

Concerning life annuities*

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1. Having established the true principle for which one must base the calculation for life annuities, I believe that the development of this calculation will not fail to be very interesting, indeed as much for those who wish to run such an establishment as for those who will wish to profit from it. I worked on this matter in my “Recherches générales sur la mortalité et la multiplication du genre humain”, where I showed the correct method of determining by calculation how much a man of a given age must pay in order to enjoy for his whole life a specific annual income. However, since this calculation seemed to me at the time very cumbersome, I could not convince myself to carry it out. Then, on a given occasion I was forced to undertake this calculation, for which by means of certain tricks to abbreviate the calculation, I was fortunately able to complete it.

2. There are two things on which the calculation of the income from life annuities must be based: one of them is good data on mortality, which tells us for each age how many will probably die during the course of one or many years; the other is the way in which the manager can make the money grow in value that he will have received from the annuitants, or at which interest rates he is able to place it. Together, these two things essentially determine what returns the manager will be able to commit himself to, as much with respect to the amount initially deposited as with respect to the age of the

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annuitant, because it is evident that the more the profit the manager can obtain from the capital in his hands, the more they will also be able to provide the annuitant strong returns.

3. For the mortality list, the manager would no doubt risk much if they based it on the mortality of men in general which we compile from observations made in a big city or in a whole country, where we keep track equally of all men, vigorous and infirm. Now, when the business is to procure oneself life annuities, it is very natural that necessarily excluded are all those whose constitution does not seem to promise long life, thus there is reason to regard annuitants as a more robust species. It is also with this consideration that I chose in my “Mémoire allegué la liste de M. Kerseboom”, which he generated from observations made only on persons who enjoy life annuities: and using also this same list will assist me in the elaboration of the following calculations.

4. If the manager is not well able to place the capital that is given him by the annuitants, he would be able to afford naught very mediocre returns such that no one would want them. Once the city of Amsterdam paid ten percent annuities to all persons under the age of twenty, that is for 1000 florins they are paid 100 per year, which is an annuity so rich that the city would have suffered considerable loss if it had not generated almost 10 percent per year of the fund which this enterprise had provided it. Thus, if we can only count on 5 percent interest, the annuities must become far less considerable; however, it is on this which it seems annuities must be fixed, provided that those who will have occasion to obtain greater profit from it will hardly participate in such an enterprise, which could only be worthwhile after a large number of years.

5. For determining the cost of these annuities, we fix for each age an average lifespan, which one is equally likely to survive as to die before having reached it; that is, this term is chosen such that as many men of the same age die before this term as die after. Thus we presume that all men of this age reach exactly this term, and that they then die; on this we believe we can fix with certainty the price of the annuities, since the value of the annuities must be payable during a given number of consecutive years: and we estimate that the profit that the manager obtains on the side of those who die before their projected term is exactly compensated by the loss caused by those annuitants who survive this term. However, we will easily understand

that this reasoning is flawed, since it does not take into account the reduction in current price of an annuity which will not be paid out before many years. Given this circumstance, it will be necessary to base the calculation on true principles, as I explained in my aforementioned “Mémoire”, without making use of any reasoning which might seem suspect.

6. To achieve this, we consider a number of 1000 children born at the same time, and that the symbols (1), (2), (3), (4), etc. indicate the number of those who still live at the end of 1, 2, 3, 4, etc. years, such that in general (m) represents the number of those who will obtain the age of m years. Now let r be the annual annuity that a man of m years wishes to receive, and let x be the price which he must then pay at the present to the manager, which must be a just equivalent of the spending to which the manager engages himself by this agreement. To determine this price x , many men of the same age m must be considered, and those who reach this age. Let (m) be this number of men, and the sum they will presently pay to the manager will be equal to $(m)x$, which must be sufficient to provide for all the annuities which he will have to pay afterwards.

7. Of these (m) men, there will remain alive after one year $(m + 1)$, after two years $(m + 2)$, after three years $(m + 3)$, and so on. Thus the manager will have to pay after one year $(m + 1)r$, after two years $(m + 2)r$, after three years $(m + 3)r$, etc., until all of these annuitants will be extinguished. We thus only have to reduce all of the payments at the present time by the amount of 5 percent, and make the sum equal to $(m)x$ to determine the value of x . Now to make the calculation more general, instead of $\frac{105}{100}$ or $\frac{21}{20}$, let us write the letter λ , and the sum of all the annuities which the manager must pay successively will now be:

$$\frac{(m + 1)r}{\lambda} + \frac{(m + 2)r}{\lambda^2} + \frac{(m + 3)r}{\lambda^3} + \frac{(m + 4)r}{\lambda^4} + \text{etc.}$$

which being equal to $(m)x$, will give:

$$x = \frac{r}{(m)} \left(\frac{(m + 1)}{\lambda} + \frac{(m + 2)}{\lambda^2} + \frac{(m + 3)}{\lambda^3} + \frac{(m + 4)}{\lambda^4} + \text{etc.} \right)$$

8. Then the exact price is found which a man of m years must pay in order to enjoy an annual annuity r during his whole life, this one having been

initially placed at 5 percent, puts the manager precisely within the means of paying from that point the annuities as long of the number of annuitants is sufficiently large. We understand well, having thus placed initially all the capital which the manager will have received, the following year the interest will not be sufficient to pay the annuities but that it will be needed to employ part of the capital, hence the capital will suffer every year a diminution: however, it will only be entirely extinguished when the annuitants are dead. For this reason the manager will be well obliged to raise the price of the annuities that I have just found, according to the particular circumstances and expenses which such an establishment requires.

9. We clearly see that the determination of this price called x requires a calculation as tedious as it is unpleasant, especially for low ages, where the number of terms to be added together is very considerable. But it is not hard to notice, that having already done a calculation for a certain age, we will from it be easily able to extract the one which corresponds to a later or earlier year. To better explain this artifice, I will employ this character $\overline{m}r$ to indicate the price which a man of age m must pay for the annuity r : in order that

$$\overline{m} = \frac{1}{(m)} \left(\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \frac{(m+3)}{\lambda^3} + \frac{(m+4)}{\lambda^4} + \text{etc.} \right),$$

from there, for men aged $m+1$ years we will have,

$$\overline{m+1} = \frac{1}{(m+1)} \left(\frac{(m+2)}{\lambda} + \frac{(m+3)}{\lambda^2} + \frac{(m+4)}{\lambda^3} + \frac{(m+5)}{\lambda^4} + \text{etc.} \right),$$

from which we conclude:

$$\lambda(m)\overline{m} = (m+1) + (m+1)\overline{m+1},$$

and starting,

$$\overline{m} = \frac{1}{\lambda} \cdot \frac{(m+1)}{(m)} (1 + \overline{m+1}),$$

such that having found the value $\overline{m+1}$, we will from it calculate easily enough the value of \overline{m} .

10. With the aid of this artifice, after having started with the age of 90 years, I calculated the price of an annuity r successively for all inferior ages,

down to those children newly born; where I obtained the following table, by fixing the annuity r at 100 crowns, and interest at 5 percent.

Table

Table which indicates the price of a life annuity of 100 crowns for all ages

age in years	number of survivors	price of an- nuity	age in years	number of survivors	price of an- nuity
0	1000	1155.50	25	552	1403.60
1	804	1409.04	26	544	1395.45
2	768	1448.84	27	535	1389.87
3	736	1487.43	28	525	1387.16
4	709	1521.27	29	516	1382.54
5	690	1541.32	30	507	1376.82
6	676	1551.90	31	499	1368.84
7	664	1558.94	32	490	1363.68
8	654	1561.92	33	482	1355.63
9	646	1560.33	34	475	1344.38
10	639	1556.29	35	468	1332.71
11	633	1549.59	36	461	1320.60
12	627	1542.64	37	454	1308.01
13	621	1535.42	38	446	1298.04
14	616	1525.28	39	439	1284.67
15	611	1514.65	40	432	1270.76
16	606	1503.50	41	426	1253.09
17	601	1491.81	42	420	1234.54
18	596	1479.54	43	413	1218.24
19	590	1469.31	44	406	1201.21
20	584	1458.63	45	400	1180.19
21	577	1450.18	46	393	1161.27
22	571	1438.68	47	386	1141.44
23	565	1426.66	48	378	1123.88
24	559	1414.07	49	370	1105.59
25	552	1403.60	50	362	1086.52
m	(m)	\overline{m}	m	(m)	\overline{m}

age in years	number of survivors	price of an- nuity	age in years	number of survivors	price of an- nuity
50	362	1086.52	70	175	638.30
51	354	1066.62	71	165	610.83
52	345	1049.17	72	155	582.75
53	336	1031.14	73	145	554.09
54	327	1012.49	74	135	524.89
55	319	989.78	75	125	495.22
56	310	969.44	76	114	470.16
57	301	948.35	77	104	441.13
58	291	929.98	78	93	417.98
59	282	907.64	79	82	397.75
60	273	884.44	80	72	375.64
61	264	860.32	81	63	350.77
62	254	838.90	82	54	329.69
63	245	813.21	83	46	309.38
64	235	790.20	84	39	279.44
65	225	766.59	85	32	257.60
66	215	742.30	86	26	232.90
67	205	717.43	87	20	217.91
68	195	691.93	88	15	205.07
69	185	665.14	89	11	193.62
70	175	638.30	90	8	179.54
m	(m)	\bar{m}	m	(m)	\bar{m}

11. M. Kerseboom only continued the table on mortality up to 95 years, and for this reason I did not judge it convenient to continue this one beyond 90 years, since probably at this age nobody will desire life annuities. At the least, in almost all plans, such ancients find themselves filed into the same class as those of 60 or 70 years, notwithstanding that it would be very unjust if we wanted to demand of a nonagenarian more than a third of the price which a septuagenarian must pay or a quarter of what a sexagenarian must pay. However, if we are curious to see the continuation of my table, here it is:

m	90	91	92	93	94
(m)	8	6	4	3	2
\bar{m}	179.54	151.35	138.38	93.73	47.62

But I would not advise a manager to get involved with such ancients unless their number be sufficiently considerable; which is a general rule for all establishments founded on probabilities.

12. From there we will conclude easily how much the manager should pay in interest for each age for a given sum which we will have initially given. It is not necessary to enter here in as much detail, and it suffices to mark for every five years which the annuitants might expect.

age	percent	age	percent	age	percent
0	$8\frac{2}{3}$	30	$7\frac{1}{4}$	60	$11\frac{1}{3}$
5	$6\frac{1}{2}$	35	$7\frac{1}{2}$	65	13
10	$6\frac{1}{3}$	40	8	70	$15\frac{2}{3}$
15	$6\frac{1}{2}$	45	$8\frac{1}{2}$	75	20
20	$6\frac{3}{4}$	50	9	80	$25\frac{2}{3}$
25	7	55	10	85	$38\frac{1}{2}$
30	$7\frac{1}{4}$	60	$11\frac{1}{3}$	90	$55\frac{1}{2}$

On this basis, the manager will obtain no profit unless he is able to appreciate his money at more than 5 percent.

13. Thus, if a state has need of money and it can find at better than 5 percent interest as much as it needs, then it would be assuredly very bad if it wished to establish such life annuities that I have just determined on this basis of 5 percent, since, with regard to the burden which such an establishment necessarily imposes, it would always be better to borrow the sum which it needs at 5 percent which it would then be able to satisfy according to circumstance, in the place of life annuities which would remain in its charge during a very long period. Or else, the price on the annuities would have to be raised beyond that which I have just described to bring it any benefit, but then it might be difficult to find any more annuitants unless it be ancients over 60 years old, who could be amazed by interests of 10 percent or more.

14. But wanting to establish more advantageous life annuities for annuitants, it would be a project hardly proper for satisfying a state, since this would return to the same, then if one wanted to burden one's self with debts at 6 or more percent: while we could borrow at 5 percent without being subjected to the trouble which life annuities would incur. Indeed, if a state wants to establish the life annuities described here and calculated on the

basis of 5 percent, this charge need only be regarded as a loan taken at 6 percent, because of the number of arrangements which would be required for it. Thus I hardly see a case where the establishment of life annuities can be advantageous to a state, as long as we can borrow money at 5 percent and perhaps less. But we can imagine a different type annuity which may be better tailored, although still based on 5 percent. I want to speak of annuities which must not begin to run before 10 or 20 years; and we easily understand that the price of such annuities will be quite mediocre, capable of attracting the public only poorly.

15. Consider this question also in general, and ask how much a man of age m years must pay presently to obtain an annual annuity r which will not begin to be paid to him before n years, such that after this time he is able to enjoy it regularly until his death. Let x be the current price of this annuity, and we will find as here below:

$$x = \frac{r}{(m)} \left(\frac{(m+n)}{\lambda^n} + \frac{(m+n+1)}{\lambda^{n+1}} + \frac{(m+n+2)}{\lambda^{n+2}} + \text{etc.} \right).$$

Thus to calculate the previous explained ordinary annuities, we will have:

$$\overline{m+n-1} = \frac{1}{(m+n+1)} \left(\frac{(m+n)}{\lambda} + \frac{(m+n+1)}{\lambda^2} + \frac{(m+n+2)}{\lambda^3} + \text{etc.} \right),$$

from which we conclude,

$$x = \frac{r}{(m)} \cdot \frac{(m+n-1)}{\lambda^{n-1}} \cdot \overline{m+n-1} = \frac{r}{\lambda^{n-1}} \cdot \frac{(m+n-1)}{(m)} \cdot \overline{m+n-1},$$

where $\overline{m+n-1}r$ expresses the current price of the ordinary annuity for a man of age $m+n-1$ years.

16. Thus if we ask the current price of an annual annuity of 100 crowns which will not begin to be paid for 10 years, for a man of age m years we will take from the table developed in §10 the price of the ordinary annuity which is suited to the age $m+9$ years, and we will multiply by $\left(\frac{20}{21}\right)^9 \frac{(m+9)}{(m)}$ to obtain the value sought for x . From there I calculated the following tables for every 5 years:

Table

Prices of life annuities of 100 crowns which will not start to run for 10

<i>years</i>					
age years	price of the annnuity	age year	price of the annuity	age year	price of the annuity
0	649.75	30	717.05	60	290.55
5	877.77	35	671.73	65	203.11
10	874.50	40	610.40	70	120.14
15	833.95	45	533.55	75	56.20
20	787.43	50	455.78	80	19.07
25	745.72	55	375.25		
30	717.05	60	290.55		

Table

Price of a life annuity of 100 crowns will not begin to run for 20 years

age year	price of the annuity	age year	price of the annuity	age year	price of the annuity
0	343.06	30	319.30	60	47.28
5	453.36	35	272.96	65	19.17
10	441.81	40	234.47	70	4.82
15	413.60	45	183.72		
20	382.17	50	134.52		
25	349.63	55	87.91		
30	319.30	60	47.28		

17. Maybe such a project of life annuities will better succeed in spite of being fixed on a basis of 5 percent. It seems that it will always be advantageous for a newborn child to ensure it, assuming a price of 343 or 350 crowns, a fixed annuity of 100 crowns per year, though it will not start to be paid before the child attains the age of 20: and if we wish to use the sum of 3500 crowns, it will always be a nice establishment to be able to enjoy from the age of 20 years a fixed pension of 1000 crowns. However, it is doubtful that there will be many parents who will well be willing to make such a sacrifice for the good of their children. Perhaps there will be more men of 60 years who will not mind paying first 3000 crowns to be assured a fixed pension of 1000 crowns per year once they have passed their 70-th year.