt{-1})}\cdot etc...$$

Multiplying (29.2) times (29.3) he gets

$$A^2 = (ff+bb)\frac{(ff+bb)((f+2a)^2+bb)}{((f+a)^2+bb)((f+a)^2+bb}\cdot\frac{((f+2a)^2+bb)((f+4a)^2+bb)}{((f+3a)^2+bb)((f+3a)^2+bb)}\cdot etc...$$

which is the same as his product in section 25 with $c = b^2$.

<u>Section 30.</u>

Euler replaces b by bi in

$$A = (f-b)\cdot\int\frac{x^{f+a-b-1}\partial x}{\sqrt{1-x^{2a}}} : \int\frac{x^{f+b-1}\partial x}{\sqrt{1-x^{2a}}}.$$

from section 27 and he gets

$$(30.1)\quad A=(f-b\sqrt{-1})\int\frac{x^{f+a-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}:\int\frac{x^{f-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}.$$

Next he replaces b by $-bi$ to get

$$(30.2)\quad A=(f+b\sqrt{-1})\int\frac{x^{f+a-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}:\int\frac{x^{f-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}.$$

Section 31.

Euler multiplies (30.1) times (30.2) to get,

$$(31.1)\quad A^2=(ff+bb)\frac{\int\frac{x^{f+a-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}\cdot\int\frac{x^{f+a-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}}{\int\frac{x^{f-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}\cdot\int\frac{x^{f-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}}$$

Section 32.

In the denominator of (31.1) Euler sets $\frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}}=\partial V$ and gets

$$\int x^{+b\sqrt{-1}}\partial V\cdot\int x^{-b\sqrt{-1}}\partial V.$$

Now he makes the substitutions

The sum: $\int(x^{b\sqrt{-1}}+x^{-b\sqrt{-1}}\partial V=p$

Difference: $\int(x^{b\sqrt{-1}}-x^{-b\sqrt{-1}}\partial V=q$

and gets

$$(32.1)\quad \int(x^{b\sqrt{-1}}\partial V\cdot\int x^{-b\sqrt{-1}}\partial V=\frac{pp-qq}{4}$$

Section 33.

Euler sets $x=\exp(\log x)$ in p and q of the previous section to get

$$p = \int (e^{b\log x\sqrt{-1}} + e^{-b\log x\sqrt{-1}})\partial V,$$

$$q = \int (e^{b\log x\sqrt{-1}} - e^{-b\log x\sqrt{-1}})\partial V,$$

Using the identities

$$e^{\phi\sqrt{-1}} + e^{-\phi\sqrt{-1}} = 2\cos\phi \text{ and } e^{\phi\sqrt{-1}} - e^{-\phi\sqrt{-1}} = 2\sqrt{-1}\sin\phi$$

with $\phi = b\log x$ he has

$$p = 2\int \partial V\cos\phi \text{ and } q = 2\sqrt{-1}\int \partial V\sin\phi\,.$$

Using (32.1) he gets

(33.1) $$\frac{pp-qq}{4} = (\int \partial V\cos\phi)^2 + (\int \partial V\sin\phi)^2$$

which is the denominator of (31.1).

Section 34.

The same argument with numerator of (31.1) would have given Euler

$$(\int x^a\partial V\cos\phi)^2 + (\int x^a\partial V\sin\phi)^2$$

and thus (31.1) can be expressed as

(34.1) $$A^2 = (ff + bb)\frac{(\int x^a\partial V\cos\phi)^2 + (\int x^a\partial V\sin\phi)^2}{(\int \partial V\cos\phi)^2 + (\int \partial V\sin\phi)^2}$$

where we recall $\partial V = \dfrac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}}$ and $\phi = b\log x$.

Section 35.

Euler remarks briefly on the integrals

$$\int \frac{x^{f-1}\partial x\cos(b\log x)}{\sqrt{1-x^{2a}}} \quad \textit{and} \quad \int \frac{x^{f-1}\partial x\sin(b\log x)}{\sqrt{1-x^{2a}}}$$

Section 36.

Using integration by parts Euler gets

(36.1) $\int x^{f-1}\partial x \cos blx = \frac{x^f}{ff+bb}(f\cos(b\log x) + b\sin(b\log x));.$

and

(36.2) $\int x^{f-1}\partial x \sin(b\log x) = \frac{x^f}{ff+bb}(f\sin(b\log x) - b\cos(b\log x))$

References

[1] Euler, L., *De Fractionibus Continuis Wallisii*, (E745), on the web at the Euler Archive, http://www.math.dartmouth.edu/~euler/.

[2] Khrushchev, S., *A recovery of Brouncker's proof for the quadrature continued fractions,* Publicacions matermatiques, 50(2006), pp. 2-42.

[3] Nunn, T. Percy, *The Arithmetic of Infinites.: A School Introduction to the Integral Calculus*, (Part 1), The Mathematical Gazette, 5(1910), pp. 345-356.

[4] Nunn, T. Percy*, The Arithmetic of Infinites.: A School Introduction to the Integral Calculus*, (Part 2), The Mathematical Gazette, 5(1911), pp. 377-386.

[5] Stedall, Jacqueline A., *Catching Proteus: The Collaborations of Wallis and Brouncker. I. Squaring the Circle,* Notes and Records of the Royal Society of London, Vol. 54, No. 3, (Sep., 2000), pp. 293 -316

[6] Wallis, John, *The Arithmetic of Infinitesimals*, (Translated from Latin by Jacqueline A. Stedall), Springer Verlag, New York, 2004.

.