On two numbers, the sum of which either increased or decreased by the square of one of them produces a square*

Leonhard Euler

Translated by Joseph P. Dexter

1. If two particular numbers $\frac{x}{z}$ and $\frac{y}{z}$ are specified, these two two-part formulas, $\frac{x+y}{z} \pm \frac{xx}{zz}$ and $\frac{x+y}{z} \pm \frac{yy}{zz}$, need to be made into squares. From this, after multiplying through by *zz*, these two formulas, $(x+y)z \pm xx$ and $(x+y)z \pm yy$, both ought to be equal to squares. However, the following numbers, which are clearly the smallest, satisfy these conditions:

$$x = 9028 = 4.37.61,\ \ y = 3124 = 4.11.71, \text{ and } z = \frac{5.37^2.61^2}{2.49.31}.$$

Then,

$$(x+y)z = 20.37^2.61^2 \text{ and } xx = 16.37^2.61^2,$$

the sum of which numbers is $6^2.37^2.61^2$ and the difference of which is $2^2.37^2.61^2$; then it is true that $yy = 16.11^2.71^2$, from which after dividing by 4 the sum and the difference of these numbers,

$$5.37^2.61^2 \text{ and } 4.11^2.71^2,$$

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must be shown to be squares. However, since $5 = 2^2 + 1^2$, $37^2 = 35^2 + 12^2$ and $61^2 = 60^2 + 11^2$ yield $5.37^2 = 82^2 + 11^2$, and hence $5.37^2.61^2 = 5041^2 + 242^2$; to this sum of squares is either added or subtracted twice the product of the roots of this square, yielding

$$2.242.5041 = 4.11^2.71^2 .$$

This is itself the number to be either added or subtracted.

Analysis leading to this solution.

2. Let a number $(x + y)z$ be specified to be exactly double the sum of two squares ($= A^2 + B^2$ and $= C^2 + D^2$). Then the question to be satisfied, whether $xx = 2AB$ and $yy = 2CD$, becomes clear. Let it also be specified that $(x + y)z = (aa + bb)(cc + dd)$, from which it is deduced that $A = ac + bd$ and $B = ad - bc$. Then $C = ad + bc$ and $D = ac = bd$, so we have

$$xx = 2(ac + bd)(ad - bc) \text{ and } yy = 2(ad + bc)(ac - bd) .$$

Now, so that these formulas do not become squares, it is specified that $x = (ac + bd) f$ and $y = (ad + bd) ff$, and from this the following equations are produced:

$$2(ad - bc) = (ac + bd) ff \text{ and } 2(ac - bd) = (ad + bc)gg .$$

It is deduced from the first equation that $\frac{a}{b} = \frac{2c + dff}{2d - cff}$, and from the second it is true that $\frac{a}{b} = \frac{2d + cgg}{2c - dgg}$. However, setting these values equal to each other yields the equation

$$cc(4 + ffgg) + 4cd(ff - gg) = dd(4 + ffgg),$$

from which, by extracting the root, is obtained either

$$\frac{d}{c} = \frac{2(ff-gg) \pm \sqrt{4(ff-gg)^2 + (4+ffgg)^2}}{4+ffgg} \text{ or}$$

$$\frac{d}{c} = \frac{2(ff-gg) \pm \sqrt{(4+f^4)(4+g^4)}}{4+ffgg}.$$

3. Therefore, the whole matter returns to this problem--to reduce this product $(4+f^4)(4+g^4)$ to a square. Because the product contains two quantities *f* and *g*, one may make one of the quantities any value he wishes. Therefore, let us suppose that $g=1$, yielding $\frac{d}{c} = \frac{2(ff-3) \pm \sqrt{5(4+f^4)}}{4+ff}$; then, by unraveling, $\frac{a}{b} = \frac{2d+c}{2c-d}$, and further, $(ac+bd)f$ and $y = ad+bc$. However, we then have $z = \frac{(aa+bb)(cc+dd)}{x+y}$.

4. Now, let it be that established that $\sqrt{5(4+f^4)} = 5v$, so that

$$\frac{d}{c} = \frac{2(ff-1) \pm 5v}{4+ff}.$$

Thus, $25vv = 20 + 5f^4$, so this formula, which conveniently is a square for the case $f=1$, in reality leads to an unsuitable solution. Therefore, to elicit other values for *f*, let us set $f = 1+t$, yielding $25vv = 25 + 20t + 30tt + 20t^3 + 5t^4$, the root of which is $5 + \alpha t + \beta tt$; then,

$$20 + 30t + 20tt + 5t^3 = 10\alpha + (10\beta + \alpha\alpha)t + 2\alpha\beta tt + \beta\beta t^3.$$

Now, so that the two prior terms cancel themselves out, one ought to make $\alpha = 2$, and, so that the second terms also cancel themselves out, one ought to obtain $\beta = \frac{13}{5}$; therefore, we have $5v = 5 + 2t + \frac{13}{5}tt$. Now, the third and fourth terms, which yield

$t = \frac{60}{11}$ after dividing through by *tt*, still remain. From the value determined, it is computed that $5v = \frac{11285}{121}$. Then, it is true that $f = \frac{71}{11}$, from which values it is computed that $\frac{d}{c} = \frac{2(71^2 - 11^2) \pm 11285}{4.11^2 + 71^2}$, from which, by the last accepted proof, it is derived that $\frac{d}{c} = \frac{21125}{5525} = \frac{846}{221} = \frac{5.169}{13.17}$. Therefore, let us select $d = 5.169$ and $c = 13.17$, yielding $\frac{a}{b} = \frac{147}{31}$. By supposing that $a = 147$ and $b = -31$, $x = 4.11.74$ and $y = 4.37.61$; from here, it is computed that $x + y = 4.13.2.7^2.31$ and $z = \frac{5.13.37^2.61^2}{2.49.31}$, because of the fact that $aa + bb = 10.37.61$and $cc + dd = 13^2.2.37.61$. Since these three numbers *x*, *y*, and *z* have a common factor of 13, the values of theses variables therefore become simpler by dividing through:

$$x = 4.11.74, \; y = 4.37.61, \text{ and } z = \frac{5.37^2.61^2}{2.49.31}.$$

The numbers desired are now

$$\frac{x}{z} = \frac{8.11.31.49.71}{5.37^2.61^2} \text{ and } \frac{y}{z} = \frac{8.31.49}{5.37.61},$$

which themselves are the numbers stated at the beginning. However, just as these numbers are deduced from the hypothesis $g = 1$, in a similar way solutions from any other value can be investigated by assuming a number for *g*. However, such solutions will soon grow to an infinite number. Moreover, it must be observed that by assuming that $g = 2$, the same solution is produced, since $g^4 + 4 = 4.5$. When there are four values, it follows from the calculation to do the series of operations a second time.