

A commentary on the continued fraction by which the illustrious La Grange has expressed the binomial powers *

Leonhard Euler[†]

I. This illustrious man has converted the Binomial power $(1+x)^n$, by his most singular method of logarithmic differentials, into this continued fraction:

$$(1+x)^n = 1 + \frac{nx}{1 + \frac{(1-n)x}{2 + \frac{(1+n)x}{3 + \frac{(2-n)x}{2 + \frac{(2+n)x}{5 + \frac{(3-n)x}{2 + \frac{(3+n)x}{7 + \text{etc.}}}}}}}}$$

which expression celebrates the marvelous property that whenever the exponent n is an integral number, either positive or negative, it is halted and is reduced to a finite form.

II. Seeing that this continued fraction does not proceed by a uniform law, but rather is interrupted, we may bring it to a uniform law, as it would be most desirable, if we should represent it in the following way by parts:

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[†]Date of translation: July 21, 2005. Translated from the Latin by Jordan Bell, 3rd year undergraduate in Honours Mathematics, School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada. Email: jbell3@connect.carleton.ca. This translation was written during an NSERC USRA supervised by Dr. B. Stevens.

$$\begin{aligned}
(1+x)^n &= 1 + \frac{nx}{A}; \\
A &= 1 + \frac{(1-n)x}{2 + \frac{(1+n)x}{B}}; \\
B &= 3 + \frac{(2-n)x}{2 + \frac{(2+n)x}{C}}; \\
C &= 5 + \frac{(3-n)x}{2 + \frac{(3+n)x}{D}}; \\
D &= 7 + \frac{(4-n)x}{2 + \frac{(4+n)x}{E}}; \\
&\text{etc.}
\end{aligned}$$

From here, we therefore will have by reducing:

$$\begin{aligned}
A &= 1 + \frac{(1-n)Bx}{2B+(1+n)x} = 1 + \frac{(1-n)x}{2} - \frac{(1-nn)xx \cdot 2}{2B+(1+n)x} \\
&= 1 + \frac{(1-n)x}{2} + \frac{(nn-1)xx \cdot 4}{B+(\frac{1+n}{2})x}.
\end{aligned}$$

In a similar way it will be:

$$\begin{aligned}
B &= 3 + \frac{(2-n)Cx}{2C+(2+n)x} = 3 + \frac{(2-n)x}{2} - \frac{(4-nn)xx \cdot 2}{2C+(2+n)x} \\
&= 3 + \frac{(2-n)x}{2} + \frac{(nn-4)xx \cdot 4}{C+(\frac{2+n}{2})x}.
\end{aligned}$$

In the very same way we shall have:

$$\begin{aligned}
C &= 5 + \frac{(3-n)Dx}{2D+(3+n)x} = 5 + \frac{(3-n)x}{2} - \frac{(9-nn)xx \cdot 2}{2D+(3+n)x} \\
&= 5 + \frac{(3-n)x}{2} + \frac{(nn-9)xx \cdot 4}{D+(\frac{3+n}{2})x},
\end{aligned}$$

and so on.

III. Now if we were to substitute these values in succession in the place of A, B, C , etc., the continued fraction will be induced to the following form:

$$(1+x)^n = 1 + \frac{nx}{1 + \frac{(1-n)x}{2} + \frac{\frac{(nn-1)xx \cdot 4}{(nn-4)xx \cdot 4}}{3(1+\frac{1}{2}x) + \frac{\frac{(nn-9)xx \cdot 4}{(nn-16)xx \cdot 4}}{5(1+\frac{1}{2}x) + \frac{\frac{(nn-16)xx \cdot 4}{etc.}}{7(1+\frac{1}{2}x) + \dots}}}}},$$

IV. So that we can remove these partial fractions, we set $x = 2y$, so that this expression shall be obtained:

$$(1 + 2y)^n = 1 + \frac{2ny}{1 + (1 - n)y + \frac{(nn-1)yy}{3(1+y) + \frac{(nn-4)yy}{5(1+y) + \frac{(nn-9)yy}{7(1+y) + \text{etc.}}}}},$$

which form can easily be transformed into this:

$$\frac{2ny}{(1 + 2y)^n - 1} = 1 + (1 - n)y + \frac{(nn - 1)yy}{3(1 + y) + \frac{(nn-4)yy}{5(1+y) + \text{etc.}}}.$$

Then ny is added to both sides, so that it will emerge

$$\frac{ny(1 + (1 + 2y)^n)}{(1 + 2y)^n - 1} = 1 + y + \frac{(nn - 1)yy}{3(1 + y) + \frac{(nn-4)yy}{5(1+y) + \text{etc.}}};$$

this expression is ordered enough that it may proceed to be regulated.

V. We will now divide both sides by $1 + y$, and the left member will come out as: $\frac{ny}{1+y} \cdot \frac{(1+2y)^n + 1}{(1+2y)^n - 1}$. From the right side moreover, each of the fractions on the top and bottom should be divided by $1 + y$, and this form will extend:

$$1 + \frac{(nn - 1)yy \cdot (1 + y)^2}{3 + \frac{(nn-4)yy \cdot (1+y)^2}{5 + \frac{(nn-9)yy \cdot (1+y)^2}{7 + \frac{(nn-16)yy \cdot (1+y)^2}{9 + \frac{(nn-25)yy \cdot (1+y)^2}{11 + \text{etc.}}}}},$$

VI. Then we may reduce this expression again for greater elegance, with it being set $\frac{y}{1+y} = z$, so that it may thus be $y = \frac{z}{1-z}$. Then presently the left member, on account of $1 + 2y = \frac{1+z}{1-z}$, will admit this form: $\frac{nz[(1+z)^n + (1-z)^n]}{(1+z)^n - (1-z)^n}$, because of which it may therefore be equated to this continued fraction:

$$1 + \frac{(nn - 1)zz}{3 + \frac{(nn-4)zz}{5 + \frac{(nn-9)zz}{7 + \frac{(nn-16)zz}{9 + \text{etc.}}}}},$$

which, by its elegance, merits the highest attention.

VII. Now, it is therefore manifest for this expression to always be halted whenever n is an integral number, either positive or negative. It is moreover

evident for the left member to retain the same value even if for n is written $-n$. Namely, this fact comes forth from:

$$\frac{-nz[(1+z)^{-n} + (1-z)^{-n}]}{(1+z)^{-n} - (1-z)^{-n}},$$

which fraction, if multiplied above and below by $(1-zz)^n$, induces this form:

$$\frac{-nz[(1-z)^n + (1+z)^n]}{(1-z)^n - (1+z)^n} = \frac{nz[(1+z)^n + (1-z)^n]}{(1+z)^n - (1-z)^n},$$

which is the same as the preceding expression. Thus it is the same whether the positive or negative of the letter n is taken.

VIII. Thus if we were to take $n = \pm 1$, the left member would be equal to 1, which is moreover the value of the right. Furthermore, by putting $n = \pm 2$, the left member will come forth as equal to $1 + zz$, and indeed the right member will also be equal to $1 + zz$. In a similar way, by taking $n = \pm 3$, the left part, and in turn the right, becomes $\frac{3(1+3zz)}{3+zz}$.

IX. Here one may deduce several conclusions of great importance, depending on whenever a vanishing or infinite value is taken for the exponent n , but first of all for the case in which an imaginary value is taken for the letter z , which leads to an outstanding conclusion, since this continued fraction shall nevertheless remain real, for which conclusion we will therefore take up first.

Conclusion I. where $z = t\sqrt{-1}$

X. In this case therefore the continued fraction will have this form:

$$1 - \frac{(nn-1)tt}{3 - \frac{(nn-4)tt}{5 - \frac{(nn-9)tt}{7 - \frac{(nn-16)tt}{9 - \text{etc.}}}}},$$

and to be sure the left part will now be:

$$\frac{nt\sqrt{-1}[(1+t\sqrt{-1})^n + (1-t\sqrt{-1})^n]}{(1+t\sqrt{-1})^n - (1-t\sqrt{-1})^n},$$

which not having been opposed by imaginary parts ought certainly to have a real value, which we shall now investigate. Then to this end we put $t = \frac{\sin \phi}{\cos \phi}$, so that it will thus be $t = \text{tang } \phi$; then it will therefore be:

$$(1 + t\sqrt{-1})^n = \frac{(\cos \phi + \sqrt{-1} \sin \phi)^n}{\cos \phi^n} = \frac{\cos n\phi + \sqrt{-1} \sin n\phi}{\cos \phi^n},$$

and in a similar way:

$$(1 - t\sqrt{-1})^n = \frac{(\cos \phi - \sqrt{-1} \sin \phi)^n}{\cos \phi^n} = \frac{\cos n\phi - \sqrt{-1} \sin n\phi}{\cos \phi^n}.$$

Therefore by substituting these values our left member will come forth:

$$\frac{2n\sqrt{-1} \cdot \text{tg } \phi \cos n\phi}{2\sqrt{-1} \sin n\phi} = \frac{n \text{tg } \phi \cos n\phi}{\sin n\phi} = \frac{n \text{tg } \phi}{\text{tg } n\phi}.$$

XI. Therefore by putting $\text{tg } \phi = t$ we will have the following most remarkable continued fraction;

$$\frac{nt}{\text{tg } n\phi} = 1 - \frac{(nn-1)tt}{3 - \frac{(nn-4)tt}{5 - \frac{(nn-9)tt}{7 - \text{etc.}}}}$$

which then will be able to be represented in this way:

$$\text{tg } n\phi = \frac{nt}{1 - \frac{(nn-1)tt}{3 - \frac{(nn-4)tt}{5 - \frac{(nn-9)tt}{7 - \text{etc.}}}}}$$

which expression therefore is able to be helpfully applied to the tangents of multiplied angles which are to be expressed by the tangent of the single angle t . Thus if it were $n = 2$, we will have $\text{tg } 2\phi = \frac{2t}{1-tt}$. In the very same way if $n = 3$, it will be:

$$\text{tg } 3\phi = \frac{3t}{1 - \frac{8tt}{3-tt}} = \frac{3t - t^3}{1 - 3tt}.$$

In the most notable case presented whenever the exponent n is taken as less than infinite, it will then be $\text{tg } n\phi = n\phi$, therefore, by dividing both sides by n , this form arises:

$$\phi = \frac{t}{1 + \frac{tt}{3 + \frac{4tt}{5 + \frac{9tt}{7 + \text{etc.}}}}}$$

where the continued fraction is expressed by the tangent t of the angle itself.

XII. We will now consider the case in which an infinite magnitude is taken for the exponent n , but when the angle ϕ less than infinite, and then too for the tangent t of it less than infinite, so that it would thus be $n\phi = \theta$, and then also $nt = \theta$; then we will therefore have such a continued fraction:

$$\operatorname{tg} \theta = \frac{\theta}{1 - \frac{\theta\theta}{3 - \frac{\theta\theta}{5 - \frac{\theta\theta}{7 - \text{etc.}}}}},$$

by which formula, from the given angle θ , the tangent of it will be able to be determined, which expression will be able to be seen just as the reciprocal of the preceding.

Conclusion II. where a vanishing exponent n is taken:

XIII. Therefore in this case the continued fraction will be:

$$1 - \frac{zz}{3 - \frac{4zz}{5 - \frac{9zz}{7 - \frac{16zz}{9 - \text{etc.}}}}}$$

It is moreover to be noted for the left part to be $\frac{(1+z)^n - 1}{n} = l(1+z)$, and for that reason $(1+z)^n = 1 + nl(1+z)$; in a similar way it will be: $(1-z)^n = 1 + nl(1-z)$, from which the left member comes forth as

$$nz \frac{[2 + nl(1+z) + nl(1-z)]}{nl(1+z) - nl(1-z)} = \frac{2z}{l \frac{1+z}{1-z}};$$

here therefore we will have such a form:

$$\frac{2z}{l \frac{1+z}{1-z}} = 1 - \frac{zz}{3 - \frac{4zz}{5 - \frac{9zz}{7 - \frac{16zz}{9 - \text{etc.}}}}}$$

and then this logarithm may be expressed in the following way:

$$l \frac{1+z}{1-z} = \frac{2z}{1 - \frac{zz}{3 - \frac{4zz}{5 - \text{etc.}}}}$$

Conclusion III.

where an infinite magnitude is taken for the exponent n

XIV. Here therefore, so that a finite value may be obtained for this continued fraction, of which having been advanced a quantity less than infinity is to be taken for z , it is put $nz = v$, so that it would be $z = \frac{v}{n}$, and thus our continued fraction will be:

$$1 + \frac{vv}{3 + \frac{vv}{5 + \frac{vv}{7 + \frac{vv}{9 + \text{etc.}}}}}$$

Also, it is apparent for the left member to be $(1 + \frac{v}{n})^n = e^v$, and in a similar way $(1 - \frac{v}{n})^n = e^{-v}$; therefore the left member will have this form:

$$\frac{v(e^v + e^{-v})}{e^v - e^{-v}} = \frac{v(e^{2v} + 1)}{e^{2v} - 1},$$

and from this fact we will have this remarkable continued fraction:

$$\frac{v(e^{2v} + 1)}{e^{2v} - 1} = 1 + \frac{vv}{3 + \frac{vv}{5 + \frac{vv}{7 + \frac{vv}{9 + \text{etc.}}}}}$$

whose transcendent value is actually able to be exhibited in this way by a series:

$$\frac{1 + \frac{vv}{1 \cdot 2} + \frac{v^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{v^6}{1 \cdot 2 \cdot \dots \cdot 6} + \text{etc.}}{1 + \frac{vv}{1 \cdot 2 \cdot 3} + \frac{v^4}{1 \cdot 2 \cdot \dots \cdot 5} + \frac{v^6}{1 \cdot 2 \cdot \dots \cdot 7} + \text{etc.}}$$