Reflections and Notes on Euler's paper E745:

De fractionibus continuis Wallisii

(On the continued fractions of Wallis)

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Section 1.

Euler refers to the book by John Wallis "Arithmetica Infinitorum" in which we find a sequence of continued fractions due to Lord Brouncker. These are given without a complete derivation and Euler wishes to provide this. Euler states that Brouncker discovered the first continued fraction in this sequence and Wallis discovered the remaining fractions. This is contrary to recent investigations [5] by Jacqueline A. Stedall which show that Brouncker obtained the entire sequence himself, but never published it. It was Wallis who made the sequence known to the world. The importance of Euler's paper is that he finds a nice derivation and generalization of this sequence for the first time.

Section 2.

Euler examines the list of integrals

$$\int \frac{x \partial x}{\sqrt{1-xx}} = 1 = 1$$

$$\int \frac{x^3 \partial x}{\sqrt{1-xx}} = \frac{2}{3} = \frac{2 \cdot 2}{2 \cdot 3}$$

$$\int \frac{x^5 \partial x}{\sqrt{1-xx}} = \frac{2\cdot 4}{3\cdot 5} = \frac{2\cdot 2\cdot 4\cdot 4}{2\cdot 3\cdot 4\cdot 5}$$

$$\int \frac{x^7 \partial x}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$$

$$\int \frac{x^9 \partial x}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}$$, etc…

Using the substitution $t^2 = 1 - x^2$ we can convert these integrals to the form

$\int_0^1 \frac{x^P dx}{\sqrt{1-x^2}} = \int_0^1 (1-t^2)^P dt$. In Wallis, we find a table of the reciprocal integrals

$1/\int_0^1 (1-x^{1/Q})^P dx$. Wallis after much work obtains the following table:

	$P=0$	$P=1/2$	$P=1$	$P=3/2$	$P=2$	$P=5/2$	$P=3$
$Q=0$	1	$1\cdot\frac{1}{1}$	$1\cdot\frac{2}{2}$	$1\cdot\frac{1}{1}\cdot\frac{3}{3}$	$1\cdot\frac{2}{2}\cdot\frac{4}{4}$	$1\cdot\frac{1}{1}\cdot\frac{3}{3}\cdot\frac{5}{5}$	$1\cdot\frac{2}{2}\cdot\frac{4}{4}\cdot\frac{6}{6}$
$Q=1/2$	1	$\frac{2}{\pi}\cdot\frac{2}{1}$	$1\cdot\frac{3}{2}$	$\frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{4}{3}$	$1\cdot\frac{3}{2}\cdot\frac{5}{4}$	$\frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}$	$1\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}$
$Q=1$	1	$\frac{1}{2}\cdot\frac{3}{1}$	$1\cdot\frac{4}{2}$	$\frac{1}{2}\cdot\frac{3}{1}\cdot\frac{5}{3}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}$	$\frac{1}{2}\cdot\frac{3}{1}\cdot\frac{5}{3}\cdot\frac{7}{5}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}\cdot\frac{8}{6}$
$Q=3/2$	1	$\frac{4}{3\pi}\cdot\frac{4}{1}$	$1\cdot\frac{5}{2}$	$\frac{4}{3\pi}\cdot\frac{4}{1}\cdot\frac{6}{3}$	$1\cdot\frac{5}{2}\cdot\frac{7}{4}$	$\frac{4}{3\pi}\cdot\frac{4}{1}\cdot\frac{6}{3}\cdot\frac{8}{5}$	$1\cdot\frac{5}{2}\cdot\frac{7}{4}\cdot\frac{9}{6}$
$Q=2$	1	$\frac{3}{8}\cdot\frac{5}{1}$	$1\cdot\frac{6}{2}$	$\frac{3}{8}\cdot\frac{5}{1}\cdot\frac{7}{3}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}$	$\frac{3}{8}\cdot\frac{5}{1}\cdot\frac{7}{3}\cdot\frac{9}{5}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}\cdot\frac{10}{6}$
$Q=5/2$	1	$\frac{16}{15\pi}\cdot\frac{6}{1}$	$1\cdot\frac{7}{2}$	$\frac{16}{15\pi}\cdot\frac{6}{1}\cdot\frac{8}{3}$	$1\cdot\frac{7}{2}\cdot\frac{9}{4}$	$\frac{16}{15\pi}\cdot\frac{6}{1}\cdot\frac{8}{3}\cdot\frac{10}{5}$	$1\cdot\frac{7}{2}\cdot\frac{9}{4}\cdot\frac{11}{6}$
$Q=3$	1	$\frac{5}{8}\cdot\frac{7}{1}$	$1\cdot\frac{8}{2}$	$\frac{5}{8}\cdot\frac{7}{1}\cdot\frac{9}{3}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}$	$\frac{5}{8}\cdot\frac{7}{1}\cdot\frac{9}{3}\cdot\frac{11}{5}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}\cdot\frac{12}{6}$

The list of integrals that Euler is studying, $\int_0^1 \frac{x^P dx}{\sqrt{1-x^2}}$, are from the row where $Q = 1/2$.

Sections 3 and 4.

Euler now wishes to find expressions, *A, B, C, D, E* so that

$AB = 1^2$, $BC = 2^2$, $CD = 3^2$, $DE = 4^2$, etc. Then he can write the above list of integrals as

$$\int \frac{x\partial x}{\sqrt{1-xx}} = 1 = \frac{1}{A}\cdot\frac{A}{1},$$

$$\int \frac{x^3\partial x}{\sqrt{1-xx}} = \frac{BC}{2\cdot 3} = \frac{1}{A}\cdot\frac{ABC}{1\cdot 2\cdot 3},$$

$$\int \frac{x^5\partial x}{\sqrt{1-xx}} = \frac{BCDE}{2\cdot 3\cdot 4\cdot 5} = \frac{1}{A}\cdot\frac{ABCDE}{1\cdot 2\cdot 3\cdot 4\cdot 5},$$

$$\int \frac{x^7\partial x}{\sqrt{1-xx}} = \frac{BCDEFG}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = \frac{1}{A}\cdot\frac{ABCDEFG}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}, \text{ etc}\ldots$$

The above integrals have odd powers in the numerators. Euler now wishes to find expressions for integrals with even powers. For this purpose, he studies the third column of expressions above and "interpolates" to get

$$\int \frac{\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot 1,$$

$$\int \frac{xx\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot\frac{AB}{1\cdot 2},$$

$$\int \frac{x^4 \partial x}{\sqrt{1-xx}} = \frac{1}{A} \cdot \frac{ABCD}{1 \cdot 2 \cdot 3 \cdot 4},$$

$$\int \frac{x^6 \partial x}{\sqrt{1-xx}} = \frac{1}{A} \cdot \frac{ABCDEF}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}, \text{ etc}\ldots$$

This is a conjecture totally in the spirit of Wallis.

For convenience, we also use the modern notation $L(0) = A$, $L(1) = B$, $L(2) = C$, etc.

Section 5.

Euler knows that $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$. He calls this value *q*. From the first integral above we see that $q = \frac{1}{A}$. Since $AB = 1$ we now have $B = \frac{1}{A} = q$. From $BC = 4$ we get $C = \frac{4}{B} = \frac{4}{q}$, etc. Euler lists these values and there numerical equivalents in a table.

	Difference
$A = \frac{1}{q} = 0{,}636620$	
	0,934176
$B = q = 1{,}570796$	
	0,975683
$C = \frac{4}{q} = 2{,}546479$	
	0,987813
$D = \frac{9q}{4} = 3{,}534292$	
	0,992782

$$E = \frac{4 \cdot 16}{9q} = 4{,}527074$$

0,995257

$$F = \frac{9 \cdot 25}{4 \cdot 16} q = 5{,}522331$$

Euler now makes the following troubling statement:

"Here I attached a third column, which exhibits the numeric values of these letters to show more clearly the extent that these numbers increase according to the law of uniformity. This does not happen if I take a false value in the place of *q*."

Just what is this *law of uniformity*? Euler hints that the value $q = \frac{2}{\pi}$ has a special value. Somehow it creates some *uniform* flow to the numbers A, B, C as they increase.

We can start with *any* value for *A*, then all the remaining letters are determined by this choice. However, there is something special about the choice $q = \frac{2}{\pi}$. Let us study this as apparently Euler did. (We use our notation: $L(0) = A$, $L(1) = B$, $L(2) = C$, etc.). We calculate the following:

$$L(2) = \frac{1^2}{L(1)}$$

$$L(3) = \frac{2^2}{L(2)} = \frac{2^2 L(1)}{1^2}$$

$$L(4) = \frac{3^2}{L(3)} = \frac{1^2 \cdot 3^2}{2^2 L(1)}$$

$$L(5) = \frac{4^2}{L(4)} = \frac{2^2 \cdot 4^2 L(1)}{1^2 \cdot 3^2}$$

$$L(6) = \frac{5^2}{L(5)} = \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 L(1)}$$

Let us denote the partial product in the Wallis product as

$$W(n)=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots\frac{(2n-1)(2n+1)}{(2n)(2n)}\ .$$

Then from the above calculations we can write in general

$$L(2n)=\frac{1^2\cdot 3^2\cdot 5^2\cdots(2n-1)^2}{2^2\cdot 4^2\cdot 6^2\cdots(2n-2)^2\,L(1)}=W(n-1)\frac{(2n-1)}{L(1)}\text{, and}$$

$$L(2n+1)=\frac{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2\,L(1)}{1^2\cdot 3^2\cdot 5^2\cdots(2n-1)^2}=\frac{(2n+1)L(1)}{W(n)}\ .$$

Since $\lim W(n)=\frac{2}{\pi}$ we get the two limits

(5.1) $\lim\frac{L(2n)}{2n-1}=\lim\frac{L(2n)}{2n}=\frac{2}{\pi L(1)}$ and

(5.2) $\lim\frac{L(2n+1)}{2n+1}=\frac{\pi L(1)}{2}$.

From these last two results we see that $\lim\frac{L(n)}{n}$ exists and equals 1 if and only if $L(1)=\frac{2}{\pi}$. Otherwise, if $L(1)\neq\frac{2}{\pi}$, the limit does not exist, but the separate limits through even and odd values on n do exist. Thus it is clear that the choice $q=A=L(1)=\frac{2}{\pi}$ is very special.

Euler will work intuitively in the next sections to move “uniformly” from A to B to C and so on in the hope of getting the right result. Later in the paper, he will demonstrate that his intuitive guess is correct.

Section 6.

Euler now hunts for *A, B, C, D, E.* He states his problem in a more general form:

Problem:

To find a series of letters A, B, C, D, etc…progressing by the law of uniformity in such a way that AB =*ff*; BC= $(f+a)^2$; CD= $(f+2a)^2$; etc.

Given *f* and *a* we wish to have $AB = f^2$, $BC = (f+a)^2$, $CD = (f+2a)^2$, etc.

Section 7.

Euler begins with the problem of finding expressions *A, B, C, D,* etc., such that $AB = f^2$; $BC = (f+a)^2$; $CD = (f+2a)^2$**;** etc.

Noticing that $\left(f-\frac{a}{2}\right)\left(f+\frac{a}{2}\right) = f^2 - \frac{a^2}{4}$, Euler asks how he might increase these two factors on the LHS so that the product is exactly f^2.

We start with

(7.1) $AB = f^2$,

and assume that the desired increase is given by

$$A = f - \frac{a}{2} + \frac{s/2}{A'}, \qquad B = f + \frac{a}{2} + \frac{s/2}{B'},$$

where s, *A'* and *B'* are to be determined. Multiplying by 2 we get

(7.2) $2A = 2f - a + \frac{s}{A'}, \qquad 2B = 2f + a + \frac{s}{B'}.$

Multiplying the above together we get using (7.1)

(7.3) $\left(2f - a + \frac{s}{A'}\right)\left(2f + a + \frac{s}{B'}\right) = 4f^2,$

which can also be written as

$$4AB = 4f^2 - a^2 + \frac{s}{B'}(2f - a) + \frac{s}{A'}(2f + a) + \frac{s^2}{A'B'}.$$

Since $4AB = 4f^2$ we have

$$a^2 = \frac{s}{B'}(2f - a) + \frac{s}{A'}(2f + a) + \frac{s^2}{A'B'},$$

and multiplying by $A'B'$ we get

(7.4) $\quad a^2 A'B' = A's(2f - a) + B's(2f + a) + s^2.$

We are free to choose *s* as we like, and a reasonable choice to simplify (7.4) is

(7.5) $\quad s = a^2.$

After we divide (7.4) by a^2 we get

$$A'B' = A'(2f - a) + B'(2f + a) + a^2.$$

Notice that this last expression can be written as

$$(A' - (2f + a))(B' - (2f - a)) = 4f^2$$

or

(7.6) $\quad (A' - 2f - a)(B' - 2f + a) = 4f^2.$

We also notice that because of (7.5), (7.2) has become the important equations

(7.7) $\quad 2A = 2f - a + \frac{a^2}{A'}$, and $\qquad 2B = 2f + a + \frac{a^2}{B'}.$

<u>Section 8.</u>

We now wish to determine appropriate expressions for *A'* and *B'*. We see from (7.6) that these expressions should start as $A' = 4f + \cdots$ and $B' = 4f + \cdots$. How does a fit with $4f$? Since we have been examining the three equally spaced numbers $f - \frac{a}{2}$, f, and $f+\frac{a}{2}$, it seems reasonable to think of $4f-2a$, $4f$, and $4f+2a$. Thus we now try the equations

(8.1) $$A' = 4f - 2a + \frac{s'}{A''}, \text{ and } B' = 4f + 2a + \frac{s'}{B''},$$

where *s'* , *A''*, and *B''* are to be found. Using (8.1) to remove *A'* and *B'* from (7.6) we get

(8.2) $$\left(2f - 3a + \frac{s'}{A''}\right)\left(2f + 3a + \frac{s'}{B''}\right) = 4f^2.$$

Now compare (8.2) with (7.3). If in (7.3) we replace $s \to s'$, $A' \to A''$, $B' \to B''$, and $a \to 3a$ we get (8.2). Therefore (7.4) becomes after these substitutions

(8.3) $$3^2 a^2 A'' B'' = A'' s'(2f - 3a) + B'' s'(2f + 3a) + s'^2.$$

We are free to select a value for *s'* and a nice choice to simplify (8.3) is

(8.4) $$s' = 3^2 a^2.$$

Dividing (8.3) by by *s'* get

$$A'' B'' = A''(2f - 3a) + B''(2f + 3a) + 3^2 a^2.$$

Corresponding to (7.6), this last expression can be written as

(8.5) $$(A'' - 2f - 3a)(B'' - 2f + 3a) = 4f^2.$$

Notice that equations (8.1) are now the important relations

(8.6) $$A' = 4f - 2a + \frac{3^2 a^2}{A''}, \text{ and } B' = 4f + 2a + \frac{3^2 a^2}{B''}.$$

<u>Section 9 and 10.</u>

Next we seek appropriate expressions for *A''* and *B''*. As we reasoned above to get (8.1) we now try

(9.1) $$A'' = 4f - 2a + \frac{s''}{A'''} \text{ and } B'' = 4f + 2a + \frac{s''}{B'''}.$$

Using (9.1) to remove *A''* and *B''* from (8.5) we now have (corresponding to (8.2))

(9.2) $$\left(2f-5a+\frac{s''}{A'''}\right)\left(2f+5a+\frac{s''}{B'''}\right)=4f^2,$$

and by a similar reasoning we will arrive at

(9.3) $$A''=4f-2a+\frac{5^2a^2}{A'''} \quad\text{and}\quad B''=4f+2a+\frac{5^2a^2}{B'''}.$$

Summarizing our results for (7.7), (8.6) and (9.3) we see

$$2A=2f-a+\frac{a^2}{A'},\ \text{and}\qquad 2B=2f+a+\frac{a^2}{B'},$$

$$A'=4f-2a+\frac{3^2a^2}{A''},\ \text{and}\quad B'=4f+2a+\frac{3^2a^2}{B''},$$

$$A''=4f-2a+\frac{5^2a^2}{A'''} \quad\text{and}\quad B''=4f+2a+\frac{5^2a^2}{B'''}.$$

Section 11.

The general pattern is now emerging and we can write our continued fractions as

(11.1) $$2A=2f-a+\frac{a^2}{4f-2a\,+}\;\frac{3^2a^2}{4f-2a\,+}\;\frac{5^2a^2}{4f-2a\,+}\;\cdots,$$

and

(11.2) $$2B=2f+a+\frac{a^2}{4f+2a\,+}\;\frac{3^2a^2}{4f+2a\,+}\;\frac{5^2a^2}{4f+2a\,+}\;\cdots.$$

Notice that (11.2) is obtained from (11.1) by replacing f by $f+a$. In a similar way, replacing f by $f+2a$ in (11.1) gives us

$$2C=2f+3a+\frac{a^2}{4f+6a\,+}\;\frac{3^2a^2}{4f+6a\,+}\;\frac{5^2a^2}{4f+6a\,+}\;\cdots,$$

replacing f by $f+3a$ in (11.1) yields

$$2D = 2f + 5a + \frac{a^2}{4f+10a\,+}\;\frac{3^2a^2}{4f+10a\,+}\;\frac{5^2a^2}{4f+10a\,+}\;\cdots,$$

and replacing f by $f+4a$ in (11.1) yields

$$2E = 2f + 7a + \frac{a^2}{4f+14a\,+}\;\frac{3^2a^2}{4f+14a\,+}\;\frac{5^2a^2}{4f+14a\,+}\;\cdots$$

and so forth.

Section 12.

The continued fractions of Brouncker are obtained from the above by writing $f = a = 1$. We get

$$2A = 1 + \frac{1^2}{2\,+}\;\frac{3^2}{2\,+}\;\frac{5^2}{2\,+}\;\cdots = \frac{4}{\pi},$$

$$2B = 3 + \frac{1^2}{6\,+}\;\frac{3^2}{6\,+}\;\frac{5^2}{6\,+}\;\cdots = \pi,$$

$$2C = 5 + \frac{1^2}{10\,+}\;\frac{3^2}{10\,+}\;\frac{5^2}{10\,+}\;\cdots = \frac{16}{\pi},$$

$$2E = 7 + \frac{1^2}{14\,+}\;\frac{3^2}{14\,+}\;\frac{5^2}{14\,+}\;\cdots = \frac{9\pi}{4},$$

$$2F = 9 + \frac{1^2}{18\,+}\;\frac{3^2}{18\,+}\;\frac{5^2}{18\,+}\;\cdots = \frac{256}{9\pi}.$$

Section 13.

Euler remind us that he previously derived $2A = 1 + \frac{1^2}{2\,+}\;\frac{3^2}{2\,+}\;\frac{5^2}{2\,+}\;\cdots = \frac{4}{\pi}$ from the Gregory Leibniz series

(13.1) $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Euler has a neat way of doing this. From (13.1) we have

(13.2) $\frac{\pi}{4} = 1 - \alpha$.

So

(13.3) $\frac{4}{\pi} = \frac{1}{1-\alpha} = \frac{1-\alpha+\alpha}{1-\alpha} = 1 + \frac{\alpha}{1-\alpha} = 1 + \frac{1}{-1+\frac{1}{\alpha}}$.

From (13.1) and (13.2) we have

(13.4) $\alpha = \frac{1}{3} - \beta$.

So proceeding as before

(13.5) $\frac{1}{\alpha} = \frac{1}{\frac{1}{3}-\beta} = \frac{3}{1-3\beta} = \frac{3-9\beta+9\beta}{1-3\beta} = 3 + \frac{9\beta}{1-3\beta} = 3 + \frac{9}{-3+\frac{1}{\beta}}$.

Combining (13.3) with (13.5) we get

(13.6) $\frac{4}{\pi} = 1 + \frac{1}{-1+\frac{1}{\alpha}} = 1 + \frac{1}{-1+3+\frac{9}{-3+\frac{1}{\beta}}} = 1 + \frac{1}{2+\frac{9}{-3+\frac{1}{\beta}}}$.

From 13.1) and (13.4) we have

(13.7) $\beta = \frac{1}{5} - \gamma$.

So proceeding as before

(13.8) $\frac{1}{\beta} = \frac{1}{\frac{1}{5}-\gamma} = \frac{5}{1-5\gamma} = \frac{5-25\gamma+25\gamma}{1-5\gamma} = 5 + \frac{25\gamma}{1-5\gamma} = 5 + \frac{25}{-5+\frac{1}{\gamma}}$.

Combining (13.6) with (13.8) we get

$$\frac{4}{\pi}=1+\cfrac{1}{2+\cfrac{9}{-3+\cfrac{1}{\beta}}}=1+\cfrac{1}{2+\cfrac{9}{-3+5+\cfrac{25}{-5+\cfrac{1}{\gamma}}}}=1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{-5+\cfrac{1}{\gamma}}}},$$

etc.

Section 14.

Euler now poses the problem to find representations for *A, B, C, …,* as infinite products and ratios of integrals.

Problem.

To investigate the values of the individual letters, expressed first (1) through continual products, then (2) however expressed through integral formulas, for the series *A*, *B*, *C*, *D*, *etc…*that continues according to the law of uniformity in such a way that

$AB = ff; BC = (f+a)^2; CD = (f+2a)^2; etc...$

Section 15.

Since $A=\frac{f^2}{B}$, $B=\frac{(f+a)^2}{C}$, $C=\frac{(f+2a)^2}{D}$, etc., we have

$$A=\frac{f^2}{B}=\frac{f^2}{\frac{(f+a)^2}{C}}=\frac{f^2C}{(f+a)^2}=\frac{f^2}{(f+a)^2}\frac{(f+2a)^2}{D}.$$

<u>(Notice the morphing of the continued fraction *A* into the Wallis like product!)</u>

Continuing in this way we get the infinite product

$$A = \frac{ff(f+2a)^2(f+4a)^2(f+6a)^2(etc...}{(f+a)^2(f+3a)^2(f+5a)^2(etc...}$$

which can be written for the purpose of convergence as

(15.1)

$$A = f \cdot \frac{f(f+aa)}{(f+a)(f+a)} \cdot \frac{(f+2a)(f+4a)}{(f+3a)(f+3a)} \cdot \frac{(f+4a)(f+6a)}{(f+5a)(f+5a)} \cdot etc...$$

The above product and the products for *B*, *C*, *D*, have been checked with Mathematica

($a = 1$, and $f = 1$, 2, 3, 4.).

<u>Section 16.</u>

Euler has previously derived the following lemma:

Integrating from 0 to 1 we have

$$\int \frac{x^{m-1}\partial x}{\sqrt[n]{(1-x^n)^{n-k}}} = \frac{m+k}{m} \cdot \frac{m+k+n}{m+n} \cdot \frac{m+k+2n}{m+2n} \cdot \frac{m+k+3n}{m+3n} \cdot \frac{m+k+4n}{m+4n} \cdots \int \frac{x^{\infty}\partial x}{\sqrt[n]{(1-x^n)^{n-k}}}.$$

(Here we see eighteenth century mathematics. In the above expression, the infinite product on the RHS l diverges to infinity while the integral tends to zero. No doubt Euler is aware of this, and the results he will obtain are valid.)

We will be more careful than Euler and avoid x^{∞}. To derive an improvement on the above expression for the integral $I = \int_0^1 \frac{x^{m-1}dx}{(1-x^n)^{(n-k)/n}}$ we make the substitution

(16.1) $t = x^n$

and get

$$I = \frac{1}{n}\int t^{\frac{m}{n}-1}(1-t)^{\frac{k}{n}-1}\,dt = \frac{1}{n}\frac{\Gamma\left(\frac{m}{n}\right)\Gamma\left(\frac{k}{n}\right)}{\Gamma\left(\frac{m+k}{n}\right)}.$$

Using $z\Gamma(z) = \Gamma(z+1)$ we get

$$I = \frac{1}{n}\frac{\Gamma\left(\frac{m}{n}\right)\Gamma\left(\frac{k}{n}\right)}{\Gamma\left(\frac{m+k}{n}\right)} = \frac{\left(\frac{m+k}{n}\right)\Gamma\left(\frac{m}{n}+1\right)}{\left(\frac{m}{n}\right)\Gamma\left(\frac{m+k}{n}+1\right)}\cdot\frac{\Gamma\left(\frac{k}{n}\right)}{n}$$

$$= \frac{(m+k)}{m} \frac{\left(\frac{m+k+n}{n}\right)\Gamma\left(\frac{m+n}{n}+1\right)}{\left(\frac{m+n}{n}\right)\Gamma\left(\frac{m+k+n}{n}+1\right)} \cdot \frac{\Gamma\left(\frac{k}{n}\right)}{n}.$$

Continuing in this way we get

$$I = \frac{(m+k)(m+k+n)(m+k+2n)}{m(m+n)(m+2n)(m+3n)} \cdots \frac{(m+k+rn)}{(m+rn)} \frac{\Gamma\left(\frac{m+(r+1)n}{n}\right)}{\Gamma\left(\frac{m+k+(r+1)n}{n}\right)} \cdot \frac{\Gamma\left(\frac{k}{n}\right)}{n},$$

Using (16.1) and the beta integral we can finally write our improved form of Euler's lemma

$$\text{(16.2)} \quad \int_0^1 \frac{x^{m-1}dx}{(1-x^n)^{(n-k)/n}} = \frac{(m+k)(m+k+n)(m+k+2n)}{m(m+n)(m+2n)(m+3n)} \cdots \frac{(m+k+rn)}{(m+rn)} \int_0^1 \frac{x^{m+(r+1)n+1}dx}{(1+x^n)^{(n-k)/n}}.$$

Next Euler lets $n = 2a$, $m = f$ and $k = a$ in (16.2) to get

$$\int \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}} = \frac{f+a}{f} \cdot \frac{f+3a}{f+2a} \cdot \frac{f+5a}{f+3a} \ldots \int \frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}}$$

(Note the typo in the third factor in the denominator of the RHS, $3a$ should be $4a$.)

$$\int_0^1 \frac{x^{f-1}dx}{(1-x^{2a})^{1/2}} = \frac{(f+a)(f+3a)(f+5a)}{f(f+2a)(f+4a)(f+6a)} \cdots \frac{(f+(2r+1)a)}{(f+2ra)} \int_0^1 \frac{x^{f+2(r+1)a+1}dx}{(1+x^{2a})^{1/2}}.$$

$$= \frac{(f+a)(f+3a)(f+5a)}{f(f+2a)(f+4a)(f+6a)} \cdots \frac{(f+(2r+1)a)}{(f+2ra)} \frac{\Gamma\left(\frac{f}{2a}+r+1\right)}{\Gamma\left(\frac{f}{2a}+r+3/2\right)} \cdot \frac{\Gamma\left(\frac{1}{2}\right)}{2a}$$

Using $\frac{\Gamma(x+n+1/2)}{\Gamma(x+n+1)} = \lim_{n\to\infty}(x+n)^{(-1/2)}$ (see (16.5) below) we get

(16.3)

$$\int_0^1 \frac{x^{f-1}dx}{(1-x^{2a})^{1/2}} = \lim_{r\to\infty} \frac{(f+a)(f+3a)(f+5a)}{f(f+2a)(f+4a)(f+6a)} \cdots \frac{(f+(2r+1)a)}{(f+2ra)} \frac{\sqrt{\pi}}{2a\sqrt{\frac{f}{2a}+r+1/2}}.$$

Next Euler replaces f by $f+a$ in the last expression to get

(16.4)

$$\int_0^1 \frac{x^{f+a-1}dx}{(1-x^{2a})^{1/2}} =$$

$$\lim_{r\to\infty} \frac{(f+2a)(f+4a)(f+6a)}{(f+a)(f+3a)(f+5a)} \cdots \frac{(f+(2r+2)a)}{(f+(2r+1)a)} \frac{\sqrt{\pi}}{2a\sqrt{\frac{f}{2a}+r+1}}$$

Lemma on limits

(16.5) $\lim_{n\to\infty} \frac{\Gamma(x+n+1/2)}{\Gamma(x+n+1)} = (x+n)^{(-1/2)}$

Proof:

Using Stirlings formula $\Gamma(m+1)=\lim_{m\to\infty}\sqrt{2\pi m}\; m^m e^{-m}$ for large m we get

$$\frac{\Gamma(x+n+1/2)}{\Gamma(x+n+1)}\lim_{n\to\infty}\frac{\sqrt{(x+n-1/2)}(x+n-1/2)^{(x+n-1/2)}e^{-(x+n-1/2)}}{\sqrt{(x+n)}(x+n)^{(x+n)}e^{-(x+n)}}.$$

Now the numerator can be written as

$$\sqrt{(x+n-1/2)}(x+n-1/2)^{(x+n-1/2)}e^{-(x+n-1/2)} =$$
$$\sqrt{(x+n)}\sqrt{1-\frac{1}{2(x+n)}}(x+n)^{(x+n-1/2)}\left((1-\frac{1}{2(x+n)}\right)^{(x+n-1/2)} e^{-(x+n-1/2)}$$

and so we have for large n

$$\frac{\Gamma(x+n+1/2)}{\Gamma(x+n+1)}=\lim_{n\to\infty}\frac{\sqrt{(x+n)}\sqrt{1-\frac{1}{2(x+n)}}(x+n)^{(x+n-1/2)}\left((1-\frac{1}{2(x+n)}\right)^{(x+n-1/2)} e^{-(x+n-1/2)}}{\sqrt{(x+n)}(x+n)^{(x+n)}e^{-(x+n)}}$$
$$=\lim_{n\to\infty}\sqrt{1-\frac{1}{2(x+n)}}(x+n)^{(-1/2)}\left((1-\frac{1}{2(x+n)}\right)^{(x+n-1/2)} e^{1/2}$$
$$=\lim_{n\to\infty}(x+n)^{(-1/2)}$$

This checks with Mathematica and completes our lemma.

Section 17.

Euler gets the ratios of integrals

$$A=\int\frac{x^{f+a-1}\partial x}{\sqrt{1-x^2}^{\,a}}:\int\frac{x^{f-1}\partial x}{\sqrt{1-x^2}^{\,a}}.$$

$$B = \int \frac{x^{f+2a-1}\partial x}{\sqrt{1-x^2}^{\,a}} : \int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^2}^{\,a}}$$

$$C = \int \frac{x^{f+3a-1}\partial x}{\sqrt{1-x^2}^{\,a}} : \int \frac{x^{f+2a-1}\partial x}{\sqrt{1-x^2}^{\,a}} \quad \text{etc…}$$

We will show that Euler has made a rare error here. The right hand sides should be multiplied by *f*, $f+a$, and $f+2a$. This error has been checked with Mathematica.

Euler now divides (16.4) by (16.3)

$$\frac{\displaystyle\int_0^1 \frac{x^{f+a-1}dx}{\left(1-x^{2a}\right)^{1/2}}}{\displaystyle\int_0^1 \frac{x^{f-1}dx}{\left(1-x^{2a}\right)^{1/2}}} = \lim_{r\to\infty} \frac{f(f+2a)}{(f+a)(f+a)} \frac{(f+2a)(f+4a)}{(f+3a)(f+3a)} \cdots \frac{(f+(2r+2)a)(f+(2r+2)a)}{(f+(2r+1)a)(f+(2r+1)a)} \times \frac{\sqrt{\dfrac{f}{2a}+r+1/2}}{\sqrt{\dfrac{f}{2a}+r+1}}.$$

Since $\dfrac{\sqrt{\dfrac{f}{2a}+r+1/2}}{\sqrt{\dfrac{f}{2a}+r+1}} \to 1$ we have

(17.1)

$$\frac{\int_0^1 \frac{x^{f+a-1}dx}{(1-x^{2a})^{1/2}}}{\int_0^1 \frac{x^{f-1}dx}{(1-x^{2a})^{1/2}}} = \lim_{r\to\infty} \frac{f(f+2a)}{(f+a)(f+a)} \frac{(f+2a)(f+4a)}{(f+3a)(f+3a)} \cdots \frac{(f+(2r)a)(f+(2r+2)a)}{(f+(2r+1)a)(f+(2r+1)a)}$$

Comparing this with Euler's (15.1)

$$A = f \cdot \frac{f(f+aa)}{(f+a)(f+a)} \cdot \frac{(f+2a)(f+4a)}{(f+3a)(f+3a)} \cdot \frac{(f+4a)(f+6a)}{(f+5a)(f+5a)} \cdot etc...$$

we see that they differ by the factor *f*.

This shows that

(17.2) $$\frac{\int_0^1 \frac{x^{f+a-1}dx}{(1-x^{2a})^{1/2}}}{\int_0^1 \frac{x^{f-1}dx}{(1-x^{2a})^{1/2}}} = \frac{A}{f} = \frac{L(0)}{f} \quad .$$

(For convenience, we use the modern notation $L(0) = A$, $L(1) = B$, $L(2) = 3$, etc.)

Replacing f by $f + ma$ in (17.2) we get

(17.3)

$$\frac{\int_0^1 \frac{x^{f+ma+a-1}dx}{(1-x^{2a})^{1/2}}}{\int_0^1 \frac{x^{f+ma-1}dx}{(1-x^{2a})^{1/2}}} = \lim_{r\to\infty} \frac{(f+ma)(f+ma+2a)}{(f+ma+a)(f+ma+a)} \frac{(f+ma+2a)(f+ma+4a)}{(f+ma+3a)(f+ma+3a)} \cdots$$

$$\frac{(f+ma+(2r)a)(f+ma+(2r+2)a)}{(f+ma+(2r+1)a)(f+ma+(2r+1)a)}$$

and

(17.4) $$\frac{\int_0^1 \frac{x^{f+ma+a-1}dx}{(1-x^{2a})^{1/2}}}{\int_0^1 \frac{x^{f+ma-1}dx}{(1-x^{2a})^{1/2}}} = \frac{L(m)}{f+ma}.$$

(This completes our demonstration of Euler' omission of the factor $f+ma$ from (17.4) . In section 27, equation (27.3) as well as in section 30 Euler does not omit this factor in essentially the same equations.)

Euler now poses a generalization of his previous problem.

A MORE GENERAL PROBLEM

To find a series *A*, *B*, *C*, *D*, etc. that proceeds uniformly in such a way that

$AB = ff + c; BC = (f+a)^2 + c; CD = (f+2a)^2 + c; DE = (f+3a)^2 + c;$

where, in the case of single products, the letter f is increased by a quantity a.

PREVIOUS SOLUTION FOR CONTINUAL FRACTIONS

Section 18.

Euler begins with the problem of finding expressions *A, B, C, D,* etc., such that $AB = f^2 + c$; $BC = (f+a)^2 + c$; $CD = (f+2a)^2 + c$ **;** etc. The solution is an alteration on the previous work. The following sections 19 to 23 are a very minor variation on sections 7 to 11.

Section 19.

We start with

(19.1) $AB = f^2 + c$,

and assume that the desired increase is given by

$$A = f - \frac{a}{2} + \frac{s/2}{A'}, \qquad B = f + \frac{a}{2} + \frac{s/2}{B'},$$

where s, *A'* and *B'* are to be determined. Multiplying by 2 we get

(19.2) $2A = 2f - a + \dfrac{s}{A'}, \qquad 2B = 2f + a + \dfrac{s}{B'}$.

Multiplying the above together we get using (19.1)

(19.3) $\left(2f - a + \dfrac{s}{A}\right)\left(2f + a + \dfrac{s}{B}\right) = 4f^2 + 4c$,

which can also be written as

$$4AB = 4f^2 - a^2 + \frac{s}{B'}(2f-a) + \frac{s}{A'}(2f+a) + \frac{s^2}{A'B'}.$$

Since $4AB = 4f^2 + 4c$ we have

$$a^2 + 4c = \frac{s}{B'}(2f - a) + \frac{s}{A'}(2f + a) + \frac{s^2}{A'B'},$$

and multiplying by $A'B'$ we get

(19.4) $(a^2 + 4c)A'B' = A's(2f - a) + B's(2f + a) + s^2$.

We are free to choose *s* as we like, and a reasonable choice to simplify (19.4) is

(19.5 divide (19.4) by $a^2 + 4c$ we get

$$A'B' = A'(2f - a) + B'(2f + a) + a^2 + 4c\,.$$

Notice that this last expression can be written as

$$(A' - (2f + a))(B' - (2f - a)) = 4f^2 + 4c$$

or

(19.6) $(A' - 2f - a)(B' - 2f + a) = 4f^2 + 4c$.

We also notice that because of (19.5), (19.2) has become the important equations

(19.7) $2A = 2f - a + \dfrac{a^2 + 4c}{A'}$, and $\quad 2B = 2f + a + \dfrac{a^2 + 4c}{B'}$.

Section 20.

We now wish to determine appropriate expressions for *A'* and *B'*. We see from (19.6) that these expressions should start as $A' = 4f + \cdots$ and $B' = 4f + \cdots$. How does *a* fit with $4f$? Since we have been examining the three equally spaced numbers $f - \dfrac{a}{2}$, *f* , and $f + \dfrac{a}{2}$, it seems reasonable to think of $4f - 2a$, $4f$, and $4f + 2a$. Thus we now try the equations

(20.1) $A' = 4f - 2a + \dfrac{s'}{A''}$, and $\quad B' = 4f + 2a + \dfrac{s'}{B''}$, where *s'* , *A''*, and *B''* are to be found. Using (20.1) to remove *A'* and *B'* from (19.6) we get

(20.2) $\left(2f-3a+\frac{s'}{A''}\right)\left(2f+3a+\frac{s'}{B''}\right)=4f^2+4c$.

Now compare (20.2) with (19.3). If in (19.3) we replace

$s\to s'$, $A'\to A''$, $B'\to B''$, and $a\to 3a$ we get (20.2). Therefore (19.4) becomes after these substitutions

(20.3) $(3^2a^2+4c)A''B''=A''s'(2f-3a)+B''s'(2f+3a)+s'^2$.

We are free to select a value for *s'* and a nice choice to simplify (20.3) is

(20.4) $s'=3^2a^2+4c$.

Dividing (20.3) by by *s'* get

$$A''B''=A''(2f-3a)+B''(2f+3a)+3^2a^2+4c\ .$$

Corresponding to (19.6), this last expression can be written as

(20.5) $(A''-2f-3a)(B''-2f+3a)=4f^2+4c$.

Notice that equations (20.1) are now the important relations

(20.6) $A'=4f-2a+\frac{3^2a^2+4c}{A''}$, and $B'=4f+2a+\frac{3^2a^2+4c}{B''}$.

<u>Section 21 and 22.</u>

Next we seek appropriate expressions for *A''* and *B''*. As we reasoned above to get (20.1) we now try

(21.1) $A''=4f-2a+\frac{s''}{A'''}$ and $B''=4f+2a+\frac{s''}{B'''}$.

Using (9.1) to remove *A''* and *B''* from (20.5) we now have (corresponding to (20.2))

(21.2) $$\left(2f-5a+\frac{s''}{A'''}\right)\left(2f+5a+\frac{s''}{B'''}\right)=4f^2+4c,$$

and by a similar reasoning we will arrive at

(21.3) $$A''=4f-2a+\frac{5^2a^2+4c}{A'''}\quad\text{and}\quad B''=4f+2a+\frac{5^2a^2+4c}{B'''}.$$

Summarizing our results for (19.7), (20.6) and (21.3) we see

$$2A=2f-a+\frac{a^2+4c}{A'},\ \text{and}\qquad 2B=2f+a+\frac{a^2+4c}{B'},$$

$$A'=4f-2a+\frac{3^2a^2+4c}{A''},\ \text{and}\quad B'=4f+2a+\frac{3^2a^2+4c}{B''},$$

$$A''=4f-2a+\frac{5^2a^2+4c}{A'''}\quad\text{and}\quad B''=4f+2a+\frac{5^2a^2+4c}{B'''}.$$

Section 23.

The general pattern is now emerging and we can write our continued fractions as

(23.1) $$2A=2f-a+\frac{a^2+4c}{4f-2a\ +}\ \frac{3^2a^2+4c}{4f-2a\ +}\ \frac{5^2a^2+4c}{4f-2a\ +}\cdots,$$

and

(23.2) $$2B=2f+a+\frac{a^2+4c}{4f+2a\ +}\ \frac{3^2a^2+4c}{4f+2a\ +}\ \frac{5^2a^2+4c}{4f+2a\ +}\cdots.$$

Notice that (23.2) is obtained from (23.1) by replacing f by $f+a$. In a similar way, replacing f by $f+2a$ in (23.1) gives us

$$2C=2f+3a+\frac{a^2+4c}{4f+6a\ +}\ \frac{3^2a^2+4c}{4f+6a\ +}\ \frac{5^2a^2+4c}{4f+6a\ +}\cdots,$$

replacing f by $f+3a$ in (23.1) yields

$$2D=2f+5a+\frac{a^2+4c}{4f+10a\ +}\ \frac{3^2a^2+4c}{4f+10a\ +}\ \frac{5^2a^2+4c}{4f+10a\ +}\cdots,$$

and replacing f by $f+4a$ in (23.1) yields

$$2E = 2f + 7a + \frac{a^2+4c}{4f+14a\,+}\;\frac{3^2a^2+4c}{4f+14a\,+}\;\frac{5^2a^2+4c}{4f+14a\,+}\cdots$$

and so forth.

Section 24.

With $AB + f^2 + c$, $BC = (f+a)^2 + c$ $CD = (f+2a)^2 + c$, $DE = (f+3a)^2 + c$

we have

$$A = \frac{f^2+c}{B},\; B = \frac{(f+a)^2+c}{C},\; C = \frac{(f+2a)^2+c}{D}, \text{ etc., so}$$

$$A = \frac{f^2+c}{B} = \frac{f^2+c}{\frac{(f+a)^2+c}{C}} = \frac{(f^2+c)C}{(f+a)^2+c} = \frac{(f^2+c)}{\left((f+a)^2+c\right)}\frac{(f+2a)^2+c}{D}.$$

Section 25.

The product emerging is

(25.1)

$$A = \frac{(f^2+c)}{\left((f+a)^2+c\right)}\cdot\frac{(f+2a)^2+c}{(f+3a)^2+c}\cdot\frac{(f+4a)^2+c}{(f+5a)^2+c}\cdot\frac{(f+6a)^2+c}{(f+7a)^2+c}\cdots.$$

This product will not converge as written. Squaring we can write

(25.2) $$A^2 = (ff+c)\cdot\frac{(ff+c)((f+2a)^2+c)}{((f+a)^2+c)((f+a)^2+c)}\cdot\frac{((f+2a)^2+c)((f+4a)^2+c)}{((f+3a)^2+c)((f+3a)^2+c)}\cdot etc...$$

which does converge.

Case 1 in which c = - bb

Section 26.

Replacing c by $-b^2$ in (23.1) we get

(26.1) $$2A = 2f - a + \frac{a^2-4b^2}{4f-2a} + \frac{3^2a^2-4b^2}{4f-2a} + \frac{5^2a^2-4b^2}{4f-2a} + \cdots$$

which can also be written as

$$2A = 2f - a + \cfrac{(a+2b)(a-2b)}{4f-2a+\cfrac{(3a+2b)(3a-2b)}{4f-2a+\cfrac{(5a+2b)(5a-2b)}{4f-2a+\cfrac{(7a+2b)(7a-2b)}{4f-2a+etc...}}}}.$$

Returning to (25.1) we have

$$A = \frac{(f^2-b^2)}{\left((f+a)^2-b^2\right)}\cdot\frac{(f+2a)^2-b^2}{(f+3a)^2-b^2}\cdot\frac{(f+4a)^2-b^2}{(f+5a)^2-b^2}\cdot\frac{(f+6a)^2-b^2}{(f+7a)^2-b^2}\cdots$$

which we rewrite in the convergent form

(26.2) $$A = (f-b)\cdot\frac{(f+b)(f+2a-b)}{(f+a+b)(f+a-b)}\cdot\frac{(f+2a-b)(f+4a-b)}{(f+3a+b)(f+3a-b)}\cdot etc...$$

<u>There is a misprint in the last factor.</u> It should be $\frac{(f+2a+b)(f+4a-b)}{(f+3a+b)(f+3a-b)}$.

<u>Section 27.</u>

This section is a slight variation on section 16. Euler begins again with his lemms:

Integrating from 0 to 1 we have

$$\int\frac{x^{m-1}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-k}}} = \frac{m+k}{m}\cdot\frac{m+k+n}{m+n}\cdot\frac{m+k+2n}{m+2n}\cdot\frac{m+k+3n}{m+3n}\cdot$$
$$\frac{m+k+4n}{m+4n}\ldots\int\frac{x^{\infty}\partial x}{\sqrt[n]{\left(1-x^n\right)^{n-k}}}.$$

Again he lets $n = 2a$ and $k = a$, but he changes m from f to $m = f+b$ and gets

(27.1) $$\frac{f+a+b}{f+b}\cdot\frac{f+3a+b}{f+2a+b}\cdot\frac{f+5a+b}{f+4a+b}\cdots\cdot\int\frac{x^{\infty}\partial x}{\sqrt{(1-x^{2a})}} = \int\frac{x^{f+b-1}\partial x}{\sqrt{1-x^{2a}}}.$$

Next he lets $n = 2a$ and $k = a$, but now he changes m from f to $m = f + a - b$ and gets

(27.2) $\frac{f+2a-b}{f+a-b} \cdot \frac{f+4a-b}{f+3a-b} \cdot \frac{f+6a-b}{f+5a-b} \cdots \int \frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}} = \int \frac{x^{f+a-b-1}\partial x}{\sqrt{1-x^{2a}}}.$

Dividing (27.2) by (27.1) he gets the product in (26.2). Therefore

(27.3) $A = (f-b) \cdot \int \frac{x^{f+a-b-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f+b-1}\partial x}{\sqrt{1-x^{2a}}}.$

Note that this time he has the correct factor $(f-b)$ which he neglected in (17.2) when $b = 0$.

Section 28.

Euler exhibits the example of (26.1)

$$2A = 2f - a + \frac{a^2 - 4b^2}{4f - 2a \; +} \; \frac{3^2 a^2 - 4b^2}{4f - 2a \; +} \; \frac{5^2 a^2 - 4b^2}{4f - 2a \; +} \cdots$$

With $f = 2$, $a = b = 1$ we have $AB = f^2 - b^2 = 3$, $BC = (f+a)^2 - b^2 = 8$,

$CD = (f+2a)^2 - b^2 = 15$, $DE = (f+3a)^2 - b^2 = 24$, etc.

(28.1) $$2A = 3 - \frac{3}{6 \; +} \; \frac{5}{6 +} \; \frac{21}{6 \; +} \; \frac{45}{6 \; +} \; \frac{77}{6 \; +} \cdots.$$

Euler writes:

"But it will be this in continual products: $A = \frac{3\cdot 3}{2\cdot 4} \cdot \frac{5\cdot 5}{4\cdot 6} \cdot \frac{7\cdot 7}{6\cdot 8} \cdot \frac{9\cdot 9}{8\cdot 10} \cdot etc.$..

Then it will be considered this way in integral formulas: $A = \int \frac{x\partial x}{\sqrt{1-xx}} : \int \frac{xx\partial x}{\sqrt{1-xx}}$."

We can also write the infinite product using (26.2)

$$A=(f-b)\cdot\frac{(f+b)(f+2a-b)}{(f+a+b)(f+a-b)}\cdot\frac{(f+2a-b)(f+4a-b)}{(f+3a+b)(f+3a-b)}\cdot etc...$$

(28.2) $$A=1\cdot\frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdots.$$

Using (27.3) we get the ratio of integrals

(28.3) $$A=(f-b)\cdot\int\frac{x^{f+a-b-1}\partial x}{\sqrt{1-x^{2a}}}:\int\frac{x^{f+b-1}\partial x}{\sqrt{1-x^{2a}}}.$$

$$A=\int\frac{x\partial x}{\sqrt{1-xx}}:\int\frac{xx\partial x}{\sqrt{1-xx}}.$$

Since the above two integrals are 1 and $\pi/4$ we get

(28.4) $$A=4/\pi.$$

Case 2 in which c = + bb

<u>Section 29.</u>

Now we let $c=b^2$ in (23.1)

$$2A=2f-a+\cfrac{a^2+4c}{4f-2a+\cfrac{3^2a^2+4c}{4f-2a+\cfrac{5^2a^2+4c}{4f-2a+\cdots}}}$$

to get

(29.1) $$2A=2f-a+\cfrac{a^2+4b^2}{4f-2a+\cfrac{3^2a^2+4b^2}{4f-2a+\cfrac{5^2a^2+4b^2}{4f-2a+\cdots}}}.$$

In the infinite product (26.2)

$$A=(f-b)\cdot\frac{(f+b)(f+2a-b)}{(f+a+b)(f+a-b)}\cdot\frac{(f+2a-b)(f+4a-b)}{(f+3a+b)(f+3a-b)}\cdot etc...$$

we can replace b by either $+bi$ or $-bi$ to get

(29.2)

$$A=(f-b\sqrt{-1})\cdot\frac{(f+b\sqrt{-1})(f+2a-b\sqrt{-1})}{(f+a+b\sqrt{-1})(f+a-b\sqrt{-1})}\cdot\frac{(f+2a+b\sqrt{-1})(f+4a-b\sqrt{-1})}{(f+3a+b\sqrt{-1})(f+3a-b\sqrt{-1})}\cdot etc...$$

and

(29.3)

$$A=(f+b\sqrt{-1})\cdot\frac{(f-b\sqrt{-1})(f+2a+b\sqrt{-1})}{(f+a-b\sqrt{-1})(f+a+b\sqrt{-1})}\cdot\frac{(f+2a-b\sqrt{-1})(f+4a+b\sqrt{-1})}{(f+3a-b\sqrt{-1})(f+3a+b\sqrt{-1})}\cdot etc...$$

Multiplying (29.2) times (29.3) we get

$$A^2=(ff+bb)\frac{(ff+bb)((f+2a)^2+bb)}{((f+a)^2+bb)((f+a)^2+bb}\cdot\frac{((f+2a)^2+bb)((f+4a)^2+bb)}{((f+3a)^2+bb)((f+3a)^2+bb)}\cdot etc...$$

which is the same as our (25.1) with $c=b^2$.

Section 30.

If we replace *b* by bi in (27.3)

$$A=(f-b)\cdot\int\frac{x^{f+a-b-1}\partial x}{\sqrt{1-x^{2a}}}:\int\frac{x^{f+b-1}\partial x}{\sqrt{1-x^{2a}}}.$$

we get

(30.1) $$A=(f-b\sqrt{-1})\int\frac{x^{f+a-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}:\int\frac{x^{f-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}.$$

and if we replace *b* by $-bi$ we get

(30.2) $$A=(f+b\sqrt{-1})\int\frac{x^{f+a-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}:\int\frac{x^{f-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}.$$

Section 31.

If we multiply (30.1) times (30.2) we get, (31.1) $$A^2 = (ff+bb)\frac{\int \frac{x^{f+a-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} \cdot \int \frac{x^{f+a-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}}{\int \frac{x^{f-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} \cdot \int \frac{x^{f-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}}$$

Section 32.

In the denominator of (31.1) we set $\frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}} = \partial V$ and get

$$\int x^{+b\sqrt{-1}}\partial V \cdot \int x^{-b\sqrt{-1}}\partial V.$$

Now make the substitutions

The sum: $\int (x^{b\sqrt{-1}} + x^{-b\sqrt{-1}}\partial V = p$

Difference: $\int (x^{b\sqrt{-1}} - x^{-b\sqrt{-1}}\partial V = q$

and get

(32.1) $$\int (x^{b\sqrt{-1}}\partial V \cdot \int x^{-b\sqrt{-1}}\partial V = \frac{pp-qq}{4}$$

Section 33.

If we replace x by $\exp(\log x)$ in p and q of the previous section we get

$$p = \int (e^{b\log x\sqrt{-1}} + e^{-b\log x\sqrt{-1}})\partial V,$$

$$q = \int (e^{b\log x\sqrt{-1}} - e^{-b\log x\sqrt{-1}})\partial V,$$

(Here Euler writes *lx* for our $\log x$.) Using the identities

$$e^{\phi\sqrt{-1}} + e^{-\phi\sqrt{-1}} = 2\cos\phi \text{ and } e^{\phi\sqrt{-1}} - e^{-\phi\sqrt{-1}} = 2\sqrt{-1}\sin\phi$$

with $\phi = b\log x$ we have

$$p = 2\int \partial V\cos\phi \text{ and } q = 2\sqrt{-1}\int \partial V\sin\phi .$$

Using (32.1) we see

$$(33.1)\quad \frac{pp-qq}{4} = (\int \partial V \cos\phi)^2 + (\int \partial V \sin\phi)^2$$

which is the denominator of (31.1).

Section 34.

The same argument with numerator of (31.1) would have given Euler

$$(\int x^a \partial V \cos\phi)^2 + (\int x^a \partial V \sin\phi)^2$$

and thus (31.1) can be expressed as

$$(34.1)\quad A^2 = (ff + bb)\frac{(\int x^a \partial V \cos\phi)^2 + (\int x^a \partial V \sin\phi)^2}{(\int \partial V \cos\phi)^2 + (\int \partial V \sin\phi)^2}$$

where we recall $\partial V = \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}}$ and $\phi = b\log x$.

Section 35.

Euler remarks briefly on the integrals (I need help with the Latin here.)

$$\int \frac{x^{f-1}\partial x \cos(b\log x)}{\sqrt{1-x^{2a}}} \;\; and \;\; \int \frac{x^{f-1}\partial x \sin(b\log x)}{\sqrt{1-x^{2a}}}.$$

Section 36.

Euler begins by writing the formula for integration by parts

$$\int P\partial Q = PQ - \int Q\partial P.$$

He then takes $P = \cos(b\log x)$ and $\partial Q = x^{f-1}\partial x$ and gets

$$(36.1)\quad \int x^{f-1}\partial x \cos(b\log x) = \frac{x^f}{f}\cos(b\log x) + (b\log x)$$

A second integration by parts uses $P = \sin(b\log x)$ and $\partial Q = x^{f-1}\partial x$ and we get

(36.2) $\int x^{f-1}\partial x \sin(b\log x) = \frac{x^f}{f}\sin(b\log x) - \frac{b}{f}\int x^{f-1}\partial x \cos(b\log x)$

Now, starting with (36.1) we can replace the integral on the RHS with (36.2) to obtain

$$\int_0^1 x^{f-1}\cos(b\log x)dx = \frac{x^f}{f}\cos(b\log x) + \frac{b}{f}\left[\frac{x^f}{f}\sin(b\log x) - \frac{b}{f}\int_0^1 x^{f-1}\cos(b\log x)dx\right]$$

which simplifies to

$$\int_0^1 x^{f-1}\cos(b\log x)dx = \frac{x^f}{f}\cos(b\log x) + \frac{bx^f}{f^2}\sin(b\log x) - \frac{b^2}{f^2}\int_0^1 x^{f-1}\cos(b\log x)dx\,.$$

Solving for the integral we get

(36.3) $\int x^{f-1}\partial x \cos blx = \frac{x^f}{ff+bb}(f\cos(b\log x) + b\sin(b\log x));.$

In a similar way, starting with (36.2) we get

(36.4) $\int x^{f-1}\partial x \sin(b\log x) = \frac{x^f}{ff+bb}(f\sin(b\log x) - b\cos(b\log x))$

(Note that neither of the above integrals connect with the integrals in previous sections. All of those had $\sqrt{1-x^{2a}}$ in the denominator.) Perhaps Euler was showing his skill in the use of complex numbers.

References

[1] Euler, L., *De Fractionibus Continuis Wallisii*, (E745), on the web at the Euler Archive, http://www.math.dartmouth.edu/~euler/.

[2] Khrushchev, S., *A recovery of Brouncker's proof for the quadrature continued fractions*, Publicacions matermatiques, 50(2006), pp. 2-42.

[3] Nunn, T. Percy, *The Arithmetic of Infinites.: A School Introduction to the Integral Calculus*, (Part 1), The Mathematical Gazette, 5(1910), pp. 345-356.

[4] Nunn, T. Percy, *The Arithmetic of Infinites.: A School Introduction to the Integral Calculus*, (Part 2), The Mathematical Gazette, 5(1911), pp. 377-386.

[5] Stedall, Jacqueline A., *Catching Proteus: The Collaborations of Wallis and Brouncker. I. Squaring the Circle*, Notes and Records of the Royal Society of London, Vol. 54, No. 3, (Sep., 2000), pp. 293 -316

[6] Wallis, John, *The Arithmetic of Infinitesimals*, (Translated from Latin by Jacqueline A. Stedall), Springer Verlag, New York, 2004.