Proofs of certain arithmetic theorems *

Leonhard Euler

Arithmetic theorems, of which Fermat and others unveiled many, are worthy of greater attention because of this, but instead their truth is hidden and difficult to prove. Indeed, Fermat has left behind a sufficiently great abundance of such theorems, but never published the proofs, even though he may have steadfastly claimed for himself to have established their validity with great certainty. Therefore, it is very painful to write of the tragedy that, thus far, all of these proofs still remain unknown. The idea behind propositions in common knowledge is similar: in them it is claimed that neither the sum nor the difference of fourth powers can be a square; though no one doubts the truth of this, a rigorous proof is still not available anywhere, as far as I know, except in a certain pamphlet once edited by Frénicle, the title of which is *Traité des triangles rectangles en nombres*. Here, the author proves among other things that in no right triangle whose sides are expressed as rational numbers can the area be a square, from which the truth of the previously mentioned propositions about the sums and differences of two fourth powers is deduced. But that proof is very much enmeshed in the properties of triangles, so that unless the greatest attention is employed, it can be understood clearly only with great difficulty. On account of this, I believe these works to be of value, if I will have drawn the proofs of these propositions away from right triangles and put them forward clearly and analytically. Now my strategy offers a great advantage in that many other much more difficult theorems can be elicited from the proofs. Here, of course, the theorem belongs to that celebrated Fermat, who has established that no triangular number can be a fourth power except for 1, the proof of which has fallen to me to produce from those proofs. But that proof seems more difficult, because the proposition can have exceptions and belongs only to the whole numbers; that is, for infinitely many fractional numbers, it is possible to make $\frac{x(x+1)}{2}$ a fourth power. Therefore, for this and several other theorems to be proved, it will be necessary to provide some lemmas which the following proofs rely on; but beforehand, it is always necessary to point out that, for me, all letters represent whole numbers.

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Lemma 1

The product of two or more relatively prime numbers cannot be a square or a cube or any other power unless the individual factors are squares or cubes or another such power.

The proof of these lemmas is easy and has already been done by Euclid, so it would be superfluous to expound upon it here.

Lemma 2

If $a^2 + b^2$ is a square, and a and b are relatively prime numbers, then $a = pp - qq$ and $b = 2pq$, where p and q are relatively prime numbers, one even and the other odd.

Proof

Because $a^2 + b^2$ is a square, its square root can be assumed to be equal to $a + \frac{bq}{p}$, where the fraction $\frac{q}{p}$ is expressed in lowest terms, so that p and q are numbers that are relatively prime. But the products will be equations $a^2 + b^2 = a^2 + \frac{2abq}{p} + \frac{bbqq}{pp}$. So,

$$a : b = pp - qq : 2pq.$$[1]

Now either the numbers $pp - qq$ and $2pq$ are relatively prime or they have common divisor 2. Thus, in the case in which $pp - qq$ and $2pq$ are relatively prime numbers, which occurs if one of the numbers p or q is even and the other is odd, it is necessary that

$$a = pp - qq \text{ and } b = 2pq$$

because a and b were assumed to be relatively prime numbers. But in the case in which the numbers $pp-qq$ and $2pq$ have common divisor 2, which will happen if the numbers p and q are either both odd or both even (which, of couse, cannot be, since they are assumed to be relatively prime), then $a = \frac{pp-qq}{2}$ and $b = pq$. Now set $p + q = 2r$ and $p - q = 2s$; r and s will be relatively prime numbers; one of them will be even and the other will be odd, so that

$$a = 2rs \text{ and } b = rr - ss;$$

[1] That is, $\frac{a}{b} = \frac{pp - qq}{2pq}$. Euler chooses to express this as a ratio.

this expression, because it agrees with what came before, indicates that if $aa+bb$ is a square and the numbers a and b are coprime, one of them will be the difference of two coprime squares, one even and one odd, and the other will certainly be a number equal to twice the product of the roots of those squares. That is,

$$a = pp - qq \text{ and } b = 2pq,$$

where p and q are coprime numbers, one even and the other odd. Q. E. D.

Corollary 1

Therefore, if the sum of two coprime squares is a square, it is necessary that one square is even and the other is certainly odd; from this it follows that the sum of two odd squares cannot be a square.

Corollary 2

Thus, if $aa + bb$ is a square, one of the numbers a and b, say a, will be odd and the other, b will certainly be even. Surely, the odd number $a = pp - qq$ and the even number $b = 2pq$.

Corollary 3

Furthermore, because one of the numbers p and q is even and the other is odd, the number b will be divisible by 4. Then, if neither p nor q is divisible by 3, it is necessary that either $p - q$ or $p + q$ can be divided by 3. From this, it follows that one of the two numbers a or b, the sum of whose squares is a square, is divisible by 3.

Corollary 4

Since $a = pp - qq$ and $b = 2pq$ if $aa + bb$ constitutes a square, it is easily understood that the numbers p and q are less than a and b. That is, seeing that $a = (p + q)(p - q)$, a will be greater than $p + q$, unless $p - q = 1$; and since $b = 2pq$, b will be greater than p or q. Therefore, by means of superior reasoning, the numbers a and b will be greater than the numbers p and q. If $p = q$, certainly a would have to be 0, but this case is impossible, because p and q were assumed to be coprime numbers, of which one is even and the other is odd.

Scholium

In the proof of this lemma, from the ratio $a : b = pp - qq : 2pq$ it therefore follows that $a = pp - qq$ and $b = 2pq$, [2] because a and b are coprime numbers, and they are equivalent to the numbers $pp - qq$ and $2pq$. Indeed, if $a : b = c : d$, and the numbers a and b are coprime as are the numbers c and d, it is necessary that $a = c$ and $b = d$, as is easily established from the nature of proportions.

Lemma 3

If $aa - bb$ is a square, where a and b are coprime numbers, then $a = pp + qq$ and either $b = pp - qq$ or $b = 2pq$, where the numbers p and q are coprime and one of them is even while the other is odd.

Proof

Because $aa - bb$ is a square, $a^2 - b^2$ can be set equal to c^2, so $a^2 = b^2 + c^2$, where b and c are coprime numbers. Therefore, since by Corollary 1 of the preceding lemma one of the numbers b and c is even and the other is odd, it is necessary that a be an odd number; certainly b can be either odd or even.

First, let b be odd and c be even; by the preceding lemma, $b = pp - qq$ and $c = 2pq$, where p and q are coprime numbers with one even and the other odd. Consequently, let $a = pp + qq$. And if b is even and c is odd, then $b = 2pq$ and $c = pp - qq$, so that again $a = pp + qq$. Therefore, if $aa - bb$ is a square, then

$$a = pp + qq \text{ and either } b = pp - qq \text{ or } b = 2pq.$$

Q. E. D.

Corollary 1

Therefore, if the difference of two squares is a square number, the larger square should be an odd number, if indeed those squares are coprime numbers.

Corollary 2

Furthermore, in the same way, it is understood that the numbers p and q are less than the numbers a and b, since $a = pp + qq$ and b is either $pp - qq$ or $2pq$.

Corollary 3

[2]Translator: In the *Opera Omnia*, this is erroneously stated as $b = 2bq$.

If $aa - bb = cc$, one of the numbers a, b, c will always be divisible by 5. For since $a = pp + qq, b = pp - qq$ and $c = 2pq$, either one of the numbers p or q is divisible by 5 or neither is; now in the first case, c would be divisible by 5. Surely in the second case, pp and qq will be numbers of the form $5n \pm 1$; thus, either $pp - qq$ or $pp + qq$ will be divisible by 5.

Theorem 1

The sum of two fourth powers $a^4 + b^4$ cannot be a square unless one of the fourth powers is zero.

Proof

In the proof of this theorem that I will unfurl, I will show that if in one case $a^4 + b^4$ were a square, then, however great the larger numbers a and b are, continually smaller numbers could be assigned in place of a and b and eventually one should arrive at smaller whole numbers. But because among such small numbers there are none the sum of whose fourth powers constitute a square, it must be concluded that none exist among the larger numbers.

Thus, we will suppose that $a^4 + b^4$ is a square, where a and b are relatively prime numbers; for if they were not relatively prime, they could be reduced to relatively prime via division. If a is an odd number, then certainly b will be even, because it is necessary for one to be even and the other odd. Therefore,

$$aa = pp - qq \text{ and } bb = 2pq,$$

where the numbers p and q are relatively prime, one of them even, and the other odd. But since $aa = pp - qq$, it is necessary that p be the odd number, because otherwise $pp - qq$ could not be a square. Therefore, p will be an odd number, and q an even number. Furthermore, because $2pq$ needs to be a square, it is necessary that both p and $2q$ be squares, because p and q are relatively prime numbers. Certainly in order for $pp - qq$ to be a square, it is necessary that

$$p = mm + nn \text{ and } q = 2mn,$$

where, again, m and n are relatively prime numbers, one of them even, and the other odd. But since $2q$ is a square, $4mn$ will be a square, and mn, too; hence, both m and n will be squares. Therefore, if

$$m = xx \text{ and } n = yy,$$

then

$$p = mm + nn = x^4 + y^4,$$

because they likewise are squares. Hence, it follows that if $a^4 + b^4$ were a square, then $x^4 + y^4$ would also be a square; but it is clear that the numbers x and y are much smaller than a and b. Therefore, in the same way, from two fourth powers $x^4 + y^4$ again arise smaller ones, the sum of which is a square and, proceeding, eventually smaller fourth powers of whole numbers are reached. Therefore, because there would be no smallest fourth powers whose sum produces a square, there are none such among the greatest numbers. Moreover, if for the even one of the fourth powers, if that one of the two equals 0, in all of the remaining even ones, one of the two vanishes, so that no new cases would arise. Q. E. D.

Corollary 1

Therefore, since the sum of two fourth powers cannot be a square, still less can two fourth powers added together produce a fourth power.

Corollary 2

Although this proof applies only to whole numbers, still it can be concluded from it that it is certainly not possible to produce two fourth powers of fractions for which the sum is a square. For if $\frac{a^4}{m^4} + \frac{b^4}{n^4}$ were a square, then also $a^4n^4 + b^4m^4$ would be a square in whole numbers, which, by the same proof, cannot be done.

Corollary 3

By the same proof, one can conclude that there are no such numbers p and q such that $p, 2q$ and $pp - qq$ are squares; for if such numbers existed, then there would be values for a and b which would render $a^4 + b^4$ a square; for a would be $\sqrt{pp - qq}$ and b would be $\sqrt{2pq}$.

Corollary 4

Therefore, by setting $p = xx$ and $2q = 4yy$, we will have $pp - qq = x^4 - 4y^4$, Thus, not all of this can be done so that $x^4 - 4y^4$ is a square. And neither can $4x^4 - y^4$ be a square; for $16x^4 - 4y^4$ would be a square; this case reverts to the previous one for the fourth power $16x^4$.

Corollary 5

Hence, it follows that $ab(a^2 + b^2)$ can never be a square either. That is, for relatively prime factors $a, b, a^2 + b^2$, each would need to be a square, which is not possible.

Corollary 6

Similarly, neither are there such numbers a and b, relatively prime, which would make $2ab(a^2+b^2)$ a square. This follows from Corollary 3, where it was shown that there are no numbers p and q such that $p, 2q$ and $pp-qq$ are squares. Now all of these are also valid for numbers not relatively prime and thus for fractions, by Corollary 2.

Theorem 2

The difference of two fourth powers a^4-b^4 cannot be a square unless either $b=0$ or $b=a$.

Proof

I will prove the preceding theorem in like manner. Therefore, letting the fourth powers be reduced to lowest terms and assuming that a^4-b^4 is a square, a will be an odd number; certainly b will be either even or odd.

Case 1

First, let b be an even number; then

$$a^2 = pp+qq \text{ and } b^2 = 2pq,$$

where p and q are relatively prime, and one of them, p, is even, while the other, q, is odd. Because $b^2=2pq$, $2p$ and q must therefore be squares. Furthermore, because $pp+qq$ itself is equal to a^2,

$$q = mm-nn \text{ and } p = 2mn,$$

where m and n are relatively prime numbers. But since $2p$ is a square, which is $4mn$, mn is a square; and so m and n are individually squares. Therefore, setting

$$m = x^2 \text{ and } n = y^2$$

makes

$$q = x^4-y^4,$$

where, since one of the numbers m and n is even and the other is odd, one of the numbers x and y will also be even and the other will be odd. Now because q is a square, x^4-y^4 will be a square, where x will be an odd number and y will certainly be even. Therefore, if a^4-b^4 were a square, x^4-y^4 would also be a square, where x and y are much smaller than a and b. Therefore, since there are no two fourth powers having a square difference among the smallest numbers, neither will there be among the largest, at least in the case in which the smaller fourth power is an even number. This is the first case which was to be proved.

Case 2

Now let b be an odd number; then

$$a^2 = pp + qq \text{ and } b^2 = pp - qq$$

where p and q are relatively prime numbers, and one of them, p, is even, while the other, q, is odd. Because $pp + qq$ is certainly a square, p will be an odd number and thus q, even. By considering a^2 and b^2 in turn, $a^2b^2 = p^4 - q^4$ results; this expression is a square and thus, by the first case, cannot be equal to a^2b^2 itself. Therefore, the difference of two fourth powers cannot be a square in any case, unless both are equal or the lesser one equals 0. This is the other case which was to be proved.

Corollary 1

Since $a^2 = pp + qq$ and $b^2 = 2pq$, and similarly $q = mm - nn$ and $p = 2mn$, and furthermore $m = x^2$ and $n = y^2$, therefore $a^2 = (x^4 + y^4)^2$ and $b^2 = 4x^2y^2(x^4 - y^4)$. From this, we will have $a = x^4 + y^4$ and $b = 2xy\sqrt{x^4 - y^4}$.

Corollary 2

Therefore, if among small numbers x and y there are such numbers that the difference of their fourth powers constitutes a square, then at once from them can be found much larger numbers a and b enjoying the same property.

Corollary 3

Hence it can be seen more clearly that the case in which fourth powers are equal or one of them is 0 do not present new cases; that is, by setting either $x = y$ or $y = 0$ also makes $b = 0$, from which the validity of the proof is better seen through it.

Corollary 4

Moreover, it follows from the proof that there are no numbers p and q with the characteristic that $2p, q$ and $pp + qq$ are squares. Therefore, if one sets $2p = 4xx$ and $q = yy$, there cannot be a square of the form $4x^4 + y^4$.

Corollary 5

It also follows from this formula that neither $ab(aa-bb)$ nor $2ab(aa+bb)$ can ever be a square; this is valid, not only if a and b are relatively prime numbers, but also if they are not relatively prime and, thus, even fractional. That is, fractions of this kind are easily reduced to whole numbers, and whole numbers, to relatively prime numbers.

Corollary 6

Therefore, through these two propositions, it has been shown that the following nine expressions can never be squares:

I. $a^4 + b^4$

II. $a^4 - 4b^4$

III. $4a^4 - b^4$

IV. $ab(aa + bb)$

V. $2ab(aa - bb)$

VI. $a^4 - b^4$

VII. $4a^4 + b^4$

VIII. $ab(aa - bb)$

IX. $2ab(aa + bb)$

X. $2a^4 \pm 2b^4$

Accordingly, I have added the tenth expression because its validity will be proved next.

Theorem 3

The sum of two fourth powers multiplied by 2, that is, $2a^4 + 2b^4$, cannot be a square unless $a = b$.

Proof

First, I suppose a and b to be relatively prime numbers; and if they were not such, their form could thus be reduced through division. Now it is easily seen that both of the numbers a and b must be odd; for if one of them were even, then the number the number $2a^4 + 2b^4$ would be 2 more than a multiple of 4, which cannot be a square. Moreover, this form corresponds with $(aa+bb)^2+(aa-bb)^2$, which one can therefore prove cannot be a square unless $a = b$. But because a and b are odd numbers, $(aa + bb)^2$ and $(aa - bb)^2$ will be even numbers, the former certainly 2 more than a multiple of 4, and the latter surely a multiple of 4. Thus, one has arrived at the form $\left(\frac{aa+bb}{2}\right)^2 + \left(\frac{aa-bb}{2}\right)^2$ in which $\frac{aa+bb}{2}$ and $\frac{aa-bb}{2}$ are relatively prime numbers, the former odd and the latter certainly even; on account of this, if the given form were a square, then

$$\frac{aa+bb}{2} = pp - qq \text{ and } \frac{aa-bb}{2} = 2pq$$

so that $a^2 = pp + 2pq - qq$ and $b^2 = pp - 2pq - qq$ are obtained, of which the difference is

$$4pq = aa - bb;$$

and therefore $a + b = \frac{2mp}{n}$ and $a - b = \frac{2nq}{m}$, so that

$$a = \frac{mp}{n} + \frac{nq}{m} \text{ and } b = \frac{mp}{n} - \frac{nq}{m}.$$

Now with this substitution, the products will be

$$\frac{mm}{nn}pp + \frac{nn}{mm}qq = pp - qq \text{ and } \frac{pp}{qq} = \frac{nn(mm+nn)}{mm(nn-mm)} = \frac{nn(n^4-m^4)}{mm(nn-mm)^2}.$$

Therefore, it would be necessary for $n^4 - m^4$ to be a square, which, by the preceding theorem, cannot be.

Corollary 1

Therefore, if a and b are odd numbers, $2ab(aa+bb)$ cannot be a square either; for a, b, and $2aa + 2bb$ would be squares, which, by this theorem, cannot be.

Corollary 2

Therefore, the proof also could have been crafted from the ninth formula $2ab(aa + bb)$; but here one of the numbers a and b will be assumed even, and the other odd; if nothing else were a hindrance, this still plays a role in giving this particular proof.

Corollary 3

Therefore, by this proof, the very truth of the ninth formula is very much confirmed, since based on this it is evident that $2ab(aa+bb)$ cannot be a square, even if the numbers a and b are both odd.

Corollary 4

Surely the truth of this theorem can also be made clear quickly from the form $(a^2+b^2)^2+(a^2-b^2)^2$, which thus cannot be a square, because $(a^2+b^2)^2-(a^2-b^2)^2$ is a square. Now it is impossible that the sum of two squares is a square if the difference of the same squares is a square. For if both $pp+qq$ and $pp-qq$ were squares, p^4-q^4 would be a square, which is impossible.

Corollary 5

In the same way, $a^4-6aabb+b^4$ cannot be a square. For

$$a^4-6aabb+b^4=(aa-bb)^2-4aabb,$$

which is the difference of this kind of squares, squares for which the sum is a square.

Corollary 6

And in the same way, $a^4+6a^2b^2+b^4$ cannot be a square, because it is equal to $(a^2+b^2)^2+4aabb$; the sum of these squares cannot be a square, because the difference of the same squares $(a^2+b^2)^2-4aabb$ is a square.

Theorem 4

Twice the difference of two fourth powers, that is, $2a^4-2b^4$, cannot be a square unless $a=b$.

Proof

Let us suppose that a and b are coprime numbers and $2a^4-2b^4$ is a square; a and b will be odd numbers. Then $2(a-b)(a+b)(aa+bb)$ would be a square and therefore one-sixteenth of this as well, that is, $\left(\frac{a-b}{2}\right)\left(\frac{a+b}{2}\right)\left(\frac{aa+bb}{2}\right)$; the factors, since they are relatively prime, should each be squares. Thus,

$$\frac{a-b}{2}=pp \text{ and } \frac{a+b}{2}=qq;$$

then,

$$a=pp+qq \text{ and } b=qq-pp$$

so that

$$\frac{aa+bb}{2}=p^4+q^4.$$

Therefore, since p^4+q^4 cannot be a square, $\frac{aa+bb}{2}$ and thus $2a^4-2b^4$ cannot be squares either. Q. E. D.

Theorem 5

Neither $ma^4 - m^3b^4$ nor $2ma^4 - 2m^3b^4$ can be a square.

Proof

Let us suppose that a and b are coprime numbers and that m is a number that is neither a square nor divisible by a square; for if m were divisible by a square, then a square factor could be removed from it by division. Next, suppose m is a number relatively prime to both a and b; then because

$$ma^4 - m^3b^4 = m(aa - mbb)(aa + mbb)$$

and all the factors are relatively prime, individual terms must therefore be squares. Thus, with $m = pp$, $(aa - ppbb)(aa + ppbb)$ must be a square, which is impossible.

In the same way, because $2ma^4 - 2m^3b^4 = 2m(aa - mbb)(aa + mbb)$ and the factors are either relatively prime or there will be two having common measure 2, either $2m$ or m will be a square; certainly in the first case, it is necessary that $2m = 4pp$, so that $a^4 - 4p^4b^4$ is a square, which is equally impossible. But if $m = pp$, then $2a^4 - 2p^4b^4$ would be a square, which is impossible by the preceding theorem.

But if m is not prime with respect to a, let us set $m = rs$ and $a = rc$, where r and s are relatively prime numbers, because m is assumed to have no square factor. Therefore, the squares would be of the form $r^5sc^4 - r^3s^3b^4$ and $2r^5sc^4 - 2r^3s^3b^4$, or $r^3sc^4 - rs^3b^4$ and $2r^3sc^4 - 2rs^3b^4$. Now because the factors of those formulas are relatively prime, either rs or $2rs$ must be squares and thus r and either s or $2s$ individually, from which the formulas arise. It has now been established that these cannot be squares. Q. E. D.

Corollary 1

Therefore, squares cannot be of the form $mn(m^2a^4 - n^2b^4)$ or $2mn(m^2a^4 - n^2b^4)$, even for any numbers chosen in the place of m, n, a, and b.

Corollary 2

Therefore, if $maa + nbb$ is a square, neither $m^2naa - mn^2bb$ nor $2m^2naa - 2mn^2bb$ can be a square. And if $maa - nbb$ is a square, neither $m^2naa + mn^2bb$ nor $2m^2naa + 2mn^2bb$ can be a square.

Corollary 3

Let us set $maa + nbb = cc$; then $m = \frac{cc - nbb}{aa}$; therefore, neither $n(cc - nbb)(cc - 2nbb)$ nor $2n(cc - nbb)(cc - 2nbb)$ can be a square. And if $m = \frac{cc + nbb}{aa}$; then neither of the forms $n(cc + nbb)(cc + 2nbb)$ nor $2n(cc + nbb)(cc + 2nbb)$ can be a square.

Corollary 4

If $c = \pm pp + nqq$ and $b = 2pq$, the following formulas will be obtained, which can in no way ever produce squares[3]: $n(p^4 \pm 6nppqq + n^2q^4)$ and $2n(p^4 \pm 6nppqq + n^2q^4)$.

Theorem 6

Neither $ma^4 + m^3b^4$ nor $2ma^4 + 2m^3b^4$ can be a square.

Proof

First I note that if $mp^2 - mq^2$ is a square, then neither $mp^2 + mq^2$ nor $2mp^2 + 2mq^2$ can be a square in any way, for this would make either $m^2(p^4 - q^4)$ or $2m^2(p^4 - q^4)$ a square, contrary to what has just been proved. Now let us make $mp^2 - mq^2$ a square, with its root set equal to $\frac{(p-q)a}{b}$; then $mp + mq = \frac{a^2p - a^2q}{bb}$, from which it is found that $q = \frac{p(aa - mbb)}{aa + mbb}$. Therefore, let $p = a^2 + mb^2$; then $q = a^2 - mb^2$ and therefore $p^2 + q^2 = 2a^4 + 2m^2b^4$. Thus, first, $mp^2 + mq^2 = 2ma^4 + 2m^3b^4$ will not be a square; next, $2mp^2 + 2mq^2 = 4ma^4 + 4m^3b^4$. From these, it is concluded that neither $ma^4 + m^3b^4$ nor $2ma^4 + 2m^3b^4$ can be a square.

Corollary

Therefore, in these two theorems it has been proven irrefutably that no numbers of the forms $ma^4 \pm m^3b^4$ or $2ma^4 \pm 2m^3b^4$ can be squares. In these formulas are contained all the preceding ones.

[3]Translator: In the *Opera Omnia*, Ferdinand Rudio points out that the *Commentationes arithmeticae collectae* have $n(p^6 \pm 6nppqq + n^2q^4)$ and $2n(p^6 \pm 6nppqq + n^2q^4)$, which each have an error in an exponent. Rudio corrects these.

Theorem 7

from Fermat [4]

No triangular whole number can be a fourth power except 1.

Proof

Every triangular number is contained in the form $\frac{x(x+1)}{2}$. Thus, it is to be proved that this formula can never be a fourth power if whole numbers are substituted in place of x, except in the case when $x = 1$. But it must be noted that the number x is either even or odd; thus, in the first case $\frac{x}{2}(x+1)$ must be a fourth power, and in the second case, certainly $x\left(\frac{x+1}{2}\right)$; in both of the products the two factors are relatively prime and therefore both must be fourth powers. Thus, in the first case let $\frac{x}{2} = m^4$ (that is, $x = 2m^4$); $x + 1 = 2m^4 + 1$ must be a fourth power. Certainly, in the second case, let $\frac{x+1}{2} = m^4$ so that $x = 2m^4 - 1$, which also must be a fourth power. On account of this, $2m^4 \pm 1$ should be a fourth power. Let $2m^4 \pm 1 = n^4$; then $4m^4 = 2n^4 \mp 2$; thus, $2n^4 \mp 2$ would be $4m^4$, which is a square. But above we have proved that $2a^4 \pm 2b^4$ and thus also $2n^4 \pm 2$ can never be a square except in the case $n = 1$. But setting $n = 1$ makes $m = 0$ or $m = 1$, and so x is 0 or 1. Therefore, there is no whole number which, substituted in place of x, would make $\frac{x(x+1)}{2}$ a fourth power, apart from the cases $x = 0$ and $x = 1$. Consequently, no triangular whole number which is a fourth power exists apart from 1 and 0.

Corollary 1

If $\frac{xx+x}{2}$ is set equal to y^4, then $4xx + 4x + 1 = 8y^4 + 1 = (2x+1)^2$. From this it follows that the form $8y^4 + 1$, with whole numbers substituted in place of y, can never be a square except in the cases $y = 0$ and $y = 1$.

Corollary 2

[4] Translator: In the *Opera Omnia*, Ferdinand Rudio notes, "This theorem of Fermat had first been stated without proof in the margins of the book of Diophantus [translated] by Bachet in observations edited by his son S. Fermat; the book is inscribed, *Cum Commentariis C.G. Bacheti V. C. et observationibus D.P. de Fermat Senatoris Tolosani. Accessit Doctrinae Analyticae inventum novum, collectum ex variis eiusdem D. de Fermat Epistolis. Tolosae 1670.* Here the theorem is found discussed in an observation to problem XX of the commentary about the last question of *Arithmeticorum* by Diophantus, p.338, and in *Oeuvres de Fermat,* vol.I, p.340."

If $8y^4+1$ is set equal to z^2, then $16y^4 = 2z^2 - 2$. Therefore, $2z^2 - 2$ can never be a fourth power for any whole numbers substituted in place of z except the cases $z = 1$ and $z = 3$.

Theorem 8

The sum of three fourth powers of which two are equal, that is, a form of the type $a^4 + 2b^4$, cannot be a square unless $b = 0$.

Proof

We suppose that a^4+2b^4 is a square and its root is $a^2 + \frac{m}{n}b^2$, where a and b are relatively prime numbers, as are m and n. Now the results of the equations will be $2n^2b^2 = 2mna^2 + m^2b^2$ and

$$\frac{b^2}{a^2} = \frac{2mn}{2n^2 - m^2};$$

this fraction either has simplest form already or will be reduced to simplest form after division by 2.

We suppose first that $2mn$ and $2n^2-m^2$ are relatively prime numbers, which will happen if m is an odd number and

$$b^2 = 2mn \text{ and } a^2 = 2n^2 - m^2.$$

Here two cases need to be rolled out, one of them in which n is an odd number, and the other in which n is even; in the first case, where n is odd, it needs to be shown that, because m is also odd, $2mn$ cannot be a square; certainly in the second case, in which n is an even number, $a^2 = 2n^2 - m^2$, that is, $a^2 + m^2 = 2n^2$, is impossible, because a and m are odd numbers and $2n^2$ is a multiple of 4.

Therefore, $2mn$ and $a^2 = 2n^2 - m^2$ have common divisor 2, which happens if m is an even number, say $m = 2k$; n will be an odd number; thus, $\frac{b^2}{a^2} = \frac{4kn}{2n^2-4k^2} = \frac{2kn}{n^2-2k^2}$, where $2kn$ and $n^2 - 2k^2$ are relatively prime numbers. Thus, because b^2 and a^2 are also relatively prime,

$$b^2 = 2kn \text{ and } a^2 = n^2 - 2k^2.$$

But here $2kn$ cannot be a square unless k is an even number. So let k be an even number; now n and $2k$ must both be squares; thus, let $n = cc$ and $2k = 4dd$, where c is an odd number; with this done, we will have

$$a^2 = c^4 - 8d^4.$$

Therefore, let us investigate whether $c^4 - 8d^4$ can be a square; let us set its root equal to $c^2 - \frac{2p}{q}d^2$, and so $2q^2d^2 = pqc^2 - p^2d^2$; that is,

$$\frac{dd}{cc} = \frac{pq}{pp + 2qq},$$

where, again, c and d are relatively prime numbers, as are p and q. Here, once more, two cases are to be noted: let p be either an odd number or an even number. Therefore, first let p be an odd number; because pq and $pp + 2qq$ are relatively prime numbers,

$$dd = pq \text{ and } cc = pp + 2qq.$$

Therefore, it is necessary that both p and q be squares; consequently, I set $p = x^2$ and $q = y^2$, and

$$cc = x^4 + 2y^4$$

will appear; therefore, if $a^4 + 2b^4$ were a square, then $x^4 + 2y^4$ would also be a square, and the numbers x and y will be much smaller than a and b; and in turn it would be possible to find smaller ones from these, which is impossible among the whole numbers. For the second case, in which p is an even number, let us set $p = 2r$; then[5] $\frac{dd}{cc} = \frac{2qr}{4rr + 2qq} = \frac{qr}{2rr + qq}$, and because q is odd, qr and $2rr + qq$ will be relatively prime numbers. Thus,

$$dd = qr \text{ and } cc = 2rr + qq;$$

as a consequence, each of the numbers q and r must be a square; thus, setting $q = x^2$ and $r = y^2$ will make

$$cc = 2y^4 + x^4;$$

from this it is evident that if $a^4 + 2b^4$ were a square, then the similar form $x^4 + 2y^4$ in even smaller numbers would also be a square. Therefore, $a^4 + 2b^4$ cannot be a square unless $b = 0$. Q. E. D.

Corollary 1

Since we find that $\frac{b^2}{a^2} = \frac{2mn}{2n^2 - m^2}$ in making $a^4 + 2b^4$ a square, it follows that $2mn(2n^2 - m^2)$ cannot be a square for any numbers substituted in place of m and n.

Corollary 2

[5]Translator: The *Commentationes arithmeticae collectae* and the *Opera Omnia* have $\frac{dd}{cc} = \frac{2qr}{2rr + 2qq} = \frac{qr}{2rr + qq}$, which is a mistake.

Therefore given $m = x^2$ and $n = y^2$, a square will not be of the form $4y^4 - 2x^4$, In the same way, given $2m = 4x^2$ and $n = y^2$, a square will not be of the form $2y^4 - 4x^4$. And given $m = x^2$ and $2n = 4y^2$, the formula $8y^4 - x^4$ cannot be a square.

Corollary 3

More generally, if one sets $m = \alpha x^2$ and $n = \beta y^2$, the formula $2\alpha\beta(2\beta^2 y^4 - \alpha^2 x^4)$, that is, $4\alpha\beta^3 y^4 - 2\alpha^3\beta x^4$ arises, which cannot be a square in any way.

Theorem 9

If the form $a^4 + kb^4$ cannot be a square, then the form $2k\alpha\beta^3 y^4 - 2\alpha^3\beta x^4$ cannot produce a square in any way either.

Proof

Let us suppose that the given form $a^4 + kb^4$ is a square, and its root is $a^2 + \frac{m}{n}b^2$; then $kn^2b^2 = 2mna^2 + m^2b^2$ and $\frac{b^2}{a^2} = \frac{2mn}{kn^2 - m^2}$. Therefore, because $a^4 + kb^4$ cannot be a square, then neither $\frac{2mn}{kn^2 - m^2}$ nor $2mn(kn^2 - m^2)$ will be a square either. Let $m = \alpha x^2$ and $n = \beta y^2$; then $2\alpha\beta(k\beta^2 y^4 - \alpha^2 x^4)$, that is, $2k\alpha\beta^3 y^4 - 2\alpha^3\beta x^4$ will appear; this formula can thus never be a square for any numbers, positive or negative, which are substituted in place of α and β.

Corollary 1

Let either α or β be negative, so that the form $2\alpha^3\beta x^4 - 2k\alpha\beta^3 y^4$ appears, and let $2\alpha^3\beta = p^2$; then $\beta = \frac{p^2}{2\alpha^3}$, so that the form becomes $p^2 x^4 - \frac{kp^6}{4\alpha^8}y^4$. Therefore, the form $x^4 - 4ky^4$ with $4y^4$ substituted for $\frac{p^4}{4\alpha^8}y^4$, cannot be a square. Thus, from this formula, it furthermore follows that the expression $2\alpha^3\beta x^4 + 8k\alpha\beta^3 y^4$ cannot be a square.

Corollary 2

In the formula that was obtained, $2k\alpha\beta^3 y^4 - 2\alpha^3\beta x^4$, let $2k\alpha\beta^3 = pp$ so that $\alpha = \frac{pp}{2k\beta^3}$; that will become $p^2 y^4 - \frac{p^6}{4k^3\beta^8}x^4$, from which it follows that $a^4 - kb^4$ cannot be a square; from this it follows that the $2\alpha^3\beta x^4 + 8k\alpha\beta^3 y^4$ from before will not be a square.

Corollary 3

Therefore, if $x^4 - 4ky^4$ cannot be a square, then neither the formula

$$2k\alpha\beta^3y^4 - 2\alpha^3\beta x^4$$

nor

$$\alpha^3\beta x^4 + k\alpha\beta^3y^4$$

will be a square, the latter of which follows from the preceding corollaries with 2α written in place of α.

Corollary 4

Therefore, since a^4+b^4 cannot be a square, the two formulas $\alpha^3\beta x^4+\alpha\beta^3y^4$ and $2\alpha\beta^3y^4 - 2\alpha^3\beta x^4$ will definitely not be squares.

Corollary 5

And because $a^4 - b^4$ cannot be a square, two new formulas arise, $\alpha^3\beta x^4 - \alpha\beta^3y^4$ and $2\alpha^3\beta x^4 + 2\alpha\beta^3y^4$, which cannot in any way yield squares.

Corollary 6

Finally, since a^4+2b^4 cannot be a square, the formulas $\alpha^3\beta x^4+2\alpha\beta^3y^4$ and $4\alpha\beta^3y^4 - 2\alpha^3\beta x^4$ will not produce squares either.

Scholium

Therefore, from what I have proved thus far arise the following six more general formulas which cannot be transformed into squares in any way:

I. $\alpha^3\beta x^4 + \alpha\beta^3y^4$

II. $\alpha^3\beta x^4 - \alpha\beta^3y^4$

III. $\alpha^3\beta x^4 + 2\alpha\beta^3y^4$

IV. $2\alpha^3\beta x^4 - 2\alpha\beta^3y^4$

V. $2\alpha^3\beta x^4 + 2\alpha\beta^3y^4$

VI. $2\alpha^3\beta x^4 - 4\alpha\beta^3y^4$

And everything which we drew out in the previous formulas is contained in these six formulas. Now from these formulas, before I have already done it, trinomial formulas could have been drawn out; in the same way it is certain that these cannot be rendered squares; but concerning things to be proved, I refrain from advancing several other theorems which are about cubes and cannot be obtained from those formulas.

Theorem 10

No cube plus one, not even fractional numbers, can produce a square, apart from one case: when the cube is 8.

Proof

The proposition comes to this, that $\frac{a^3}{b^3}+1$ can never be a square, except in the case when $\frac{a}{b}=2$. Therefore, it must be proved that the formula a^3b+b^4 can never be a square unless $a=2b$. Now this expression can be broken up into the three factors $b(a+b)(aa-ab+bb)$, which initially can determine a square if $b(a+b)=aa-ab+bb$, so that $a=2b$, which is the case we have removed. So that I may proceed further, I have now set $a+b=c$, that is, $a=c-b$; with substitution this will be

$$bc(cc-3bc+3bb);$$

it must be proved this cannot be a square unless $c=3b$; Now b and c are relatively prime numbers. Here there are two cases to be considered: whether c is a multiple of 3 or not. In the first case, the factors c and $cc-3bc+3bb$ will have common divisor 3; by this certainly all three factors will be relatively prime.

First, let c not be divisible by 3; it will be necessary that each of the three factors be squares, that is, b and c and $cc-3bc+3bb$ individually. Therefore, let $cc-3bc+3bb=\left(\frac{m}{n}b-c\right)^2$; then,

$$\frac{b}{c}=\frac{3nn-2mn}{3nn-mm} \text{ or } \frac{b}{c}=\frac{2mn-3nn}{mm-3nn};$$

the terms of the fractions will be relatively prime unless m is a multiple of 3. Thus, let m not be divisible by 3; then either $c=3nn-mm$ or $c=mm-3nn$, and either $b=3nn-2mn$ or $b=2mn-3nn$. But since $3nn-mm$ cannot be a square, let $c=mm-3nn$; let the square root of this be $m-\frac{p}{q}n$, and so it arises that $\frac{m}{n}=\frac{3qq+pp}{2pq}$, and

$$\frac{b}{nn}=\frac{2m}{n}-3=\frac{3qq-3pq+pp}{pq}.$$

Thus, a square would have the formula $pq(3qq-3pq+pp)$, which is altogether the same as the given $bc(3bb-3bc+cc)$ and composed of much smaller numbers. Now let m be a multiple of 3 and set $m=3k$; then $\frac{b}{c}=\frac{nn-2kn}{nn-3kk}$, so that either $c=nn-3kk$ or $c=3kk-nn$; but because $3kk-nn$ cannot be a square, let $c = nn - 3kk$, and let its root be $n - \frac{p}{q}k$, so that $\frac{n}{k} = \frac{3qq + pp}{2pq}$; that is, $\frac{k}{n} = \frac{2pq}{3qq + pp}$, and

$$\frac{b}{nn} = 1 - \frac{2k}{n} = \frac{pp + 3qq - 4pq}{3qq + pp}.$$

Therefore, $(pp + 3qq)(p - q)(p - 3q)$ should be a square. Let $p - q = t$ and $p - 3q = u$; then $q = \frac{t - u}{2}$ and $p = \frac{3t - u}{2}$, and that formula is changed into $tu(3tt - 3tu + uu)$, which again is the same as the previous $bc(3bb - 3bc + cc)$.

Thus, the latter case remains, which is that c is a multiple of 3; let $c = 3d$, and $bd(bb - 3bd + 3dd)$; again, since this is the same as before, it is clear that in both cases, the given formula cannot be a square. Consequently, apart from 8, there is no cube, not even a fraction, which when added to 1 would make a square.

Corollary 1

In the same way, it can be proved that no cube minus 1 can be a square, not even with fractions.

Corollary 2

Hence, it follows that neither $x^6 + y^6$ nor $x^6 - y^6$ can be a square, and that no triangular number can be a cube apart from 1.