

An algebraic problem of finding four numbers  
given the products of each of the numbers with  
the sum of the other three\*

Leonhard Euler

If the four numbers which are to be found are put  $v, z, y, z$ , the following four equations are obtained

$$\begin{aligned}v(x + y + z) &= a, \\x(v + y + z) &= b, \\y(v + x + z) &= c, \\z(v + x + y) &= d.\end{aligned}$$

By the usual rules, three unknowns can be successively eliminated from these equations and the fourth can then be solved for. But indeed, there is no reason why we should favor any one of these unknowns over the others as the one to be solved for. It is appropriate that none of them be determined by the final equation, and instead that a new unknown be introduced that has the same relation with them all and from which the unknowns can be defined. Thus to this end let us take the sum of the numbers to be found

$$v + x + y + z = 2t$$

and then the above equations will turn into these

$$\begin{aligned}v(2t - v) = a = 2tv - v^2, & \quad \text{whence} \quad v = t - \sqrt{(tt - a)}, \\x(2t - x) = b = 2tx - x^2, & \quad x = t - \sqrt{(tt - b)}, \\y(2t - y) = c = 2ty - y^2, & \quad y = t - \sqrt{(tt - c)}, \\z(2t - z) = d = 2tz - z^2, & \quad z = t - \sqrt{(tt - d)}.\end{aligned}$$

Therefore we have now produced a solution to the extent that from the single quantity  $t$  we can readily determine all the four numbers that are sought; hence

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what remains is for us to investigate this quantity  $t$ , and by substituting the values in terms of  $t$  just found for  $v, x, y, z$  into the equation

$$v + x + y + z = 2t,$$

$t$  will satisfy

$$4t - \sqrt{(tt - a)} - \sqrt{(tt - b)} - \sqrt{(tt - c)} - \sqrt{(tt - d)} = 2t,$$

from which follows the equation

$$2t = \sqrt{(tt - a)} + \sqrt{(tt - b)} + \sqrt{(tt - c)} + \sqrt{(tt - d)},$$

which can indeed be made rational by the method of Newton,<sup>1</sup> but this would be very cumbersome. Therefore we seek to resolve this equation in another way.

Let us put

$$\begin{aligned} \sqrt{(tt - a)} &= p, & \text{and } v &= t - p, \\ \sqrt{(tt - b)} &= q, & x &= t - q, \\ \sqrt{(tt - c)} &= r, & y &= t - r, \\ \sqrt{(tt - d)} &= s, & z &= t - s \end{aligned}$$

and it will be

$$p + q + r + s = 2t.$$

Because of the irrationals  $p, q, r, s$ , this equation should to be transformed into another, in which only even powers of the letters occur. Then by doing the substitution for the letters  $p, q, r, s$ , a rational equation will be formed from which the value of the unknown  $t$  may be defined.

To this end let us form the equation

$$X^4 - AX^3 + BX^2 - CX + D = 0,$$

whose four roots are the given quantities  $a, b, c, d$ . Therefore by the nature of equations it will be

$$\begin{aligned} A &= a + b + c + d, \\ B &= ab + ac + ad + bc + bd + cd, \\ C &= abc + abd + acd + bcd, \\ D &= abcd. \end{aligned}$$

One may then put

$$Y = tt - X \quad \text{or} \quad X = -Y + tt;$$

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<sup>1</sup>Translator: The editors of the *Opera omnia* refer to p. 66 of the 1707 edition of Newton's *Arithmetica universalis*.

by doing this substitution we shall have the equation

$$\left. \begin{array}{r} Y^4 - 4ttY^3 + 6t^4Y^2 - 4t^6Y + t^8 \\ + AY^3 - 3AttY^2 + 3At^4Y - At^6 \\ + BY^2 - 2BttY + Bt^4 \\ + CY - Ctt \\ + D \end{array} \right\} = 0.$$

The four roots of this equation of  $Y$  will be

$$tt - a, \quad tt - b, \quad tt - c, \quad tt - d.$$

For this equation let us write for brevity

$$Y^4 - PY^3 + QY^2 - RY + S = 0,$$

so that

$$\begin{aligned} P &= 4tt - A, \\ Q &= 6t^4 - 3Att + B, \\ R &= 4t^6 - 3At^4 + 2Btt - C, \\ S &= t^8 - At^6 + Bt^4 - Ctt + D. \end{aligned}$$

Then let

$$Y = Z^2 \quad \text{or} \quad Z = \pm\sqrt{Y};$$

we will have

$$Z^8 - PZ^6 + QZ^4 - RZ^2 + S = 0$$

and the the eight roots of this equation will be the following

$$\begin{aligned} +\sqrt{(tt - a)} &= +p, & -\sqrt{(tt - a)} &= -p, \\ +\sqrt{(tt - b)} &= +q, & -\sqrt{(tt - b)} &= -q, \\ +\sqrt{(tt - c)} &= +r, & -\sqrt{(tt - c)} &= -r, \\ +\sqrt{(tt - d)} &= +s, & -\sqrt{(tt - d)} &= -s. \end{aligned}$$

One may resolve this equation of eight dimensions into two biquadratics, the roots of the first of which are  $+p, +q, +r, +s$ , the second  $-p, -q, -r, -s$ ; let them be<sup>2</sup>

$$\begin{aligned} Z^4 - \alpha Z^3 + \beta Z^2 - \gamma Z + \delta &= 0, \\ Z^4 + \alpha Z^3 + \beta Z^2 + \gamma Z + \delta &= 0, \end{aligned}$$

in which by the nature of equations it will be

$$\begin{aligned} \alpha &= p + q + r + s, \\ \beta &= pq + pr + ps + qr + qs + rs, \\ \gamma &= pqr + pqs + prs + qrs, \\ \delta &= pqrs. \end{aligned}$$

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<sup>2</sup>Translator: If  $\alpha, \beta, \gamma, \delta$  are defined by  $(Z-p)(Z-q)(Z-r)(Z-s) = Z^4 - \alpha Z^3 + \beta Z^2 - \gamma Z + \delta$ , then  $(Z+p)(Z+q)(Z+r)(Z+s) = Z^4 + \alpha Z^3 + \beta Z^2 + \gamma Z + \delta$ .

Because the product of these two biquadratic equations must be equal to the equation of eight dimensions, it will be<sup>3</sup>

$$\begin{aligned} P &= \alpha^2 - 2\beta, \\ Q &= \beta^2 - 2\alpha\gamma + 2\delta, \\ R &= \gamma^2 - 2\beta\delta, \\ S &= \delta^2. \end{aligned}$$

And since  $\alpha = p + q + r + s$ , it will be

$$\alpha = 2t$$

and hence  $\alpha^2 = 4tt$ , whence it will be

$$\alpha^2 - 2\beta = 4tt - 2\beta = P = 4tt - A,$$

therefore

$$\beta = \frac{A}{2}.$$

The second equation  $Q = \beta^2 - 2\alpha\gamma + 2\delta$  will give

$$6t^4 - 3Att + B = \frac{A^2}{4} - 4\gamma t + 2\delta$$

or

$$\delta = 3t^4 - \frac{3}{2}Att + 2\gamma t - \frac{A^2}{8} + \frac{B}{2}.$$

Indeed the third equation  $R = \gamma^2 - 2\beta\delta$  will yield

$$4t^6 - 3At^4 + 2Btt - C = \gamma^2 - A\delta$$

or

$$A\delta = -4t^6 + 3At^4 - 2Btt + C + \gamma^2;$$

this with the previous gives

$$\left. \begin{array}{l} 4t^6 \\ -\frac{3}{2}A^2tt \\ +2Btt \end{array} \right\} + \left. \begin{array}{l} +2A\gamma t \\ -\frac{A^3}{8} \\ +\frac{AB}{2} \\ -C \end{array} \right\} = \gamma^2.$$

By taking the square root we will obtain<sup>4</sup>

$$\gamma = At \pm \sqrt{\left(4t^6 + \left(2B - \frac{1}{2}A^2\right)tt - \frac{1}{8}A^3 + \frac{1}{2}AB - C\right)}$$

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<sup>3</sup>Translator: Multiplying together the two biquadratics and comparing the coefficients of powers of  $Z$  in this product and in  $Z^8 - PZ^6 + QZ^4 - RZ^2 + S = 0$ .

<sup>4</sup>Translator: Applying the quadratic formula to the above equation in terms of  $\gamma$ .

and thence<sup>5</sup>

$$\delta = 3t^4 + \frac{1}{2}Att - \frac{1}{8}A^2 + \frac{1}{2}B \pm 2t\sqrt{\left(4t^6 + (2B - \frac{1}{2}A^2)tt - \frac{1}{8}A^3 + \frac{1}{2}AB - C\right)},$$

whose square is

$$\begin{array}{cccccc} 25t^8 & +3At^6 & -\frac{5}{2}A^2t^4 & -\frac{5}{8}A^3tt & +\frac{1}{64}A^4 & \\ & & +11B & +\frac{5}{2}AB & -\frac{1}{8}A^2B & \\ & & & -4C & +\frac{1}{4}B^2 & \end{array}$$

$$\pm(12t^5 + 2At^3 - \frac{1}{2}A^2t + 2Bt)\sqrt{\left(4t^6 + (2B - \frac{1}{2}A^2)tt - \frac{1}{8}A^3 + \frac{1}{2}AB - C\right)},$$

should be equal to  $S$ , that is, to the expression  $t^8 - At^6 + Bt^4 - Ctt + D$ , from which this equation follows

$$\begin{array}{cccccc} 24t^8 & +4At^6 & -\frac{5}{2}A^2t^4 & -\frac{5}{8}A^3tt & +\frac{1}{64}A^4 & \\ & & +10B & +\frac{5}{2}AB & -\frac{1}{8}A^2B & \\ & & & -3C & +\frac{1}{4}B^2 & \\ & & & & -D & \end{array}$$

$$+(12t^5 + 2At^3 - \frac{1}{2}A^2t + 2Bt)\sqrt{\left(4t^6 + (2B - \frac{1}{2}A^2)tt - \frac{1}{8}A^3 + \frac{1}{2}AB - C\right)} = 0,$$

which rationalized gives<sup>6</sup>

$$\begin{array}{cccccc} +3A^4t^8 & +\frac{5}{4}A^5t^6 & +\frac{3}{16}A^6t^4 & +\frac{3}{256}A^7tt & +\frac{1}{4096}A^8 & \\ -12A^2B & -8A^3B & -\frac{27}{16}A^4B & -\frac{9}{64}A^5B & -\frac{1}{288}A^6B & \\ +24AC & +7A^2C & +\frac{7}{16}A^3C & +\frac{3}{32}A^4C & +\frac{1}{128}A^4B^2 & \\ -48D & +12AB^2 & +\frac{9}{2}A^2B^2 & +\frac{3}{16}A^3B^2 & -\frac{1}{32}A^4D & \\ & -8AD & +5A^2D & +\frac{5}{4}A^3D & -\frac{1}{16}A^2B^3 & \\ & -12BC & -7ABC & -\frac{3}{4}A^2BC & +\frac{1}{4}A^2BD & \\ & & -3B^3 & -\frac{3}{4}AB^3 & +\frac{1}{16}B^4 & \\ & & -20BD & -5ABD & -\frac{1}{2}B^2D & \\ & & +9C^2 & +\frac{5}{2}B^2C & +D^2 & \\ & & & +6CD & & \end{array} = 0.$$

For brevity let  $E = \frac{1}{4}A^2 - B$  and  $u = 2t$ ; it will be

$$\begin{array}{cccccc} +3A^2Eu^8 & +2A^3Eu^6 & +9A^2E^2u^4 & +12AE^3uu & +4E^4 & \\ +6AC & +4A^2C & +28ACE & +80ADE & -32DE^2 & \\ -12D & +12AE^2 & +80DE & +96CD & +64D^2 & \\ & -8AD & +36C^2 & +40CE^2 & & \\ & +12CE & +12E^3 & & & \end{array} = 0.$$

<sup>5</sup>Translator: Using the equation  $\delta = 3t^4 - \frac{3}{2}Att + 2\gamma t - \frac{A^2}{8} + \frac{B}{2}$ .

<sup>6</sup>Translator: Move the surd term to the right hand side of the equation, then square both sides, then compare coefficients of  $t$ . I've checked this but I don't see a clean way to do it.

This equation then has four positive roots and four negative roots, equal except for sign, and thus the equation can be solved as a biquadratic equation. Moreover, the quantities  $A, B, C, D$  and  $E$  are known quantities determined from the given  $a, b, c, d$ , since, of course,

$$\begin{aligned} A &= a + b + c + d, \\ B &= ab + ac + ad + bc + bd + cd, \\ C &= abc + abd + acd + bcd, \\ D &= abcd \end{aligned}$$

and also

$$E = \frac{1}{4}A^2 - B.$$

For each of the values found for  $u$ , the sought quantities will be

$$\begin{aligned} v &= \frac{u - \sqrt{(uu - 4a)}}{2}, \\ x &= \frac{u - \sqrt{(uu - 4b)}}{2}, \\ y &= \frac{u - \sqrt{(uu - 4c)}}{2}, \\ z &= \frac{u - \sqrt{(uu - 4d)}}{2}. \end{aligned}$$

### Another solution

The problem can also be solved in this way. From the original equations we have

$$\begin{aligned} a - b &= (v - x)(y + z), & b - c &= (x - y)(v + z), \\ a - c &= (v - y)(x + z), & b - d &= (x - z)(v + y), \\ a - d &= (v - z)(x + y), & c - d &= (y - z)(v + x). \end{aligned}$$

From the first and last equations we obtain

$$v - x = \frac{a - b}{y + z}, \quad v + x = \frac{c - d}{y - z}.$$

Let

$$\frac{a + b - c - d}{2} = h;$$

it will be<sup>7</sup>

$$h = vx - yz$$

and by making

$$\frac{a + b + c + d}{2} = k$$

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<sup>7</sup>Translator: From the definitions of  $a, b, c, d$  given at the beginning of the paper.

it will be<sup>8</sup>

$$k = vx + vy + vz + xy + xv + yz,$$

and therefore

$$k - h = 2yz + (v + x)(y + z)$$

or

$$k - h = 2yz + \frac{(c - d)(y + z)}{y - z} = c + d,$$

therefore

$$2yz = \frac{2dy - 2cz}{y - z} \quad \text{or} \quad yyz - yzz = dy - cz.$$

By setting  $yz = t$  this equation turns into

$$(d - t)y - (c - t)z = 0.$$

Now put

$$dy - cz = u$$

and it will be

$$y = \frac{(c - t)u}{(c - d)t} \quad \text{and} \quad z = \frac{(d - t)u}{(c - d)t}.$$

These values substituted into  $t = yz$  yield

$$t = \frac{(c - t)(d - t)uu}{(c - d)^2 tt},$$

from which follows

$$u = \frac{(c - d)t\sqrt{t}}{\sqrt{(cd - (c + d)t + tt)}},$$

and when this is substituted into the values found above for  $y$  and  $z$  we shall have

$$\text{I.} \quad y = \frac{(c - t)\sqrt{t}}{\sqrt{(cd - (c + d)t + tt)}}$$

and

$$\text{II.} \quad z = \frac{(d - t)\sqrt{t}}{\sqrt{(cd - (c + d)t + tt)}}.$$

Adding and subtracting,

$$y + z = \frac{(c + d - 2t)\sqrt{t}}{\sqrt{(cd - (c + d)t + tt)}} \quad \text{and} \quad y - z = \frac{(c - d)\sqrt{t}}{\sqrt{(cd - (c + d)t + tt)}}.$$

One then deduces that<sup>9</sup>

$$v + x = \frac{\sqrt{(cd - (c + d)t + tt)}}{\sqrt{t}} \quad \text{and} \quad v - x = \frac{(a - b)\sqrt{(cd - (c + d)t + tt)}}{(c + d - 2t)\sqrt{t}},$$

<sup>8</sup>Translator: From the definitions of  $a, b, c, d$ .

<sup>9</sup>Translator: Since  $v - x = \frac{a-b}{y+z}$  and  $v + x = \frac{c-d}{y-z}$ .

whence, again adding and subtracting and putting for the sake of brevity

$$b + c + d - a = m \quad \text{and} \quad a + c + d - b = n,$$

the following values are produced for  $v$  and  $x$ <sup>10</sup>

$$\text{III.} \quad v = \frac{(n - 2t)\sqrt{(cd - (c + d)t + tt)}}{2(c + d - 2t)\sqrt{t}}$$

and

$$\text{IV.} \quad x = \frac{(m - 2t)\sqrt{(cd - (c + d)t + tt)}}{2(c + d - 2t)\sqrt{t}}.$$

But since we have found above that  $h = vx - yz$ , substituting for  $v, x, y, z$  their values and expanding yields the following equation of four dimensions for determining the value of  $t$

$$4t(h + t)(c + d - 2t)^2 = (m - 2t)(n - 2t)(c - t)(d - t).$$

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<sup>10</sup>Translator: By adding and subtracting the equations for  $v + x$  and  $v - x$ .