Analysis of a Problem in the Probability Calculus

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E812
Opera Postuma 1, 1862, p. 336–341

There are, in an urn, four tickets $a,\ b, c, d,$ of which one draws one at random, and after having returned it to the urn, one draws from it a new one, and this to n repetitions; one requires the probability that the ticket a will never be drawn, or that it will be drawn only one time alone or twice etc., or finally, that it comes out in each of n drawings.

1. The given ticket will be removed neither in the first drawing,

$$\textit{probability:}\quad \frac{3}{4};$$

nor in the second drawing,

$$\textit{probability:}\quad \left(\frac{3}{4}\right)^2;$$

nor in the third drawing,

$$\textit{probability:}\quad \left(\frac{3}{4}\right)^3;$$

etc.;

it will not be removed at all,

$$\textit{probability:}\quad \left(\frac{3}{4}\right)^n.$$

2. it will be removed only one time in the n drawings,

$$\textit{probability:}\quad \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}\cdot n;$$

3. it will be removed two times in the n drawings,

$$\textit{probability:}\quad \left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{n-2}\cdot\frac{n(n-1)}{1\cdot 2}$$

etc.;

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4. it will be removed in each of the n drawings,

$$\textit{probability:} \quad \left(\frac{1}{4}\right)^n.$$

If of the four tickets a, b, c, d, one draws two each time, in n different repetitions,

1. the given ticket a will never be found in it,

$$\textit{probability:} \quad \left(\frac{1}{2}\right)^n;$$

2. it will be found there one time,

$$n \cdot \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1}.$$

The number of tickets a, b, c etc. being $= N$, if one draws m tickets at a time and if one repeats this operation n times:

1. the given ticket a will never be found among the drawn tickets,

$$\textit{probability:} \quad \left(1 - \frac{m}{N}\right)^n;$$

2. it will be met one time,

$$\textit{probability:} \quad n \cdot \frac{m}{N}\left(1 - \frac{m}{N}\right)^{n-1};$$

3. it will be met two times,

$$\textit{probability:} \quad \frac{n(n-1)}{1 \cdot 2} \cdot \left(\frac{m}{N}\right)^2 \left(1 - \frac{m}{N}\right)^{n-2}$$

etc.;

4. it will be met in each drawing,

$$\textit{probability:} \quad \left(\frac{m}{N}\right)^n.$$

EXAMPLE

The number of tickets being 50000 of which one draws in each repetition 8000, and this five times in sequence, there will be therefore

$$N = 50000, \quad m = 8000, \quad n = 5.$$

1. The given ticket a will not be removed at all,

$$\textit{probability:} \quad \left(\frac{21}{25}\right)^5 = 0.4182120;$$

2. it will be met in one of the five drawings,

$$\textit{probability:} \quad 5 \cdot \frac{4}{25}\left(\frac{21}{25}\right)^4 = 0.3982972;$$ [1]

[1] In the original edition: 0.3982950.

3. it will be met in two drawings,

$$\textit{probability:}\quad 10\cdot\left(\frac{4}{25}\right)^2\left(\frac{21}{25}\right)^3 = 0.115732;\,^2$$

4. it will be met in three drawings,

$$\textit{probability:}\quad 10\cdot\left(\frac{4}{25}\right)^3\left(\frac{21}{25}\right)^2 = 0.028901;\,^3$$

5. it will be met in four drawings,

$$\textit{probability:}\quad 5\left(\frac{4}{25}\right)^4\left(\frac{21}{25}\right) = 0.002752;$$

6. it will be met in all five drawings,

$$\textit{probability:}\quad \left(\frac{4}{25}\right)^4 = 0.000105.$$

The number of drawings, n, being the same, one requires the probability that two tickets, a and b, are never met together if, of N tickets, one draws each time $N-m$.

1. In the first drawing, a is not in the number of $N-m$ drawn drawn tickets,

$$\textit{probability:}\quad \frac{m}{N};$$

   b is neither,

$$\textit{probability:}\quad \frac{m(m-1)}{N(N-1)} = \nu;$$

2. the two are missing in the second drawing,

$$\textit{probability:}\quad \nu^2;$$

3. the two are missing in the third drawing,

$$\textit{probability:}\quad \nu^3;$$

etc.

4. the two are missing in all the drawings,

$$\textit{probability:}\quad \nu^n;$$

[2]In the original edition: 0.151730.
[3]In the original edition: 0.028906.

One requires the probability that λ given tickets are not met in any of n drawings. One has only what is put

$$\frac{m(m-1)(m-2)\cdots(m-\lambda+1)}{N(N-1)(N-2)\cdots(N-\lambda+1)} = \nu,$$

and the sought probability will be $= \nu^n$.

In the preceding example one will have

$$N = 50000, \quad N - m = 8000, \quad m = 42000 \quad \text{and} \quad n = 5.$$

If of 10 tickets which are found in an urn, one draws 2, there will remain 8. When, after having returned them, one repeats the operation again one time, it is certain that six tickets at least will not be drawn; but it is possible that the number of non-drawn be the same 8. The concern is to enumerate the cases where 6, 7 and 8 tickets will remain intact.

There will be six of them. We suppose that in the first drawing there be removed the numbers 1 and 2. In the second turn, the two numbers must be of 3 to 10, therefore the probability is

$$= \frac{8 \cdot 7}{10 \cdot 9}.$$

There will be seven of them, when 1 or 2 are removed anew in the second drawing, that is to say when one has in the second drawing

$$1,\, 3 \quad \text{or} \quad 1,\, 4 \quad 1,\, 5 \quad \text{etc.},$$

eight favorable chances; or else

$$2,\, 3 \quad \text{or} \quad 2,\, 4 \quad \text{or} \quad 2,\, 5 \quad \text{etc.},$$

so many chances. Now, the number of all possible cases being $= \frac{10 \cdot 9}{1 \cdot 2}$, the probability will be

$$= 2 \cdot \frac{2 \cdot 8}{10 \cdot 9}.$$

There will eight of them, if in the second turn are removed the same numbers 1 and 2 as in the first: one favorable chance, of which the probability is

$$= \frac{2 \cdot 1}{10 \cdot 9}.$$

Therefore in order that the number the tickets remaining intact be

$$6 \quad \text{or} \quad 7 \quad \text{or} \quad 8,$$

the respective probability will be

$$\frac{8 \cdot 7}{10 \cdot 9}, \quad 2 \cdot \frac{8 \cdot 2}{10 \cdot 9}, \quad \frac{1 \cdot 2}{10 \cdot 9}.$$

When one draws three times in sequence, there will be either 4 or 5 or 6 or 7 or 8 tickets of non-drawns.

I. Let the number of non-drawns be 8. In the first drawing the numbers 1 and 2 being removed, it is necessary that these same numbers are removed in the second and in the third drawing; the probability of the first of these chances being $\frac{1\cdot 2}{10\cdot 9}$, which of the two chances is

$$= \left(\frac{1\cdot 2}{10\cdot 9}\right)^2.$$

II. In order that the number of non-drawns be 4, it is necessary that in the second drawing there are removed two tickets different from the numbers 1 and 2, which gives for measure of the probability $\frac{8\cdot 7}{10\cdot 9}$; and in order that in the third turn there comes again two tickets other than the four already drawn, the probability will be

$$= \frac{8\cdot 7}{10\cdot 9}\cdot\frac{6\cdot 5}{10\cdot 9}.$$

III. In order that the number of non-drawns be 7, it is necessary to consider the following four cases:

first drawing	ab	ab	ab	ab
second drawing	ab	ac	ac	ac
third drawing	ac	ab	bc	ac

the probability of each particular case is

$$2\cdot\frac{8\cdot 2\cdot 1\cdot 2}{10\cdot 9\cdot 10\cdot 9},$$

therefore, the total probability is

$$8\cdot\frac{8\cdot 2\cdot 1\cdot 2}{10\cdot 9\cdot 10\cdot 9}.$$

IV. In order that the number of non-drawns be 6, it is necessary to consider the 7 following cases:

first drawing	ab	ab	ab	ab	ab	ab	ab
second drawing	ab	cd	cd	ac	ac	cd	ac
third drawing	cd	ab	cd	ad	cd	ac	bd

of which the respective probabilities will be

$$\frac{8\cdot 7\cdot 2\cdot 1}{10\cdot 9\cdot 10\cdot 9},\quad \frac{8\cdot 7\cdot 2\cdot 1}{10\cdot 9\cdot 10\cdot 9},\quad \frac{8\cdot 7\cdot 2\cdot 1}{10\cdot 9\cdot 10\cdot 9},$$

$$2\cdot\frac{8\cdot 7\cdot 2\cdot 2}{10\cdot 9\cdot 10\cdot 9},\quad 2\cdot\frac{8\cdot 7\cdot 2\cdot 2}{10\cdot 9\cdot 10\cdot 9},\quad 2\cdot\frac{8\cdot 7\cdot 2\cdot 2}{10\cdot 9\cdot 10\cdot 9},\quad 2\cdot\frac{8\cdot 7\cdot 2\cdot 2}{10\cdot 9\cdot 10\cdot 9};$$

and consequently, the total probability will be

$$\frac{8\cdot 7\cdot 38}{10\cdot 9\cdot 10\cdot 9}.$$

V. Let the number of non-drawns be 5; the three cases to consider are:

first drawing	ab	ab	ab
second drawing	ac	cd	cd
third drawing	de	ae	de

the probability of each of these cases is

$$2 \cdot \frac{8 \cdot 7 \cdot 6 \cdot 2}{10 \cdot 9 \cdot 10 \cdot 9}$$

and consequently the total probability

$$6 \cdot \frac{8 \cdot 7 \cdot 6 \cdot 2}{10 \cdot 9 \cdot 10 \cdot 9}$$

In summarizing all these cases, we will obtain the following table:

Number of non-drawn tickets	probability
4	$\frac{8 \cdot 7 \cdot 6 \cdot 5}{(10 \cdot 9)^2}$
5	$6 \cdot \frac{8 \cdot 7 \cdot 6 \cdot 2}{(10 \cdot 9)^2}$
6	$\frac{8 \cdot 7 \cdot 38}{(10 \cdot 9)^2}$
7	$8 \cdot \frac{8 \cdot 2 \cdot 1 \cdot 2}{(10 \cdot 9)^2}$
8	$\frac{1 \cdot 2 \cdot 1 \cdot 2}{(10 \cdot 9)^2}$

If instead of 10 tickets there are n of them, of which one draws two in each repetition, one will have

I. In drawing two times,

for the number of non-removed tickets	the probability
$n-2$	$\frac{1 \cdot 2}{n(n-1)}$
$n-3$	$2 \cdot \frac{(n-2)2}{n(n-1)}$
$n-4$	$\frac{(n-2)(n-3)}{n(n-1)}$

We will consider now the numerators of these different cases, and in putting, for more simplicity,

$$n-2=m,$$

there will be

$$2, \quad 4m \quad \text{and} \quad m(m-1);$$

their sum gives us the value m^2+3m+2; and consequently the equation

$$A+Bm+m(m-1)=m^2+3m+2,$$

which must subsist for all the values of m, we supply the values of the coefficients A and B.

II. In drawing three times, one will have

for the number of non-removed tickets	the probability
$n-2$	$\frac{1\cdot 2\cdot 1\cdot 2}{n^2(n-1)^2}$
$n-3$	$8\cdot\frac{(n-2)\cdot 2\cdot 1\cdot 2}{n^2(n-1)^2}$
$n-4$	$\frac{8(n-2)2(n-3)2+3\cdot 1\cdot 2(n-2)(n-3)}{n^2(n-1)^2}$
$n-5$	$6\cdot\frac{(n-2)(n-3)(n-4)}{n^2(n-1)^2}$
$n-6$	$\frac{(n-2)(n-3)(n-4)(n-5)}{n^2(n-1)^2}$

In setting anew $n-2=m$, the numerators of these different cases will be

$$4,\quad 32m,\quad 38m(m-1),\quad 12m(m-1)(m-2)$$

and

$$m(m-1)(m-2)(m-3),$$

of which the sum gives us the value $(m^2+3m+2)^2$; and consequently, the equation to determine the coefficients 4, 32, 38 and 12 will be

$$A+Bm+Cm(m-1)+Dm(m-1)(m-2)+m(m-1)(m-2)(m-3) \\ =(m^2+3m+2)^2$$

CONCLUSION

Thus, one is able to conclude that, if the number of tickets is n, of which one draws two at each repetition, the number of drawings being $p+1$, one will have

for the number of non-removed tickets	the probability
$n-2$	$\frac{A}{n^p(n-1)^p}$
$n-3$	$\frac{B(n-2)}{n^p(n-1)^p}$
$n-4$	$\frac{C(n-2)(n-3)}{n^p(n-1)^p}$
$n-5$	$\frac{D(n-2)(n-3)(n-4)}{n^p(n-1)^p}$
etc.	etc.

and the coefficients A, B, C, D etc. will be given by the equation

$$A+Bm+Cm(m-1)+Dm(m-1)(m-2)+\cdots \\ +m(m-1)(m-2)\cdots(m-2p-1)=(m^2+3m+2)^p$$

which is independent of $m=n-2$.

Thus, in drawing four times for two, one finds

for the number of non-removed tickets	the probability
$n-2$	$\frac{8}{n^3(n-1)^3}$
$n-3$	$\frac{208(n-2)}{n^3(n-1)^3}$
$n-4$	$\frac{652(n-2)(n-3)}{n^3(n-1)^3}$
$n-5$	$\frac{576(n-2)(n-3)(n-4)}{n^3(n-1)^3}$
$n-6$	$\frac{188(n-2)(n-3)(n-4)(n-5)}{n^3(n-1)^3}$
$n-7$	$\frac{24(n-2)(n-3)(n-4)(n-5)(n-6)}{n^3(n-1)^3}$
$n-8$	$\frac{(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{n^3(n-1)^3}$

If one has n tickets, of which one draws three at each repetition, in drawing two times in sequence, one will have

for the number of non-removed tickets	the probability
$n-3$	$\frac{1\cdot 2\cdot 3}{n(n-1)(n-2)}$
$n-4$	$\frac{2\cdot 3\cdot 3\cdot (n-3)}{n(n-1)(n-2)}$
$n-5$	$9\cdot\frac{(n-3)(n-4)}{n(n-1)(n-2)}$
$n-6$	$\frac{(n-3)(n-4)(n-5)}{n(n-1)(n-2)}$

In considering the numerators, their sum

$$6 + 18m + 9m(m-1) + m(m-1)(m-2)$$

will be presented under the form

$$(m+1)(m+2)(m+3);$$

here

$$m = n - 3$$

and consequently, the identical equation which serves to determine the coefficients 6, 18 and 9 will be

$$A + Bm + Cm(m-1) + m(m-1)(m-2) = (m+1)(m+2)(m+3).$$

In drawing three times in sequence, the identical equation to determine the coefficients will be similarly

$$\begin{gathered} A + Bm + Cm(m-1) + Dm(m-1)(m-2) + Em(m-1)(m-2)(m-3) \\ +Fm(m-1)(m-2)(m-3)(m-4) + m(m-1)(m-2)(m-3)(m-4)(m-5) \\ = \{(m+1)(m+2)(m+3)\}^2 \end{gathered}$$

and thus so on.

GENERAL RULE

All these researches lead us to the following rule.

In one has n tickets, of which one draws p at each repetition, and that q times in sequence, one requires the probabilities of the different numbers of non-removed tickets.

To this effect, one begins by seeking the coefficients A, B, C etc. of the identical equation

$$\begin{gathered} A + Bm + Cm(m-1) + Dm(m-1)(m-2) + \cdots \\ +m(m-1)(m-2)(m-3)\cdots(m-p(q-1)+1) \\ = \{(m+1)(m+2)(m+3)\cdots(m+p)\}^{q-1}; \end{gathered}$$

then the numerators of the respective probabilities will be

$$\begin{gathered} A, \quad Bm, \quad Cm(m-1), \quad Dm(m-1)(m-2), \quad \text{etc.,} \\ m(m-1)(m-2)(m-3)\cdots(m-p(q-1)+1), \end{gathered}$$

m being $= n-p$, and the denominator being the same for all

$$n^{q-1}(n-1)^{q-1}(n-2)^{q-1}\cdots(n-p+1)^{q-1}.$$

Here is the table:

for the number of non-removed tickets	the probability
$n-p$	$\frac{A}{n^{q-1}(n-1)^{q-1}(n-2)^{q-1}\cdots(n-p+1)^{q-1}}$
$n-p-1$	$\frac{B(n-p)}{n^{q-1}(n-1)^{q-1}(n-2)^{q-1}\cdots(n-p+1)^{q-1}}$
$n-p-2$	$\frac{C(n-p)(n-p-1)}{n^{q-1}(n-1)^{q-1}(n-2)^{q-1}\cdots(n-p+1)^{q-1}}$
$n-p-3$	$\frac{D(n-p)(n-p-1)(n-p-2)}{n^{q-1}(n-1)^{q-1}(n-2)^{q-1}\cdots(n-p+1)^{q-1}}$
$\cdots$	$\cdots$
$n-pq$	$\frac{(n-p)(n-p-1)(n-p-2)\cdots(n-pq+1)}{n^{q-1}(n-1)^{q-1}(n-2)^{q-1}\cdots(n-p+1)^{q-1}}$