DE

TRANSFORMATIONE

SERIEI DIVERGENTIS

$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3$$
$$+ m(m+n)(m+2n)(m+3n)x^4 \text{ etc.}$$

IN FRACTIONEM CONTINUAM

Auctore

L. EULERO.

ON THE

TRANSFORMATION

OF THE DIVERGENT SERIES

$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3$$
$$+ m(m+n)(m+2n)(m+3n)x^4 \text{ etc.}$$

INTO A CONTINUED FRACTION[*]

Author

L. EULER.[**]

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1. When once I was closely studying the nature of divergent series of this type, and the true sum of the hypergeometric series

$$1 - 1 + 2 - 6 + 24 - 120 + 720 - \text{etc.}$$

I had assigned by means of a transformation into a continued fraction, I even made mention of this much more extensively accessible series:

$$1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3$$
$$+ m(m+n)(m+2n)(m+3n)x^4 - \text{etc.}$$

which sum I found to be equal to this continued fraction:

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[*] *Nova Acta Academiae Scientarum Imperialis Petropolitanae* **2** (1784) 1788, pp. 36-45. *Opera Omnia* Series I, Volume XVI (first part), pp. 34-46. Numbered 616 in Eneström's index.

[**] Translated from the original Latin in 2003, with additional notes, by C. Edward Sandifer. Typeset and reformatted in 2020 by Erik R. Tou and Christopher Goff.

$$\cfrac{1}{1+\cfrac{mx}{1+\cfrac{nx}{1+\cfrac{(m+n)x}{1+\cfrac{2nx}{1+\cfrac{(m+2n)x}{1+\text{etc.}}}}}}}$$

[37] and the truth of this claim I deduced from the conversion of the Riccatian equation into a continued fraction. Though this demonstration can seem excessively lengthily desired, I will hand over this same reduction here from simpler principles.

2. First it will be appropriate that the general series is brought into a more convenient form by putting $mx = a$ and $nx = b$, and the proposed becomes this infinite series:

$$1 - a + a(a+b) - a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) - \text{etc.}$$

In addition, so that the subsequent calculations can be completed more conveniently, and not the need for so many clauses[1], I establish as follows:

$$a = A,\ a+b = B,\ a+2b = C,\ a+3b = D,\ \text{etc.}$$

and then this series will be obtained:

$$1 - A + AB - ABC + ABCD - \text{etc.}$$

The desired sum of which let us designate with the letter S, so that

$$S = 1 - A + AB - ABC + ABCD - \text{etc.}$$

then it follows that

$$\frac{1}{S} = \frac{1}{1 - A + AB - ABC + ABCD - \text{etc.}}$$

3. Then, since $1/S > 1$, the previous equation is reduced to this form:

$$\frac{1}{S} = 1 + \frac{A - AB + ABC - ABCD + \text{etc.}}{1 - A + AB - ABC + ABCD - \text{etc.}}$$

Now let us put $\frac{1}{S} = 1 + \frac{A}{P}$, and then

$$P = \frac{1 - A + AB - ABC + ABCD - \text{etc.}}{1 - B + BC - BCD + BCDE - \text{etc.}}$$

which expression again surpasses unity, because $B - A = b$, $C - A = 2b$, $D - A = 3b$, etc., this gives

$$P = 1 + \frac{b - 2bB + 3bBC - 4bBCD + etc.}{1 - B + BC - BCD + BCDE - etc.}.$$

Now set $P = 1 + \frac{b}{Q}$ and it gives [38]

[1] *neque tot clausulis sit opus*

$$Q = \frac{1 - B + BC - BCD + BCDE - \text{etc.}}{1 - 2B + 3BC - 4BCD + \text{etc.}}$$

and we deduce that

$$Q = 1 + \frac{B - 2BC + 3BCD - 4BCDE + \text{etc.}}{1 - 2B + 3BC - 4BCD + \text{etc.}}$$

On account of this thing, let us now put $Q = 1 + \frac{B}{R}$, and that will produce

$$R = \frac{1 - 2B + 3BC - 4BCD + \text{etc.}}{1 - 2C + 3CD - 4CDE + \text{etc.}}$$

4. Here, therefore, so much in the numerator as in the denominator the same coefficients occur, but the capital letters in the denominator are promoted by one level. Then since $C - B = b$, $D - B = 2b$, $E - B = 3b$, etc., that makes

$$R = 1 + \frac{2b - 2 \cdot 3bC + 3 \cdot 4bCD + 4 \cdot 5bCDE - \text{etc.}}{1 - 2C + 3CD - 4CDE + 5CDEF - \text{etc.}}$$

Now if we put $R = 1 + \frac{2b}{S}$, we get[2]

$$S = \frac{1 - 2C + 3CD - 4CDE + \text{etc.}}{1 - 3C + 6CD - 10CDE + \text{etc.}}$$

where the numbers that occur in the denominator are obviously triangular, which expression is reduced to this one:

$$S = 1 + \frac{C - 3CD + 6CDE - 10CDEF + \text{etc.}}{1 - 3C + 6CD - 10CDE + \text{etc.}}$$

If we therefore set $S = 1 + \frac{C}{T}$, it becomes

$$T = \frac{1 - 3C + 6CD - 10CDE + 15CDEF - \text{etc.}}{1 - 3D + 6DE - 10DEF + 15DEFG - \text{etc.}}$$

5. This form, since $D - C = b$, $E - C = 2b$, $F - C = 3b$, etc., is changed into this:

$$T = 1 + \frac{3b - 2 \cdot 6bD + 3 \cdot 10bDE - 4 \cdot 15bDEF + \text{etc.}}{1 - 3D + 6DE - 10DEF + 15DEFG - \text{etc.}}$$

Let us put $T = 1 + \frac{3b}{U}$, so that it becomes

$$U = \frac{1 - 3D + 6DE - 10DEF + 15DEFG - \text{etc.}}{1 - 4D + 10DE - 20DEF + 35DEFG - \text{etc.}}$$

where in the denominator are found the first pyramidal numbers, or sums of the triangular numbers, and we get: [39]

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[2] This new value of *S* is, of course, different from the *S* that was introduced in paragraph 2. [ES]

$$U = 1 + \frac{D - 4DE + 10DEF - 20DEFG + \text{etc.}}{1 - 4D + 10DE - 20DEF + 35DEFG - \text{etc.}}$$

where above and below occur the pyramidal numbers. Now we set $U = 1 + \frac{D}{V}$, and that makes

$$V = \frac{1 - 4D + 10DE - 20DEF + 35DEFG - \text{etc.}}{1 - 4E + 10EF - 20EFG + 35EFGH - \text{etc.}}$$

6. Proceeding this calculation as above, since $E - D = b$, $F - D = 2b$, $G - D = 3b$, it will be

$$V = 1 + \frac{4b - 2 \cdot 10bE + 3 \cdot 20bEF - 4 \cdot 35bEFG + \text{etc.}}{1 - 4E + 10EF - 20EFG + 35EFGH + \text{etc.}}$$

Let $V = 1 + \frac{4b}{X}$, so that it makes

$$X = \frac{1 - 4E + 10EF - 20EFG + 35EFGH - \text{etc.}}{1 - 5E + 15EF - 35EFG + 70EFGH - \text{etc.}}$$

which expression is reduced to this:

$$X = 1 + \frac{E - 5EF + 15EFG - 35EFGH + \text{etc.}}{1 - 5E + 15EF - 35EFG + \text{etc.}}$$

Let $X = 1 + \frac{E}{Y}$ and it becomes

$$Y = \frac{1 - 5E + 15EF - 35EFG + 70EFGH - \text{etc.}}{1 - 5F + 15FG - 35FGH + 70FGHI - \text{etc.}}$$

7. Now, therefore, since $F - E = b$, $G - E = 2b$, $H - E - 3b$, etc., it will be

$$Y = 1 + \frac{5b - 2 \cdot 15bF + 3 \cdot 35bFG - 4 \cdot 70bFGH + \text{etc.}}{1 - 5F + 15FG - 35FGH + 70FGHI - \text{etc.}}$$

Now let $Y = 1 + \frac{5B}{Z}$, so that it makes

$$Z = \frac{1 - 5F + 15FG - 35FGH + 70FGHI - \text{etc.}}{1 - 10F + 21FG - 56FGH + 126FGHI - \text{etc.}}$$

Now, since at the beginning, we put $\frac{1}{S} = 1 + \frac{A}{P}$, that makes the sum being sought $S = \frac{1}{1+\frac{A}{P}}$; and the following posits have been made:

$$P = 1 + \frac{b}{Q}, Q = 1 + \frac{B}{R}, R = 1 + \frac{2b}{S}, S = 1 + \frac{C}{T}, T = 1 + \frac{3b}{U}$$
$$U = 1 + \frac{D}{V}, V = 1 + \frac{4b}{X}, X = 1 + \frac{E}{Y}, Y = 1 + \frac{5B}{Z}, \text{etc.}$$

[40] after which values have been substituted in order, this continued fraction arises:

$$S=\cfrac{1}{1+\cfrac{A}{1+\cfrac{b}{1+\cfrac{B}{1+\cfrac{2b}{1+\cfrac{C}{1+\cfrac{3b}{1+\cfrac{D}{1+\cfrac{4b}{1+\text{etc.}}}}}}}}}}$$

If, in place of the letters A, B, C, D, etc. we restore their assumed, so that for us this divergent series becomes:

$$1-a+a(a+b)-a(a+b)(a+2b)+a(a+b)(a+2b)(a+3b)-\text{etc.}$$

its sum is expressed by the following continued fraction

$$S=\cfrac{1}{1+\cfrac{a}{1+\cfrac{b}{1+a+\cfrac{b}{1+\cfrac{2b}{1+a+\cfrac{2b}{1+\cfrac{3b}{1+a+\cfrac{3b}{1+\cfrac{4b}{1+\text{etc.}}}}}}}}}}$$

which is in the form I have given already. [41]

8. This transformation is even more worthy of note, because it opens for us the most prudent and perhaps unique way that the value of the divergent series can approximately be determined. For if the continued fraction is resolved into simple fractions in the usual custom, 1, $\frac{1}{1+a}$, $\frac{1+b}{1+a+b}$, etc., then these are alternately larger and smaller than the value of the divergent series, and they approach ever nearer to this value. Besides, I have already published these remarkable methods[3], which much more readily lead toward the true value.

9. Indeed, it will help to have noted that such a continued fraction:

$$1+\cfrac{\alpha}{1+\cfrac{\beta}{1+\cfrac{\gamma}{1+\cfrac{\delta}{1+\text{etc.}}}}}$$

in general can be reduced conveniently enough to some number of parts. For having put this value = S, one can represent it thus:

[3] Euler is likely referring to "De seriebus divergentibus" (E247), *Novi Commentarii academiae scientiarum Petropolitanae* **5** (1754-55) 1760, pp. 205-237, which appears in *Opera Omnia* Series I, Volume XIV, pp. 585-617.

$$S = 1 + \frac{\alpha}{1+\frac{\beta}{P}},\ P = 1 + \frac{\lambda}{1+\frac{\delta}{Q}},\ Q = 1 + \frac{\varepsilon}{1+\frac{\zeta}{R}}, \text{etc.}$$

Now the first of these formulas will be

$$S = 1 + \frac{\alpha P}{P+\beta} = 1 + \alpha - \frac{\alpha\beta}{\beta + P},$$

then the second of these formulas gives

$$P = 1 + \frac{\gamma Q}{Q+\delta} = 1 + \gamma - \frac{\gamma\delta}{\delta + Q}$$

[42] and the third in the same way will produce

$$Q = 1 + \frac{\varepsilon R}{R+\zeta} = 1 + \varepsilon - \frac{\varepsilon R}{\zeta + R}, \text{etc.}$$

These values substituted successively will produce this new continued fraction:

$$S = 1 + \alpha - \cfrac{\alpha\beta}{1 + \beta + \cfrac{\gamma - \lambda\delta}{1 + \delta + \varepsilon - \cfrac{\varepsilon\zeta}{1 + \zeta + \eta - \cfrac{\eta\theta}{1 + \theta + \iota + \text{etc.}}}}}$$

10. Since in our case, the divergent series

$$S = 1 - a(a+b) - a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) - \text{etc.}$$

is led to this continued fraction:

$$S = \cfrac{1}{1 + \cfrac{a}{1 + \cfrac{b}{1 + \cfrac{a+b}{1 + \cfrac{2b}{1 + \cfrac{a+2b}{1 + \cfrac{3b}{1 + \cfrac{a+3b}{1 + \text{etc.}}}}}}}}}$$

let us take

$$\alpha = a, \beta = b, \gamma = a + b, \delta = 2b, \varepsilon = a + 2b, \text{etc.},$$

it becomes [43]

$$S = 1 + a - \cfrac{ab}{1 + a + 2b - \cfrac{2b(a+b)}{1 + a + 4b - \cfrac{3b(a+2b)}{1 + a + 6b - \cfrac{4b(a+3b)}{1 + a + \text{etc.}}}}}$$

APPENDIX

On the Brounckerian Continued Fraction

11. At the time when I had been much occupied in investigating the Analysis which had led Brouncker to that singular fraction, since it seemed to me hardly likely that he, through so many details, of a type which are recalled by Wallis, was led to it;[4] finally I seemed to have shown clearly enough indeed to me that Brouncker deduced this form from the Leibnizian series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ etc., which the great Gregory had already found earlier, rather than from the interpolation of the series $1, \frac{1}{2}, \frac{1\cdot 3}{2\cdot 4}, \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}, \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}$, etc. just as Wallis suspected, if indeed consideration of this series through plain enough reckoning leads by hand to the Brounckerian form.

12. This observation indeed seems worthy of greater attention, after the famous Daniel Bernoulli in no way disdained renewing the memory of the Brounckerian form. Since therefore not so long ago I showed an easy method of deriving this form from the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ etc., I will be judged to be not ungrateful to the Geometer, if I will have brought forward an inverse method into the midst, by the power of which one may reduce the Brounckerian formula to the Leibnizian series. [44]

13. Thus I will consider now this continued fraction as if its value were not yet known, by setting:

$$S = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \text{etc.}}}}}}}$$

which I represent through parts in the following way:

$$S = \frac{1}{1 + \frac{1}{-1+P}},\ P = 3 + \frac{9}{-3+Q},\ Q = 5 + \frac{25}{-5+R},\ R = 7 + \frac{49}{-7+S},\ \text{etc.}$$

Indeed from these parts duly joined the proposed form springs plainly forth.

14. Therefore let us roll out these single parts separately, and the first having been reduced to a simple fraction, offers $S = \frac{P-1}{P}$ so that $S = 1 - \frac{1}{P}$, the second indeed will be $\frac{3Q}{Q-3}$, whence it becomes $\frac{1}{P} = \frac{1}{3} - \frac{1}{Q}$. In the same way, the third part gives $Q = \frac{5R}{R-5}$ and so $\frac{1}{Q} = \frac{1}{5} - \frac{1}{R}$, in the same way from the following parts we will find $\frac{1}{R} = \frac{1}{7} - \frac{1}{S}, \frac{1}{S} = \frac{1}{9} - \frac{1}{T}$, etc. So if these values are successively substituted, we will obtain this expression:

[4] John Wallis's *Arithmetica Infinitorum* includes a continued fraction method that he attributes to Brouckner (see especially pp. 181-82).

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

and that is certainly known to be $S = \frac{\pi}{4}$.

15. In a similar way it will be possible to investigate the value of any other continued fractions of this kind. For example, if this form were [45] proposed:

$$S = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \text{etc.}}}}}}$$

In the following way it is distributed into these members:

$$S = \frac{1}{1 + \frac{1}{-1+P}},\ P = 2 + \frac{4}{-2+Q},\ Q = 3 + \frac{9}{-3+R},\ R = 4 + \frac{16}{-4+S},\ \text{etc.}$$

For after these single parts have been rolled out, it will be discovered:

$$S = 1 - \frac{1}{P},\ \frac{1}{P} = \frac{1}{2} - \frac{1}{Q},\ \frac{1}{Q} = \frac{1}{3} - \frac{1}{R},\ \frac{1}{R} = \frac{1}{4} - \frac{1}{S},\ \text{etc.}$$

whence it will follow that

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.} = l\,2.$$ [5]

Thus this method seems in hindsight[6] to have no equal.

APPENDIX

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[5] Remember that Euler uses *l* instead of ln.

[6] *in recessu*

PHYSICO-MATHEMATICA:

PHYSICA:

ASTRONOMICA: