Coroll. 5.

§. 33. Cum sit $\cos.\frac{(2k+1)\pi}{2} = 0$ casibus, quibus est $a = 2k+1$ et $c = 2$ summa seriei est $= \frac{-1}{2aa} = \frac{1}{2(2k+1)^2}$ existente k numero quocunque integro: Quare erit:

$$\frac{1}{(2k+1)^2-4} + \frac{1}{(2k+1)^2-16} + \frac{1}{(2k+1)^2-36} + \frac{1}{(2k+1)^2-64} + \text{etc.} = \frac{-1}{2(2k+1)^2}$$

quam summationem iam alibi demonstraui. Si enim singulae fractiones in partes resoluantur, oritur

$$\frac{-1}{2k+1} = \frac{1}{2k-1} + \frac{1}{2k-3} + \frac{1}{2k-5} + \frac{1}{2k-7} + \frac{1}{2k-9} + \text{etc.}$$
$$+ \frac{1}{2k+3} + \frac{1}{2k+5} + \frac{1}{2k+7} + \frac{1}{2k+9} - \text{etc.}$$

Coroll. 6.

§. 34. Transposito termino $\frac{-1}{2k+1}$ ad alteram partem binisque terminis collectis orietur noua series, cuius summa $= 0$. Erit scilicet singulis terminis per $4k$ diuisis

$$0 = \frac{1}{4kk-1} + \frac{1}{4kk-9} + \frac{1}{4kk-25} + \frac{1}{4kk-49} + \frac{1}{4kk-81} + \text{etc.}$$

cuius veritas in singulis casibus facile elucet.

Problema. IV.

§. 35. Inuenire terminum generalem seriei, cuius quilibet terminus oritur, si antecedens per datum numerum m multiplicetur, atque ad productum datus numerus c addatur, cuiusque seriei terminus primus sit pariter datus $= a$,

Solutio.

Termini ergo, qui indicibus integris respondent, ita se habebunt:

1

a;

$\overset{1}{a}$; $\overset{2}{ma+c}$; $\overset{3}{m^2a+mc+c}$; $\overset{4}{m^3a+m^2c+mc+c}$; etc. vnde si index quicunque x sit numerus integrus, erit terminus conueniens $= m^{x-1}a + \frac{m^{x-1}-1}{m-1}c$: Sin autem x sit numerus non integer, infinitae aliae formulae praeter hanc perinde satisfacient; ad quas inueniendas sit y terminus indici x respondens, et y' sequens seu indici $x+1$ competens: erit $y' = my + c$: vnde fiet:

$$my + c = y + \frac{dy}{1dx} + \frac{ddy}{1.2dx^2} + \frac{d^3y}{1.2.3dx^3} + \frac{d^4y}{1.2.3.4dx^4} + \text{etc.}$$

Ponatur $y = v - \frac{c}{m-1}$; eritque:

$$mv = v + \frac{dv}{1dx} + \frac{ddv}{1.2dx^2} + \frac{d^3v}{1.2.3dx^3} + \text{etc.}$$

quae aequatio cum congruat cum ea, quam in problemate praecedente inuenimus; si ponatur sin. $\pi x = r$ et cos. $\pi x = s$ atque Q sumatur pro functione quacunque parium dimensionum ipsarum r et s, erit $v = m^x Q$; ideoque $y = m^x Q - \frac{c}{m-1}$. Ponatur $x = 1$, quo casu fit $r = 0$ et $s = -1$, abeatque Q in C, oportet esse $a = mC - \frac{c}{m-1}$; ideoque erit $C = \frac{a}{m} + \frac{c}{m(m-1)}$. Quare si pro Q sumatur quantitas constans C ipsa, erit $y = m^{x-1}a + \frac{(m^{x-1}-1)c}{m-1}$ pro casu simplicissimo. Atque si P sit functio parium dimensionum ipsarum r et s talis quae euanescat posito $x = 1$, poni poterit $Q = C + P$ eritque termini generalis quaesiti forma latissime patens $y = m^{x-1}a + \frac{(m^{x-1}-1)c}{m-1} + m^x P$. Q. E. I.

Proble-

Problema V.

§. 36. Inuenire terminum generalem ſerierum recurrentium ſecundi ordinis, in quibus quilibet terminus aequatur aggregato binorum terminorum antecedentium per quoscunque numeros multiplicatorum:

Solutio.

Sit terminus indici x reſpondens $= y$;
terminus indici $x-1$ reſpondens $= {'y}$
terminus indici $x-2$ reſpondens $= {''y}$
haecque propoſita ſit lex ſeriei recurrentis, vt ſit:

$$y = \alpha\, {'y} + \beta\, {''y}$$

Quia igitur eſt

$$'y = y - \frac{dy}{1dx} + \frac{ddy}{1.2dx^2} - \frac{d^3y}{1.2.3dx^3} + \frac{d^4y}{1.2.3.4dx^4} - \text{etc.}$$

$$''y = y - \frac{2dy}{1dx} + \frac{4ddy}{1.2dx^2} - \frac{8d^3y}{1.2.3dx^3} + \frac{16d^4y}{1.2.3.4dx^4} - \text{etc.}$$

erit his formulis ſubſtitutis:

$$y = \begin{array}{l} +\alpha(y - \frac{dy}{1dx} + \frac{ddy}{1.2dx^2} - \frac{d^3y}{1.2.3dx^3} + \frac{d^4y}{1.2.3.4dx^4} - \text{etc.}) \\ +\beta(y - \frac{2dy}{1dx} + \frac{4ddy}{1.2dx^2} - \frac{8d^3y}{1.2.3dx^3} + \frac{16d^4y}{1.2.3.4dx^4} - \text{etc.}) \end{array}$$

Ad quam aequationem reſoluendam ponatur ſecundum praeceptum generale 1 pro y; z pro $\frac{dy}{dx}$; z^2 pro $\frac{ddy}{dx^2}$ etc. fietque

$$1 = \begin{array}{l} +\alpha(1 - \frac{z}{1} + \frac{zz}{1.2} - \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} - \text{etc.}) \\ +\beta(1 - \frac{2z}{1} + \frac{4z^2}{1.2} - \frac{8z^3}{1.2.3} + \frac{16z^4}{1.2.3.4} - \text{etc.}) \end{array}$$

quae aequatio ad hanc formam finitam reducitur, $1 = \alpha e^{-z} + \beta e^{-2z}$: cuius factores inueſtigare oportet. Poſito ergo $e^{+z} = u$ ac reſoluatur haec aequatio $uu = \alpha u + \beta$: cuius vel ambae radices ſunt reales, vel ambae imaginariae, vel denique ambae inter ſe aequales. Hosque tres caſus ſeorſim euolue oportet.

I. Sint

I. Sint igitur primo ambae radices reales et inaequales inter se, seu sit $uu - \alpha u - \mathfrak{C} = (u - A)(u - B)$: hincque pro u ponendo e^z habebimus binos factores generales $(e^z - A)$ et $(e^z - B)$. Vidimus autem supra formulam $e^z - m$ dedisse integrale hoc:

$$y = m^x \left(\begin{matrix} C + \alpha \sin. 2\pi x + \mathfrak{C} \sin. 4\pi x + \gamma \sin. 6\pi x \\ + \mathfrak{A} \cos. 2\pi x + \mathfrak{B} \cos. 4\pi x + \mathfrak{C} \cos. 6\pi x \end{matrix} + \text{etc.} \right)$$

Ergo ambo factores $e^z - A$ et $e^z - B$ dabunt coniunctim hunc valorem pro termino generali y:

$$y = \begin{matrix} + A^x \left(\begin{matrix} C + \alpha \sin. 2\pi x + \mathfrak{C} \sin. 4\pi x + \gamma \sin. 6\pi x + \text{etc.} \\ + \mathfrak{A} \cos. 2\pi x + \mathfrak{B} \cos. 4\pi x + \mathfrak{C} \cos. 6\pi x + \text{etc.} \end{matrix} \right) \\ + B^x \left(\begin{matrix} C' + \alpha' \sin. 2\pi x + \mathfrak{C}' \sin. 4\pi x + \gamma' \sin. 6\pi x + \text{etc.} \\ + \mathfrak{A}' \cos. 2\pi x + \mathfrak{B}' \cos. 4\pi x + \mathfrak{C}' \cos. 6\pi x + \text{etc.} \end{matrix} \right) \end{matrix}$$

Vel ponatur $\sin. \pi x = r$ et $\cos. \pi x = s$, sintque P et Q functiones quaecunque parium dimensionum ipsarum r et s, eritque si fuerit $uu - \alpha u - \mathfrak{C} = (u - A)(u - B)$, seu si sint A et B radices aequationis $uu - \alpha u - \mathfrak{C} = 0$; hoc inquam casu erit:

$$y = A^x P + B^x Q.$$

II. Si ambae radices fuerint imaginariae, tum quidem eadem formula iam inuenta vsum habere poterit, quoniam quouis casu imaginaria se mutuo destruent: interim tamen formula pro y exhiberi potest ab imaginariis libera. Hoc enim casu aequatio $uu - \alpha u - \mathfrak{C} = 0$ eiusmodi induet formam $uu - 2fu \cos. \omega + ff = 0$, cuius radices sunt $u = f \cos. \omega \pm f\sqrt{-1}. \sin. \omega$, ita vt sit $A = f \cos. \omega + f\sqrt{-1}. \sin. \omega$ et $B = f \cos. \omega - f\sqrt{-1}. \sin. \omega$. Hinc autem erit $A^x = f^x \cos. \omega x + f^x \sqrt{-1}. \sin. \omega x$, atque

 $B^x =$

$B^x = f^x \cos.\ \omega x - f^x \sqrt{-1}.\sin.\ \omega x$. Hi igitur valores, si loco A^x et B^x substituantur, fiet $y = (P + Q) f^x \cos.\ \omega x + (P - Q) \sqrt{-1}.f^x \sin.\ \omega x$. Quia iam P et Q sunt functiones arbitrariae ipsarum r et s, dummodo dimensiones habeant pares: loco $P + Q$, scribatur P, et loco $(P - Q)\sqrt{-1}$, ponatur Q eritque ex aequatione $uu - \alpha u - \beta = uu - 2fu \cos.\ \omega + ff = 0$, terminus generalis quaesitus:

$$y = f^x P \cos.\ \omega x + f^x Q \sin.\ \omega x.$$

III. Si ambae aequationis $uu - \alpha u - \beta = 0$ radices A et B fuerint aequales, puta $A = B = m$, aequatio habebitur $(e^z - m)^2 = 0$. Ponatur vt in §. XXIII; $m = e^\lambda$, erit formulae $(e^z - e^\lambda)^2$ primus factor quadratus $= (z - \lambda)^2$: ex quo oritur pars integralis $(\mathfrak{A} + \mathfrak{B}x) e^{\lambda x} = (\mathfrak{A} + \mathfrak{B}x) m^x = (\mathfrak{A} + \mathfrak{B}x) A^x$. Reliqui factores omnes erunt pariter quadrati, et continebuntur in hac forma generali:

$$(\lambda\lambda + 4kk\pi\pi - 2\lambda z + zz)^2$$

ex quo oritur secundum praecepta a me tradita pars integralis:

$$A^x (\mathfrak{A} + \mathfrak{B}x) \sin.\ 2k\pi x + A^x (\mathfrak{C} + \mathfrak{D}x) \cos.\ 2k\pi x.$$

Quibus colligendis sequitur, si fuerit $uu - \alpha u - \beta = (u - A)^2 = uu - 2Au + AA$, fore terminum generalem quaesitum: $y =$

$$A^x \left\{ \begin{array}{l} \mathfrak{A} + \mathfrak{B}x + (\mathfrak{C} + \mathfrak{D}x)\sin.\ 2\pi x + (\mathfrak{G} + \mathfrak{H}x)\sin\ 4\pi x + \text{etc.} \\ \qquad + (\mathfrak{E} + \mathfrak{F}x)\cos.\ 2\pi x + (\mathfrak{J} + \mathfrak{K}x)\cos.\ 4\pi x + \text{etc.} \end{array} \right)$$

Ponatur iterum $\sin.\ \pi x = r$ et $\cos.\ \pi x = s$, sintque P et Q functiones quaecunque pares ipsarum r et s, atque terminus generalis ita exprimi poterit, vt sit $y = A^x (P + Qx)$. Q. E. I.

Coroll.

Coroll. 1.

§. 37. Si ergo in ſerie recurrente quilibet terminus y ita per binos praecedentes $'y$ et $''y$ determinetur, vt ſit $y = \alpha\, 'y + \beta\, ''y$, ſeu ſi ſecundum Moioraeum fuerit $+\alpha$, $+\beta$ ſcala relationis; acſi x fuerit index termini y: erit y functio maxime indeterminata ipſius x: cum innumerabiles formulae exhiberi queant, quae valores ſatisfacientes pro y praebeant.

Coroll. 2.

§. 38. Ad has autem omnes expreſſiones pro y inueniendas, formetur ex ſcala relationis $+\alpha$, $+\beta$ haec aequatio $uu - \alpha u - \beta = 0$: ex cuius reſolutione forma termini generalis y ſequenti modo reperietur.

Coroll. 3.

§. 39. Sint aequationis $uu - \alpha u - \beta = 0$ radices A et B, ita vt ſit $A = \frac{1}{2}\alpha + \sqrt{(\frac{1}{4}\alpha\alpha + \beta)}$ et $B = \frac{1}{2}\alpha - \sqrt{(\frac{1}{4}\alpha\alpha + \beta)}$, ſumanturque, poſito ſin. $\pi x = r$ et coſ. $\pi x = s$, functiones quaecunque pares ipſarum r et s, quae ſint P et Q, erit $y = A^x P + B^x Q = (\frac{1}{2}\alpha + \sqrt{(\frac{1}{4}\alpha\alpha + \beta)})^x P + (\frac{1}{2}\alpha - \sqrt{(\frac{1}{4}\alpha\alpha + \beta)})^x Q$.

Coroll. 4.

§. 40. Sin autem aequationis $uu = \alpha u + \beta$ ambae radices fuerint aequales, illa formula vſu caret, ob $\beta + \frac{1}{4}\alpha\alpha = 0$. Hoc autem caſu, cum vtraque radix futura ſit $\frac{1}{2}\alpha$, ſi ponatur $\frac{1}{2}\alpha = A$, erit terminus generalis $y = A^x (P + Qx)$.

Coroll. 5.

§. 41. Sin autem fit $\frac{1}{4}\alpha\alpha + \beta$ quantitas negatiua, partes ante inuentae erunt imaginariae. Ad formam ergo realem inueniendam comparetur aequatio $uu - \alpha u - \beta = 0$ cum hac $uu - 2fu \operatorname{cof.} \omega, + ff = 0$, erit $f = \sqrt{-\beta}$, et $\alpha = 2\sqrt{-\beta}$, $\operatorname{cof.} \omega$, feu $\operatorname{cof.} \omega = \frac{\alpha}{2\sqrt{-\beta}}$ et $\operatorname{fin.} \omega = \frac{\sqrt{(-4\beta - \alpha\alpha)}}{2\sqrt{-\beta}} = \sqrt{(1 + \frac{\alpha\alpha}{4\beta})}$, vnde angulus ω inuenietur, ex quo erit $y = f^x (P \operatorname{cof.} \omega x + Q \operatorname{fin.} \omega x)$

Coroll. 6.

§. 42. Si pro P et Q quantitates conftantes affumantur, prodit eadem termini generalis forma, quae vulgo exhiberi ac pro fola, quae fatisfaciat, haberi fole. Qualibet autem ferie determinata propofita, iftas binas quantitates conftantes ex duobus terminis primis, qui dati fumuntur, definiri oportet. In genere autem, cum duae ingrediantur functiones arbitrariae P et Q, quae quoties x eft numerus integer, eosdem valores conftantes inducent, patet duos feriei terminos indicibus integris refpondentes, pro lubitu affumi poffe.

Scholion.

§. 43. Methodus haec inueniendi term nos generales ferierum recurrentium ideo potiffimum eft notatu digna, quod non folum omnes poffibiles formas exhibeat, fed etiam quod a priori procedat, atque ex folis principiis analyticis negotium conficiat, cum alii, qui has feries tractauerunt, omnes per viam indirectam ad formam illam terminorum generalium fpecialem peruenerint. Haec

enim

enim eſt praecipua proprietas, et quaſi criterium methodi directae, vt non ſolum ex ipſis cuiusque rei principiis eius affectiones eruat, ſed etiam omnes determinationis modos ſimul complectatur. Methodi autem indirectae, etſi ſaepe concinnas et elegantes problematum ſolutiones ſuppeditant, tamen rariſſime naturam quaeſtionis, quae tractatur, exhauriunt. Cuius diſcriminis eximium exemplum in problemate antecedente ſpectatur; luculentius autem in problemate ſequente occurret, vbi in genere omnium ſerierum recurrentium termini generales inueſtigabuntur.

Problema VI.

§. 44. Inuenire terminum generalem ſerierum recurrentium cuiuscunque ordinis, quarum quilibet terminus aequatur aggregato aliquot terminorum antecedentium per numeros quoscunque multiplicatorum.

Solutio.

Sit terminus indici x reſpondens $= y$, termini autem antecedentes, qui indicibus $x - 1$, $x - 2$, $x - 3$, $x - 4$, etc. conueniunt, deſignentur per $'y$, $''y$, $'''y$, ${}^{\mathrm{IV}}y$, etc: propoſitaque ſit haec ſeriei lex, vt habeatur vbique:

$$y = \alpha\, 'y + \beta\, ''y + \gamma\, '''y + \delta\, {}^{\mathrm{IV}}y + \text{etc.}$$

Cum iam ſit ex natura differentialium:

$$'y = y - \frac{dy}{dx} + \frac{ddy}{1.2dx^2} - \frac{d^3y}{1.2.3dx^3} + \text{etc.}$$

$$''y = y - \frac{2dy}{1dx} + \frac{2^2ddy}{1.2dx^2} - \frac{2^3d^3y}{1.2.3dx^3} + \text{etc.}$$

$$'''y = y - \frac{3dy}{1dx} + \frac{3^2ddy}{1.2dx^2} - \frac{3^3d^3y}{1.2.3dx^3} + \text{etc.}$$

etc.

Si

Si hi valores ibi ſubſtituantur, prodibit aequatio, in cuius ſingulis terminis vnica variabilis y dimenſio occurrit, alterius vero variabilis x, nonniſi differentiale dx, quod conſtans ponitur, ingreditur. Quare ſi vbique 1 loco y, z loco $\frac{dy}{dx}$, et generatim z^m loco $\frac{d^m y}{dx^m}$ ſubſtituatur, emerget reductione facta haec aequatio:

$$1 = \alpha e^{-z} + \beta e^{-2z} + \gamma e^{-3z} + \delta e^{-4z} + \text{etc.}$$

Fiat nunc $e^z = u$, atque ſublatis fractionibus prodibit huius modi aequatio Algebraica:

$$u^n = \alpha u^{n-1} + \beta u^{n-2} + \gamma u^{n-3} \delta u^{n-4} + \text{etc.}$$

quae tot erit dimenſionum, quot termini antecedentes ad termini y determinationem requiruntur; ſeu quoti ordinis fuerit ipſa ſeries recurrens. Iam forma termini generalis y ex radicibus huius aequationis, ſeu ex factoribus huius formulae:

$$u^n - \alpha u^{n-1} - \beta u^{n-2} - \gamma u^{n-3} - \delta u^{n-4} - \text{etc.} = U$$

ſimili modo colligetur, quo in ſolutionibus problematum hactenus propoſitorum ſumus vſi: ſcilicet ſi ſit ſin. $\pi x = r$ et coſ. $\pi x = s$, ac P, Q, R, S, T, etc. denotent functiones quaſcunque parium dimenſionum ipſarum r et s, Deinde formulae U inueſtigentur omnes factores reales tam ſimplices quam trinomiales, et, ſi qui eorum fuerint aequales, ii coniunctim per poteſtates exprimantur. Singuli autem factores iſti totidem partes termini generalis y praebebunt, quae partes ope ſequentium regularum formabuntur:

I. Si

I. Si factor sit $u - A$ erit pars integralis
$$y = A^x P$$
II. Si factor sit $(u-A)^2$ erit pars integralis
$$y = A^x(P + Qx)$$
III. Si factor sit $(u-A)^3$ erit pars integralis
$$y = A^x(P + Qx + Rx^2)$$
IV. Si factor sit $(u-A)^4$ erit pars integralis
$$y = A^x(P + Qx + Rxx + Sx^3)$$
etc.

1. Si factor sit $uu - 2Au \cos. \omega + AA$ erit
$$y = A^x(P \cos. \omega x + Q \sin. \omega x)$$
2. Si factor sit $(uu - 2Au \cos. \omega + AA)^2$ erit
$$y = A^x(P + Qx) \cos. \omega x + A^x(R + Sx) \sin. \omega x$$
3. Si factor sit $(uu - 2Au \cos. \omega + AA)^3$ erit
$$y = \begin{array}{l} A^x(P + Qx + Rxx) \cos. \omega x \\ + A^x(S + Tx + Vxx) \sin. \omega x \end{array}$$
etc.

Quodsi ergo pro singulis formulae U factoribus hinc partes integralis quaerantur, eaeque in vnam summam coniiciantur, habebitur valor completus pro termino generali y quaesito. Q. E. I.

Coroll. 1.

§. 45. Hoc ergo modo obtinetur integrale completum sequentis aequationis differentialis infinitae:

$y =$

$$\begin{aligned} y = & \; y\,(\alpha + \beta + \gamma + \delta + \text{etc.}) \\ & - \frac{dy}{1dx}(\alpha + 2\beta + 3\gamma + 4\delta + \text{etc.}) \\ & + \frac{ddy}{1\cdot 2dx^2}(\alpha + 2^2\beta + 3^2\gamma + 4^2\delta + \text{etc.}) \\ & - \frac{d^3y}{1\cdot 2\cdot 3dx^3}(\alpha + 2^3\beta + 3^3\gamma + 4^3\delta + \text{etc.}) \\ & + \frac{d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}(\alpha + 2^4\beta + 3^4\gamma + 4^4\delta + \text{etc.}) \\ & \qquad \text{etc.} \end{aligned}$$

ſeu valor ipſius y exprimetur per functionem ipſius x.

Coroll. 2.

§. 46. Omnis ergo difficultas reducitur ad reſolutionem aequationis Algebraicae:

$$u^n = \alpha u^{n-1} + \beta u^{n-2} + \gamma u^{n-3} + \delta u^{n-4} + \text{etc.}$$

Eius enim radicibus ſeu factoribus inuentis facile eſt ope regularum ante traditarum valorem ipſius y determinare.

Coroll. 3.

§. 47. Quoniam per integrationem tot quantitates arbitrariae P, Q, R, S, etc. inuehuntur, quot vnitates continet exponens n, ſeu quot termini antecedentium in determinationem ſequentis ingrediuntur: manifeſtum eſt totidem terminos pro lubitu aſſumi poſſe, ex quibus reliqui omnes, quorum indices ſunt integri determinentur. Hoc tamen non obſtat, quo minus termini indicum non integrorum maneant maxime indeterminati, vti in praecedentibus problematis iam eſt notatum.

Problema VII.

§. 48. Si ſeriei quilibet terminus aequetur quantitati cuipiam conſtanti c, vna cum aggregato aliquot terminorum

rum antecedentium per datos numeros multiplicatorum, (vt in problemate praecedente): inuenire huius feriei terminum generalem.

Solutio.

Pofito vt ante termino indici indeterminato x refpondente $= y$, fint antecedentes indicibus $x-1$, $x-2$, $x-3$, etc. refpondentes $'y$, $''y$, $'''y$ etc. atque propofita fit haec lex progreffionis

$$y = c + \alpha\, 'y + \beta\, ''y + \gamma\, '''y + \delta\, {}^{\mathrm{IV}}y + \text{etc.}$$

fubftitutis ergo pro $'y$, $''y$, $'''y$, ${}^{\mathrm{IV}}y$, etc. valoribus fupra exhibitis erit:

$$\begin{aligned} y = c &+ y(\alpha + \beta + \gamma + \delta + \text{etc.}) \\ &- \tfrac{dy}{1 \cdot dx}(\alpha + 2\beta + 3\gamma + 4\delta + \text{etc.}) \\ &+ \tfrac{ddy}{1 \cdot 2 dx^2}(\alpha + 2^2\beta + 3^2\gamma + 4^2\delta + \text{etc.}) \\ &- \tfrac{d^3 y}{1 \cdot 2 \cdot 3 dx^3}(\alpha + 2^3\beta + 3^3\gamma + 4^3\delta + \text{etc.}) \\ &\text{etc.} \end{aligned}$$

Ponatur iam ad terminum conftantem c ex aequatione tollendum $y = v + g$, fiatque: $g = c + g(\alpha + \beta + \gamma + \delta + \text{etc.}$ ideoque $g = \frac{c}{1-\alpha-\beta-\gamma-\delta-\text{etc.}}$ Quo facto ob $dy = dv$, $ddy = ddv$ etc. habebitur aequatio haec:

$$\begin{aligned} v = \; & v(\alpha + \beta + \gamma + \delta + \text{etc.}) \\ &- \tfrac{dv}{1 dx}(\alpha + 2\beta + 3\gamma + 4\delta + \text{etc}) \\ &+ \tfrac{ddv}{1 \cdot 2 dx^2}(\alpha + 2^2\beta + 3^2\gamma + 4^2\delta + \text{etc.}) \\ &\text{etc.} \end{aligned}$$

Quae aequatio cum fimilis fit ei, quam in problemate praecedente

cedente resoluimus, valor ipsius v per regulas ibi datas inuenietur. Quo inuento habebitur terminus generalis quaesitus

$$y = v + \frac{c}{1-\alpha-\beta-\gamma-\delta-\text{etc.}}$$

ex quo natura seriei propositae innotescet. Q E I.

Coroll. 1.

§. 49. Quantitas igitur constans c, quae ad formulam $\alpha' y + \beta'' y + \gamma''' y +$ etc. accedit, terminum generalem y aliter non afficit, nisi quod ipsi numerum constantem adiiciat. Quaeratur ergo terminus generalis pro serie recurrente pura cuius scala relationis sit: α, $+\beta$, $+\gamma$, $+\delta$, etc. ad eumque addatur numerus $\frac{c}{1-\alpha-\beta-\gamma-\text{etc.}}$

Coroll. 2.

§. 50. Haec autem quantitas constans adiicienda $\frac{c}{1-\alpha-\beta-\gamma-\text{etc.}}$ euadit infinita ideoque incerta, si denominator euanescit, seu si $1-\alpha-\beta-\gamma-\delta-$ etc. $=0$. Hoc autem casu aequatio $u^n-\alpha u^{n-1}-\beta u^{n-2}-\gamma u^{n-3}-$etc. $=0$ radicem habebit $u-1=0$, vnde nascitur integralis pars $y=P$; quae ne omnes termini fiant infiniti, quantitas P ita debet esse infinita, vt ea cum illa constante infinita praebeat valorem finitum, qui erit $=P+Qx$.

Scholion 1.

§. 51. Quod quo clarius appareat, obseruandum est huiusmodi series, quales hic sumus contemplati, semper reduci posse ad series recurrentes puras vno gradu altiores. Si enim sit

$y =$

$$y = c + \alpha\,'y + \beta\,''y + \gamma\,'''y + {}^{IV}y,$$

erit $'y = c + \alpha\,''y + \beta\,'''y + \gamma\,{}^{IV}y + {}^{V}y,$

quarum differentia dat:

$$y = (\alpha + 1)'y + (\beta - \alpha)''y + (\gamma - \beta)'''y + (\delta - \gamma)^{IV}y - \delta\,{}^{V}y,$$

quae eſt lex pro ſerie recurrente pura: cuius terminus generalis formabitur ex reſolutione huius aequationis:

$$u^{n+1} - (\alpha + 1)u^{n} - (\beta - \alpha)u^{n-1} - (\gamma - \beta)u^{n-2} - \text{etc.} = 0.$$

Huius autem factor vnus iam conſtat, ſcilicet $u - 1$ cum ſit

$$(u - 1)(u^{n} - \alpha u^{n-1} - \beta u^{n-2} - \gamma u^{n-3} - \text{etc.}) = 0$$

Factor autem $u - 1$ dat partem integralis 1^{x} P tum ſolum, quando non ſimul factor eſt alterius formae $u^{n} - \alpha u^{n-1}$ - etc. ſin autem haec quoque factorem habeat $u - 1$ eiusue poteſtatem, iſtius exponens vnitate augeri, indeque debita integralis pars inueſtigari debet. Inuento autem hac ratione termino generali y, is, cum in eo quantitas c non inſit, nimis erit generalis; ad caſum ergo propoſitum reſtringi debebit. Ex valore ſcilicet ipſius y eruantur valores terminorum praecedentium $'y$, $''y$, $'''y$, etc. loco x ponendo $x - 1$, $x - 2$, $x - 3$, etc. vbi notandum eſt, functiones P, Q, R, etc. eosdem valores retinere, nullamque inde mutationem pati. Deinde hi valores ſubſtituantur in aequatione:

$$y = c + \alpha\,'y + \beta\,''y + \gamma\,'''y + \delta\,{}^{IV}y + \text{etc.}$$

atque hoc pacto vna functionum illarum P, Q, R, etc. determinabitur. Sic ſi propoſita ſit haec ſeriei lex:

$$y = c + 3\,'y - 2\,''y$$

hinc naſcetur aequatio $(u - 1)(u^{2} - 3u + 2) = 0$, cuius facto-

factores sunt $(u-1)^2(u-2)=0$; ex quibus colligitur terminus generalis:

$$y = P + Qx + 2^x R \text{, erit ergo}$$
$$'y = P + Qx - Q + 2^{x-1} R \text{, et}$$
$$''y = P + Qx - 2Q + 2^{x-2} R,$$

qui substituti hanc dabunt aequalitatem:

$$P + Qx + 4 \cdot 2^{x-2} R = c + P + Qx + Q + 4 \cdot 2^{x-2} R,$$

vnde reperitur $Q = -c$: sicque terminus generalis propositae legi conueniens erit $y = P - cx + 2^x R$, vbi pro P et R functiones quascunque parium dimensionum ipsarum r et s assumi possunt.

Scholion 2.

§. 52. Quoniam igitur methodum vniuersalem tradidimus inueniendi terminos generales serierum, quarum terminus quisque per praecedentes determinatur, siquidem terminorum antecedentium nullae potestates occurrant; eandem hanc methodum accommodemus ad series, quarum quisque terminus, non solum ex praecedentibus, sed etiam ex ipso indice, determinatur; in quo tertium formationis serierum genus constituimus. Quodsi vero terminorum praecedentium quadrata, altioresue potestates in determinationem sequentis ingrediantur, vt si fuerit $y' = yy + ay$, tum quidem aequatio differentialis infinita, qua terminus generalis inuenitur, facile exhibetur, quae hoc casu erit:

$$yy + ay = y + \frac{dy}{1dx} + \frac{ddy}{1 \cdot 2\, dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3\, dx^3} + \text{etc.}$$

sed quia artificium nondum constat huiusmodi aequationes resoluendi, tractationem huius generis serierum hic praetermittere cogimur.

Proble-

Problema. VIII.

§. 53. Inuenire terminum generalem seriei, cuius quilibet terminus indici x respondens, aequatur praecedentis multiplo cuicunque vna cum multiplo ipsius indicis, et quantitate quapiam constante.

Solutio.

Sit y terminus indici x respondens, et y' denotet terminum sequentem, sitque haec seriei lex proposita.

$$y' = my + a + bx,$$

ex qua valorem ipsius y definiri oporteat. Si ergo pro y' valorem suum substituamus, habebimus hanc aequationem:

$$a + bx + my = y + \frac{dy}{1dx} + \frac{ddy}{1\cdot 2dx^2} + \frac{d^3y}{1\cdot 2\cdot 3dx^3} + \text{etc.}$$

Huiusmodi autem aequationes etsi generaliter resoluere docui, tamen expediet hanc aequationem per substitutionem in aliam resoluere, in qua omnes termini vnam ipsius y complectantur dimensionem. Ponatur ergo

$$y = A + Bx + v, \text{ erit } dy = Bdx + dv; ddy = ddv \text{ etc.}$$

fietque:

$$\begin{aligned} a + bx + mv &= A + Bx + v + \frac{dv}{1dx} + \frac{ddv}{1\cdot 2dx^2} + \frac{d^3v}{1\cdot 2\cdot 3dx^3} + \text{etc.} \\ + mA + mBx &\quad + B \end{aligned}$$

Iam fiat $A + B = a + mA$ et $B = b + mB$; reperieturque $B = \frac{-b}{m-1}$; et $A = \frac{-m}{(m-1)^2} - \frac{a}{m-1}$. Restabit ergo haec aequatio:

$$mv = v + \frac{dv}{1dx} + \frac{ddv}{1\cdot 2dx^2} + \frac{d^3v}{1\cdot 2\cdot 3dx^3} + \text{etc.}$$

quae cum reducatur ad $e^z - m = u - m = 0$: erit:

$$v = m^x P, \text{ ideoque terminus generalis quaesitus,}$$

$$y = \frac{-b}{(m-1)^2} - \frac{a - bx}{m-1} + m^x P.$$

 Vnii.

Vnicus casus hic excipitur quo $m=1$, ob denominatorem $m-1$ euanescentem. Cum enim hoc casu habeatur:

$$a+bx=\frac{dy}{1\,dx}+\frac{ddy}{1\cdot 2\,dx^2}+\frac{d^3y}{1\cdot 2\cdot 3\,dx^3}+\text{etc.}$$

ad terminum bx tollendum huiusmodi valor pro y accipi debet $y=A+Bx+Cxx+v$: vnde fit $\frac{dy}{dx}=B+2Cx+\frac{dv}{dx}$ et $\frac{ddy}{dx^2}=2C+\frac{ddv}{dx^2}$: sicque habebitur:

$$a+bx=B+2Cx+\frac{dv}{1dx}+\frac{ddv}{1\cdot 2dx^2}+\frac{d^3v}{1\cdot 2\cdot 3dx^3}+\text{etc.}$$
$$+C$$

Fiat ergo $C=\frac{1}{2}b$, et $B=a-\frac{1}{2}b$; eritque $v=P$, et terminus generalis $y=A+(a-\frac{1}{2}b)x+\frac{1}{2}bxx+P$; seu cum A in functione P comprehendi possit, erit: $y=(a-\frac{1}{2}b)x+\frac{1}{2}bxx+P$. Q. E. I.

Scholion.

§. 54. Huiusmodi autem series reuocari possunt ad legem recurrentium simplicium. Cum enim sit

$y'=a+bx+my$ erit

$y''=a+b(x+1)+my'$ vnde subtrahendo

$y''-y'=b+my'-my$ simili modo erit

$y'''-y''=b+my''-my'$ denuoque subtrahendo:

$y'''-2y''+y'=my''-2my'+my$ seu

$y'''=(m+2)y''-(2m+1)y'+my$ siue pro terminis antecedentibus

$y=(m+2)'y-(2m+1)''y+m'''y$.

Hinc ergo secundum §. LI. formabitur aequatio: $u^3-(m+2)u^2+(2m+1)u-m=0$ quae habet factores: $(u-1)^2(u-m)=0$. ex quibus oritur terminus generalis: $y=P+Qx+m^xR$. Iam vt haec forma nimis late patens ad casum propositum $y'=a+bx+my$ accommodetur, ob

$y'=$

$'y = P + Qx + Q + m.m^x R$, fiet

$P + Q + Qx + m.m^x R = a + bx + mP + mQx + m.m^x R$

ideoque $P + Q = a + mP$, et $Q = b + mQ$, vnde inuenitur $Q = \frac{-b}{m-1}$ et $P = \frac{-b}{(m-1)^2} - \frac{a}{m-1}$, ita vt ſit terminus generalis vt ante eſt inuentus:

$$y = \frac{-b}{(m-1)^2} - \frac{a-bx}{m-1} + m^x R.$$

Sin autem $m = 1$, ſtatim patet aequationis $(u-1)^2(u-m) = 0$, tres factores fore aequales, fierique $(u-1)^3 = 0$, vnde fit terminus generalis $y = P + Qx + Rxx$, et propterea:

$$\begin{array}{ll} 'y = P + Qx + Rxx & = P + Qx + Rxx \\ \quad + Q + 2Rx & \quad\; a + bx \\ \quad + R & \end{array}$$

Ergo prodit: $R = \frac{1}{2}b$ et $Q = a - \frac{1}{2}b$, ita vt ſit terminus generalis $y = P + (a - \frac{1}{2}b)x + \frac{1}{2}bxx$, vt ante. Simili modo apparet, ſi lex progreſſionis in genere ſit:

$$y = X + \alpha\, 'y + \beta\, ''y + \gamma\, '''y + \delta\, ^{\text{IV}}y + \text{etc.}$$

atque X ſit functio ipſius x rationalis integra, veluti $X = a + bx + cxx + dx^3 +$ etc. per continuam ſubtractionem tandem perueniri ad legem, qua ſinguli termini per ſolos antecedentes determinentur; ſicque ſeriem ſemper fore recurrentem, cuius terminus generalis per praecepta ante data definiri queat. Hic autem terminus nimis late patebit; hancque ob rem quaerendis valoribus terminorum $'y$, $''y$, $'''y$ etc. ad legem propoſitam accommodari debebit, quo pacto tot ſemper functiones P, Q, R, etc. determinabuntur, quot litterae a, b. c, d, etc. fuerint per ſubtractionem eliminatae. Cum igitur huiusmodi

modi series nihil amplius habeant difficultatis, alias consideremus, in quibus X non sit functio, vel rationalis, vel integra ipsius x.

Problema. IX.

§. 55. Inuenire terminum generalem seriei, cuius quilibet terminus aequetur praecedenti vna cum functione quacunque ipsius indicis.

Solutio.

Sit terminus indici x respondens $= y$, eiusque antecedens

$$'y = y - \frac{dy}{1\,dx} + \frac{ddy}{1.2\,dx^2} - \frac{d^3y}{1.2.3\,dx^3} + \frac{d^4y}{1.2.3.4\,dx^4} - \text{etc.}$$

Lex autem progressionis sit $y = {}'y + X$, vnde fiet

$$X = \frac{dy}{1\,dx} - \frac{ddy}{1.2\,dx^2} + \frac{d^3y}{1.2.3\,dx^3} - \frac{d^4y}{1.2.3.4\,dx^4} + \text{etc.}$$

quae aequatio resoluetur ope regularum, quas ante aliquod tempus tradidi. Scilicet ponendo z^n pro $\frac{d^ny}{dx^n}$ formetur haec expressio:

$$Z = z - \frac{z^2}{1.2} + \frac{z^3}{1.2.3} - \frac{z^4}{1.2.3.4} + \text{etc.} = 1 - e^{-z}$$

cuius quaerantur omnes factores, qui erunt primo z; reliqui in hac forma generali continentur: $zz + 4kk\pi\pi$. Ex factore $z - 0$ orietur autem haec pars integralis:

$$y = \int X\,dx + \text{etc.}$$

Ex factore autem $zz + 4kk\pi\pi$ si comparetur cum formula $zz - 2kz\cos.\Phi + kk$, fiet $k = 2k\pi$, et $\cos.\Phi = 0$, vnde $\Phi = 90^\circ$, ideoque litterae $\mathfrak{M}$ et $\mathfrak{N}$ ita determinabuntur ob $A = 0$; $B = 1$, $C = \frac{-1}{1.2}$; $D = \frac{1}{1.2.3}$ etc.

$\mathfrak{M} =$

$$\mathfrak{M} = 1 - \frac{4k^2\pi^2}{1.2} + \frac{16k^4\pi^4}{1.2.3.4} - \frac{64k^6\pi^6}{1.2.3.4.5.6} + \text{etc.}$$

$$\mathfrak{N} = -\frac{2k\pi}{1} + \frac{8k^3\pi^3}{1.2.3} - \frac{32k^5\pi^5}{1.2.3.4.5} + \text{etc.}$$

ita vt ſit $\mathfrak{M} = \text{cof.}\, 2k\pi$, et $\mathfrak{N} = -\text{sin.}\, 2k\pi$. His valoribus inuentis, erit pars integralis ex factore $zz + 4kk\pi\pi$ oriunda:

$$y = 2\left\{\begin{array}{l}(\text{cof.}\, 2k\pi\, \text{cof.}\, 2k\pi x - \text{sin.}\, 2k\pi.\, \text{sin.}\, 2k\pi x) \int X dx\, \text{cof.}\, 2k\pi x\\ (\text{cof.}\, 2k\pi.\, \text{sin.}\, 2k\pi x + \text{sin.}\, 2k\pi.\, \text{cof}\, 2k\pi x) \int X dx\, \text{sin.}\, 2k\pi x\end{array}\right.$$

at eſt, ſi $2k = 0$, $\text{cof.}\, 2k\pi = 1$, vnde erit:

$$v = 2\, \text{cof.}\, 2k\pi x \int X dx\, \text{cof.}\, 2k\pi x + 2\, \text{sin.}\, 2k\pi x \int X dx\, \text{sin.}\, 2k\pi x.$$

Quod ſi iam omnes hi valores, ex variabilitate numeri k oriundi, in vnam ſummam colligantur, prodibit terminus generalis quaeſitus:

$$y = \int X dx \begin{array}{l} + 2\,\text{cof.}\, 2\pi x \int X dx\, \text{cof.}\, 2\pi x + 2\,\text{cof}\, 4\pi x \int X dx\, \text{cof.}\, 4\pi x + 2\,\text{cof.}\, 6\pi x \int X dx\, \text{cof.}\, 6\pi x\\ + 2\,\text{sin.}\, 2\pi x \int X dx\, \text{sin.}\, 2\pi x + 2\,\text{sin.}\, 4\pi x \int X dx\, \text{sin.}\, 4\pi x + 2\,\text{sin.}\, 6\pi x \int X dx\, \text{sin.}\, 6\pi x \end{array} \text{etc.}$$

Q. E. I.

Coroll. 1.

§. 56. Quia eſt $y = {}'y + X$, manifeſtum eſt, y exprimere terminum ſummatorium ſeriei, cuius terminus generalis ſit $= X$. Si enim ſumma omnium terminorum a primo vsque ad hunc X, cuius index eſt $= x$, ponatur $= y$, erit ſumma omnium praeter vltimum $= {}'y$, ideoque $y = {}'y + X$.

Coroll. 2.

§. 57. Expreſſio ergo inuenta y, ſeu terminus generalis ſeriei propoſitae, ſimul eſt terminus ſummatorius ſeriei, cuius terminus generalis eſt $= X$; ſicque nouam adepti ſumus expreſſionem

nem pro ſumma cuiusque ſeriei, cuius terminus generalis datur; quae autem ob multitudinem integralium infinitam rariſſime vſum aliquem praeſtabit.

Scholion.

§. 58. Si praeter functionem quamcunque indicis x, non ſolum terminus proxime praecedens, ſed plures praecedentium, ad formationem termini ſequentis adhibeantur, ſimili modo peruenietur ad reſolutionem aequationis differentialis infinitae, quae ope *methodi a me propoſitae* tractari poterit. Non ſolum igitur ſeries, quarum lex formationis ad genus pertinet ſecundum, methodo hic expoſita ad calculum reuocari, earumque termini generales inueniri poſſunt, ſed etiam ad genus tertium aeque patet, iſtarumque ſerierum veram indolem clarius ob oculos ponit.

Problema X.

§. 59. Inuenire terminum generalem ſeriei, cuius quilibet terminus aequalis ſit praecedenti per ſuum indicem multiplicato.

Solutio.

Si terminus primus vnitati aequalis ſtatuatur, orietur *Walliſii* ſeries Hypergeometrica haec:

ind:	1,	2,	3,	4,	5,	6,	7,	8,	9,	etc.
term:	1,	1,	2,	6,	24,	120,	720,	5040,	40320,	

Ponatur terminus indici x, reſpondens $= y$, eumque ſequens $= y'$, erit $y' = y.x$, vnde naſcitur haec aequatio:

$yx =$

$$y\,x = y + \frac{dy}{1\,dx} + \frac{ddy}{1.2\,dx^2} + \frac{d^3y}{1.2.3\,dx^3} + \frac{d^4y}{1.2.3.4\,dx^4} + \text{etc.}$$

pro tali autem aequatione reſoluenda regula generalis non conſtat. At leui negotio haec aequatio in aliam formam transformatur, quam reſoluere liceat. Ponatur ſcilicet $y = e^v$, erit $y' = e^{v'}$, ideoque fiet $e^{v'} = e^v x$, ſumtisque logarithmis: $v' = v + lx$, quocirca habebitur:

$$lx = \frac{dv}{1\,dx} + \frac{ddv}{1.2\,dx^2} + \frac{d^3v}{1.2.3\,dx^3} + \frac{d^4v}{1.2.3.4\,dx^4} + \frac{d^5v}{1.2.3.4.5\,dx^5} + \text{etc.}$$

quae aequatio in praecedente continetur, faciendo $X = lx$, quocirca integrale erit:

$$v = \int dx\,lx \begin{matrix} + 2\,\text{coſ.}\,2\pi x \int dx\,lx.\,\text{coſ.}\,2\pi x + 2\,\text{coſ.}\,4\pi x \int dx\,lx.\,\text{coſ.}\,4\pi x + \text{etc.} \\ + 2\,\text{ſin.}\,2\pi x \int dx\,lx.\,\text{ſin.}\,2\pi x + 2\,\text{ſin.}\,4\pi x \int dx\,lx.\,\text{ſin.}\,4\pi x + \text{etc.} \end{matrix}$$

Inuento autem valore ipſius v, erit terminus generalis quaeſitus $y = e^v$: denotante e numerum, cuius logarithmus hyperbolicus eſt $= 1$. Q. E. I.

Scholion.

§. 60. Primum huius expreſſionis membrum $\int dx\,lx$ eſt $= x\,lx - x$, reliqua vero membra ſingula per ſeries infinitas integrari poterunt. Eſt enim:

$$\int dx\,lx.\,\text{coſ.}\,mx = \frac{1}{m}\,\text{ſin.}\,mx\left(lx + \frac{1}{m^2x^2} - \frac{1.2.3}{m^4x^4} + \frac{1.2.3.4.5}{m^6x^6} - \text{etc.}\right)$$
$$+ \frac{1}{m}\,\text{coſ.}\,mx\left(\frac{1}{mx} - \frac{1.2}{m^3x^3} + \frac{1.2.3.4}{m^5x^5} - \text{etc.}\right)$$

$$\int dx\,lx.\,\text{ſin.}\,mx = -\frac{1}{m}\,\text{coſ.}\,mx\left(lx + \frac{1}{m^2x^2} - \frac{1.2.3}{m^4x^4} + \frac{1.2.3.4.5}{m^6x^6} - \text{etc.}\right)$$
$$+ \frac{1}{m}\,\text{ſin.}\,mx\left(\frac{1}{mx} - \frac{1.2}{m^3x^3} + \frac{1.2.3.4}{m^5x^5} - \text{etc.}\right)$$

Vnde colligitur, fore:

$$\left.\begin{matrix} 2\,\text{coſ.}\,mx\int dx\,lx.\,\text{coſ.}\,mx \\ + 2\,\text{ſin.}\,mx\int dx\,lx.\,\text{ſin.}\,mx \end{matrix}\right\} = \frac{2}{mmx}\left(1 - \frac{1.2}{m^2x^2} + \frac{1.2.3.4}{m^4x^4} - \frac{1.2.3.4.5.6}{m^6x^6} + \text{etc.}\right)$$
$$+ \alpha\,\text{coſ.}\,mx + \mathfrak{A}\,\text{ſin.}\,mx$$

Subſti-

Substitutis iam successiue pro m valoribus 2π, 4π, 6π, etc. cunctisque his expressionibus collectis reperietur :

$$\begin{array}{ll} v = C + x\,lx - x & + \frac{1}{2\pi^2 x}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}\right) \\ \quad \alpha \cos. 2\pi x + \mathfrak{A} \sin. 2\pi x & - \frac{1 \cdot 2}{8\pi^4 x^3}\left(1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \text{etc.}\right) \\ \quad \beta \cos. 4\pi x + \mathfrak{B} \sin. 4\pi x & \\ \quad \gamma \cos. 6\pi x + \mathfrak{C} \sin. 6\pi x & + \frac{1 \cdot 2 \cdot 3 \cdot 4}{32\pi^6 x^5}\left(1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \text{etc.}\right) \\ \qquad \text{etc.} & \qquad \text{etc.} \end{array}$$

Si nunc pro his seriebus potestatum *summae*, *a me dudum inuentae*, substituantur, habebitur :

$$\begin{aligned} v = C + x\,lx - x &+ \alpha \cos. 2\pi x + \beta \cos. 4\pi x + \gamma \cos. 6\pi x + \text{etc.} \\ &+ \mathfrak{A} \sin. 2\pi x + \mathfrak{B} \sin. 4\pi x + \mathfrak{C} \sin. 6\pi x + \text{etc.} \\ + \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2x} &- \frac{1}{3 \cdot 4 \cdot 5} \cdot \frac{1}{6x^3} + \frac{1}{5 \cdot 6 \cdot 7} \cdot \frac{1}{6x^5} - \frac{1}{7 \cdot 8 \cdot 9} \cdot \frac{3}{10x^7} + \frac{1}{9 \cdot 10 \cdot 11} \cdot \frac{5}{6x^9} - \text{etc.} \end{aligned}$$

seu si sit P functio parium dimensionum ipsarum $r = \sin. \pi x$, et $s = \cos. \pi x$, erit :

$$v = P + x\,lx - x + \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2x} - \frac{1}{3 \cdot 4 \cdot 5} \cdot \frac{1}{6x^3} + \frac{1}{5 \cdot 6 \cdot 7} \cdot \frac{1}{6x^5} - \text{etc.}$$

Cum iam posito $x = 1$, fiat $y = 1$, et $v = 0$; debebit hoc casu fieri: $P = 1 - \frac{1}{1 \cdot 2 \cdot 3 \cdot 2} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6} - \frac{1}{5 \cdot 6 \cdot 7 \cdot 6} + \frac{3}{7 \cdot 8 \cdot 9 \cdot 10} - \text{etc.}$ cuius valorem, alibi ostendi, esse $P = \frac{1}{2}\, l\, 2\pi$, huncque valorem habebit, quoties x sit numerus integer quicunque. Hinc ad numeros regrediendo inuenietur terminus generalis quaesitus :

$$y = \frac{x^x}{e^x} \cdot e^{\frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2x} - \frac{1}{3 \cdot 4 \cdot 5} \cdot \frac{1}{6x^3} + \frac{1}{5 \cdot 6 \cdot 7} \cdot \frac{1}{6x^5} - \text{etc.}} \sqrt{2\pi} \text{ seu}$$

$$y = \frac{x^x}{e^x}\, e^{\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \text{etc.}} \sqrt{2\pi}$$

Vnde

Vnde si x sit numerus valde magnus, erit proxime:

$$y = \frac{x^x}{e^x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} + \text{etc.}\right)\sqrt{2\pi}$$

sicque magnitudo cuiusuis termini, ab initio valde remoti, non difficulter proxime assignatur.