On the brachistochrone
In a resistant medium while a body is attracted to a centre of forces in one way or another
By the author
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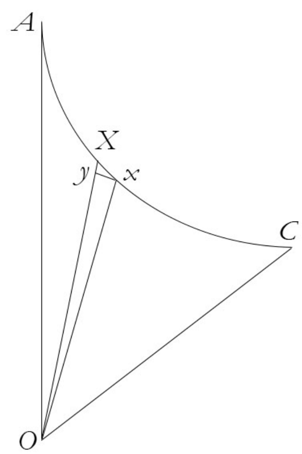

§. 1. Let O be the centre of forces, of which the attraction at a distance $= x$ is X, whichever function of that x; then, if the speed of the body will be $= v$, the friction force $= V$ is verily contrary to motion, whichever function of that v. Let the curve AXC already be the sought brachistochrone, over which a descending body traverses in the smallest time from A to C, since the descent in A begins from rest. Truly nothing hinders, that a certain speed is bestowed to it already in A. Let for the beginning of this curve A the distance $OA = a$ be posed and for the end C the distance $OC = c$ and the angle $AOC = b$. Yet verily for whichever its point X, let us pose its distance $OX = x$ and the angle $AOX = y$; and it is manifest that the curve is equally determined by the relation between x and y and by the equation between the orthogonal coordinates. Let moreover the arc $AX = s$ be posed and its element $Xx = ds$, and the drawn straight Ox and, drawn perpendicular from x to X Xy, Xy will be dx, and on account of the angle $XOx = dy$ it will hold that $xy = xdy$, whence the element is

$$Xx = ds = \sqrt{dx^2 + xxdy^2}$$

hence, if we pose $dy = pdx$, ds will be $-dx\sqrt{1+ppxx}$.

§. 2. Because now the body is disturbed in X in the direction XO, with force $=$ X, hence for the direction of the motion Xx the force appears $X \cdot \frac{Xy}{Xx} = -\frac{Xdx}{ds}$; the friction force however, having posed the speed of the body in $X = v$, is $= V$, whence the body will accelerate by the force $= -\frac{Xdx}{ds} - V$, which, drawn in the element of space ds, yields an increment of the square of the speed, whence thus $vdv = -Xdx - Vds$ will hold, and hence on account of $ds = -dx\sqrt{1+ppxx}$ becomes

$$vdv = dx\left(V\sqrt{1+ppxx} - X\right)$$

which equation expresses the relation between the speed v and the quantities particularly pertaining to the curve. Because the tiny amount of time through $Xx = ds$ is consequently $\frac{ds}{v} = -\frac{dx\sqrt{1+ppxx}}{v}$, between all curves, drawn from A to C, the one is sought, for which the value of this integral formula $\int \frac{dx\sqrt{1+ppxx}}{v}$ becomes minimum for all.

§. 3. It will help here before all to have observed, if the end C is accepted on that straight AO, that brachistochrone should coincide with that there straight, for the motion of which thus, on account of $y' = 0$ and for that reason also $p = 0$, this equation is born: $vdv = dx\,(V - X)$, which, because it can in general by no means be resolved, much less will it be possible to be postulated, that in general the determination of the motion for the brachistochrone AC is thoroughly obtained, but clearly it will have to be judged with us to be accomplished, if only we can pick out the differential equation between the three variables x, y, v, by which surely, with the connected formula: $vdv = dx\left(V\sqrt{1+ppxx} - X\right)$, it is understood that it is in itself possible that the speed v is eliminated and that for that reason the equation between both variables x and y can be obtained.

§. 4. Because consequently between all curves AG the one should be sought, for which the value of this integral formula $\int \frac{dx\sqrt{1+ppxx}}{v}$ is minimal, it will be needed to revert to the general isoperimetric problem, solved in a preceding dissertation. But, because the circumstances here are a bit varied, the decision will be to transfer the solution, invented there, within the form of a theorem here, which, if in this way, we will have:

The general isoperimetric Theorem.

§. 5. *If between all curves, which can be drawn from the point A to C, the one is sought, in which the value of the integral formula $\int Wdx$ is a maximum or a minimum, where W, besides both variables x and y and their differentials $\frac{dy}{dx}$ =*

p; $\frac{dp}{dx} = q$; $\frac{dq}{dx} = r$; etc. furthermore involves the variable v, such that in this way holds:

$$dW = Ldv + Mdx + Ndy + Pdp + etc.$$

then the quantity v is verily given by the differential equation in this way, such that, having posed $dv = \mathfrak{W}$, it holds:

$$d\mathfrak{W} = \mathfrak{L}dv + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + etc.$$

having posed this, $\Lambda = e^{\int \mathfrak{L}dx}$ is sought, and hence further the quantity $\Pi = \int L\Lambda dx$, which integral is in this way taken on, such that it vanishes for the end C, or, because it returns to itself, that end C is established there, where $\Pi = 0$ holds, having discovered which, it's assumed that

$$N' = N - \frac{\Pi\mathfrak{N}}{\Lambda}; P' = P - \frac{\Pi\mathfrak{P}}{\Lambda}; Q' = Q - \frac{\Pi\mathfrak{Q}}{\Lambda}; etc.$$

from this for the nature of the sought curve this equation is deduced:

$$0 = N' - \frac{dP'}{dx} + \frac{ddQ'}{dx^2} - \frac{d^3R'}{dx^3} + etc.$$

where the element dx is assumed constant.

§. 6. For our case then W is $\frac{\sqrt{1+ppxx}}{v}$ and $\mathfrak{W} = \frac{V\sqrt{1+ppxx}-X}{v}$, which formulae only involve three variables, of course v, x and p; and since the letters M and $\mathfrak{M}$ don't ingress in the final equation, it is also not necessary to evolve them too. Hence from the prior formula it will hold that $L = -\frac{\sqrt{1+ppxx}}{vv}, N = 0, P = \frac{pxx}{v\sqrt{1+ppxx}}$. From the other formula it verily holds that:

$$\mathfrak{L} = -\frac{V\sqrt{1+ppxx}+X}{vv} + \frac{V'\sqrt{1+ppxx}}{v}$$

having of course posed $dV = V'dv$; then verily $\mathfrak{N} = 0$ and $\mathfrak{P} = \frac{Vpxx}{v\sqrt{1+ppxx}}$ will hold, having invented which, our final equation will be $\frac{dP'}{dx} = 0$, and for that reason $P' = C$, that is $C = P - \frac{\Pi\mathfrak{P}}{\Lambda}$. Thence it is exposed, that the quantity Π vanishes, where $P = C$. Therefore the end C of the brachistochrone should be established here, where $C = \frac{pxx}{v\sqrt{1+ppxx}}$.

§. 7. Because now $\Lambda = e^{\int \mathfrak{L}dx}$ it will hold that $\frac{d\Lambda}{\Lambda} = \mathfrak{L}dx$, thus $d\Lambda = \Lambda\mathfrak{L}dx$. Hence we will moreover further have $\Pi = \int L\Lambda dx$. Therefore, because from the final equation

$$\Pi = \frac{\Lambda P}{\mathfrak{P}} - \frac{C\Lambda}{\mathfrak{P}}, \text{ that is } \Pi = \frac{\Lambda}{V} - \frac{C\Lambda v\sqrt{1+ppxx}}{Vpxx}$$

is made, let us for the grace of brevity state $\sqrt{1+ppxx} = \omega$ and $\frac{\sqrt{1+ppxx}}{pxx} = t$, such that in this way $t = \frac{\omega}{x\sqrt{\omega\omega-1}}$. Let us now differentiate the discovered equation, and because $d\Pi = L\Lambda dx$ and $d\Lambda = \Lambda\mathfrak{L}dx$, having done this substitution, the whole equation will be able to be divided by Λ, and therefore it wasn't necessary to determinate its integral value. By now substituting the found values for L and $\mathfrak{L}$ thus, we arrive at this equation:

$$0 = \frac{\omega dx}{vv} + \frac{\mathfrak{L}dx}{V} - \frac{dV}{VV} - \frac{C\mathfrak{L}tvdx}{V} - Cx\frac{(vdt+tdv)}{V} + \frac{CtvdV}{VV}$$

where $\mathfrak{L} = -\frac{V\omega+X}{vv} + \frac{V'\omega}{v}$.

§. 8. Now, because $vdv = dx\,(V\omega - X)$, it will hold that $dx = \frac{vdv}{(V\omega-X)}$, which value we substitute in our equation instead of dx, let us everywhere write vdv for dx, verily multiply the remaining terms by $V\omega - X$ and write dV instead of $V'dv$, having done which, the equation will assume the following form:

$$0 = \frac{\omega dv}{v} + \frac{\omega dV}{V} - \frac{Cv\omega tdV}{V} + \frac{V\omega - X}{VV}\left(CvtdV - CVvdt - dV - \frac{Vdv}{v}\right)$$

§. 9. Because this equation is not insufficiently complex, let us first evolve only those terms, in which the constant C does not occur, and they will be discovered

$$\frac{\omega dv}{v} + \frac{\omega dV}{V} - \frac{\omega dV}{V} + \frac{XdV}{VV} - \frac{\omega dv}{v} + \frac{Xdv}{Vv}, \text{ or } \frac{X}{V}\left(\frac{dV}{V} + \frac{dv}{v}\right)$$

And the terms, containing the constant C, are verily

$$-\frac{Cv\omega tdV}{V} + Cv\omega dt + \frac{Cv\omega tdV}{V} + \frac{CXvdt}{V} - \frac{CXtvdV}{VV}$$

or, having deleted the terms that destroy themselves

$$-\frac{CvtXdV}{VV} + \frac{CvXdt}{V} - v\omega dt$$

wherefore the whole equation will have itself in this way:

$$\frac{X}{V}\left(\frac{dV}{V} + \frac{dv}{v}\right) - Cv\omega dt + \frac{CvXdt}{V} - \frac{CvtXdV}{VV} = 0$$

§. 10. But if this equation is already divided by CvX, it will appear in this form:

$$\frac{1}{CVv}d\cdot lVv-\frac{\omega dt}{X}+\frac{dt}{V}-\frac{tdV}{VV}=0$$

of which equation the first portion as much as the following admit integration. Having consequently assumed the integral, it will be $-\frac{1}{CVv}+\frac{t}{V}-\int\frac{\omega dt}{X}=\Delta$, where in the summation sign only both variables p and x are involved, because $\omega=\sqrt{1+ppxx}$ and $t=\frac{\sqrt{1+ppxx}}{pxx}$, and besides X is a function of that x. On this account, the third variable v, with its given function V, must be assessed to be determined by this equation; but if these values were substituted in the equation $vdv\left(V\sqrt{1+ppxx}-X\right)$, an equation, involving only both variables x and p, or x and y, by which thus the nature of the sought brachistochrone curve is expressed; and nothing further can be postulated for the solution of this problem. Moreover, this invented curve should establish the end of descent C there, where, like we observed, $P=C$, or where $C=\frac{pxx}{v\sqrt{1+ppxx}}$.

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