On true Brachistochrones
Or
Lines of the fastest descent
In a resistant medium
By the author
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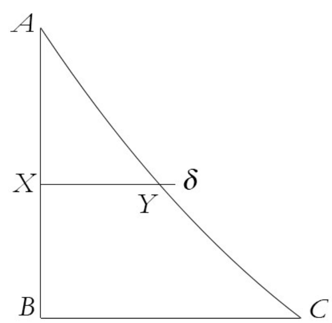
§. 1. A principle of this kind, that I taught about these curves in book II[1] of my Mechanicae, rests upon, what can't be allowed in a resistant medium. Thereafter, I tried to obtain the same argument from the first principles of maxima and minima in my isoperimetric treatment; to such a greater degree are they, which I conducted there on the brachistochrone in a resistant medium, truly involved in the excessively generalised analytic formulae, such that thence barely anyone is able to pick out a true nature of those curves. On that account I decided to expand this same argument in a bigger study here and to derive it clearly and perspicuously from the first principles.



§. 2. Here, let us finally consider whichever AYC, related to the vertical axis AB, over which a body, beginning to slip from A, descends in a resistant medium according to whichever multiplied ratio of the speed. The abscissa is already called $AX = x$ for whichever point Y of the curve, the attached {ordinal?} $XY = y$ and the arc of the curve $AY = s$. Let moreover the speed in Y be v, of which the quantity is thus expressed by such equation: $vdv = gdx - hv^{n+1}ds$, which is in this way compared, like it is able to be integrated in general in the cases $n = -1$ and $n = +1$. Meanwhile yet, having thence defined the value of that v, the element of time will be $\frac{ds}{v}$, of which thus the integral should obtain the property of the minimum, since the curve AYC will be a brachistochrone.

[1]Consult the note on page 314 [which is the footnote in the previous text, E759]

§. 3. If a motion is produced in vacuum, in which case h were 0 and $vv = 2gx$, because the speed in Y depends on only its altitude, it is evident, that the whole curve AYC turns out to be a brachistochrone, and also that single parts of this AY should be traversed in minimal time; in a resistant medium the matter yet very much has itself otherwise, where the speed no longer depends on the locus of the point Y, but simultaneously involves the whole preceding arc AY; whence it can happen that the time through the whole arc AYC becomes minimal, and also if the time through the arc AY were not minimal, it could naturally happen that a considerably larger speed, which in such size produces a shorter time through the following arc YC, were generated in Y by the descent through the arc AY; on this account our problem for the resistant medium should be proposed in this way:

Between all curves, which are allowed to be drawn from the point A all the way through C, the one is sought, over which a body, beginning a descent from A, arrives at the end C the quickest.

§. 4. With this, moreover, this investigation more widely extends the problem a lot more general, which, I will contemplate, is not restricted to only brachistochrones, because the solution further not only not becomes harder, but also more extended to be reduced to analytic formulae; on this account the following problem convenes to be procured before all:

General Problem.

Between all curves, that can be drawn from a given point A to a given point C; investigate that one, in which this integral formula: $\int V dx$ obtains a maximum or minimum value, where the letter V, besides the coordinates of x and y and the differentials of whichever ordinal, also involves the quantity v, which is determined by whichever differential equation.

Solution.

§. 5. Because the function V is also assumed to imply differentials of whichever ordinal, let us pose, having accustomed to the custom, $dy = pdx; dp = qdx; dq = rdx; etc.$ in this way, such that V besides the quantities $x, y, p, q, r, etc.$ already also involves that quantity v; whence its differential in this way will have the form:

$$dV = Ldv + Mdx + Ndy + Pdp + Qdq + etc.$$

Moreover, the quantity v is expressed by this differential equation: $dv = \mathfrak{B}dx$; where $\mathfrak{B}$ is whichever function of that v, with the quantities, pertaining to the curve, $x, y, p, q, r, etc.$ Wherefore its differential will have such form:

$$d\mathfrak{B} = \mathfrak{L}dv + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + etc.$$

§. 6. Let us use the desired method from the calculus of variation, with which the maximum or minimum value of the integral formula $\int Vdx$ can be procured, which we in the end bestow, having applied $XY = y$ as the minimal increment $Y\delta$, which we indicate by δy, such that in this way δy is the variation of that y; it is verily needed that no variation is granted to the other coordinate x, such that in this way $\delta x = 0$. To what extent thus the remaining quantities depend on the varied y, so far do they too receive certain variations, which is necessary to obtain before all.

§. 7. Let us pose with grace of brevity that the variation $\delta y = \omega$, and because $p = \frac{dy}{dx}$, δp will be $\frac{\delta dy}{dx}$. It is moreover demonstrated that $\delta dy = d\delta y = d\omega$, whence $\delta p = \frac{d\omega}{dx}$. In a similar way, because $q = \frac{dp}{dx}$, δq will be $\frac{\delta dp}{dx} = \frac{d\delta p}{dx} = \frac{dd\omega}{dx^2}$. It is equally manifest that δr will be $\frac{d^3\omega}{dx^3}$; *etc.* Here of course everywhere the letter δ prefixed to which quantities denotes its variation born from the variation of that y.

§. 8. Having posed these, let us investigate the variation of that proposed integral formula $\int Vdx$, which will thus be $= \delta \int Vdx$. From the calculus of variation however it consists that $\delta \int Vdx = \int \delta Vdx$ will hold, and because it is allowed to take the variations by the same rule, by which differentials are indicated, it will hold:

$$\delta V = L\delta v + M\delta x + N\delta y + P\delta p + Q\delta q + etc.$$

where the term $M\delta x$ vanishes; and if instead of $\delta y, \delta p, \delta q, \delta r, etc.$ the values of the just invented are written, we will have:

$$\delta V = L\delta v + N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + etc.$$

Hence the variation of the proposed integral formula will thus be:

$$\delta \int Vdx = \int Ldx \left(\delta v + N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + etc. \right)$$

or

$$\delta \int Vdx = \int L\delta vdx + \int N\omega dx + \int Pd\omega + \int \frac{Qdd\omega}{dx} + etc.$$

The whole matter reverts thus to this, such that the value of the first portion $\int L\delta vdx$ is obtained with all care.

§. 9. From §. 5. $v = \int \mathfrak{B}dx$ follows, hence it will hold that $\delta v = \delta \int \mathfrak{B}dx = \int \delta\mathfrak{B}dx$; for this reason, because $d\mathfrak{B} = \mathfrak{L}dv + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + etc.$ will in a similar way be:

$$\delta\mathfrak{B} = \mathfrak{L}\delta v + \mathfrak{M}\delta x + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + etc.$$

this is:

$$\delta\mathfrak{B} = \mathfrak{L}\delta v + \mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + etc.$$

consequently we will have:

$$\delta v = \int dx \left(\mathfrak{L}\delta v + \mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + etc. \right)$$

from which equation the value of that δv is now possible to be derived.

§. 10. To this end, where the calculus is more raised, let us pose $\delta v = u$, and with the <u>assumed</u> differentials it will hold that:

$$du = \mathfrak{L}udx + \mathfrak{N}\omega dx + \mathfrak{P}d\omega + \frac{\mathfrak{Q}dd\omega}{dx} + etc.$$

which equation is represented in this way:

$$du - \mathfrak{L}udx = \mathfrak{N}\omega dx + \mathfrak{P}d\omega + \frac{\mathfrak{Q}dd\omega}{dx} + etc.$$

which, such that it is rendered integrable, is multiplied by $e^{-\int \mathfrak{L}dx}$, instead of which for the grace of brevity we write $\frac{1}{\Lambda}$, such that in this way $\Lambda = e^{\int \mathfrak{L}dx}$ holds, and for that reason $\frac{d\Lambda}{\Lambda} = \mathfrak{L}dx$. Then consequently the integral equation will be:

$$\frac{u}{\Lambda} = \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc. \right)$$

and in this way we obtained the sought quantity δv, which will be:

$$\delta v = \Lambda \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc. \right)$$

§. 11. Now we will therefore have for the first term of the formula, by which the variation $\delta \int V dx$ is expressed:

$$\int L\Lambda dx \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc. \right)$$

where after the integration sign $\int$ further another is involved, whence it will need to be attacked in it, such that all are revoked to simple integration.

§. 12. Let us to this end state $L\Lambda dx = d\Pi$, and it will be

$$\int d\Pi \int \frac{dx}{\Lambda}\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + etc.\right) = \Pi \int \frac{dx}{\Lambda}(\mathfrak{N}\omega + etc.) - \int \frac{\Pi dx}{\Lambda}(\mathfrak{N}\omega + etc.)$$

Because Π is already $\int L\Lambda dx$, the constant, to be attached to this integral, is abandoned by our arbitrarity; whence this constant is determined in this way, such that for the whole curve AYC, where x becomes $AB = a$, this quantity Π vanishes, surely, having agreed to this, the prior part $\Pi \int \frac{dx}{\Lambda}(\mathfrak{N}\omega + etc.)$ for the whole curve, to which it is necessary to devise a calculus, spontaneously vanishes, since that integral formula, otherwise joined to nothing, cannot be reduced. Therefore, having taken the integral $\int L\Lambda dx = \Pi$ in this way, such that, having posed $x = a$, it vanishes, it will hold:

$$\int L dx \delta v = -\int \frac{\Pi dx}{\Lambda}\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc.\right)$$

§. 13. This sought variation $\delta \int V dx$, having already invented the value, will be expressed in the following way:

$$-\int \frac{\Pi dx}{\Lambda}\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc.\right) + \int dx\left(N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + etc.\right)$$

which expression, to be posed for the grace of brevity:

$$N - \frac{\Pi\mathfrak{N}}{\Lambda} = N'; P - \frac{\Pi\mathfrak{P}}{\Lambda} = P'; Q - \frac{\Pi\mathfrak{Q}}{\Lambda} = Q'; etc.$$

is reduced to this plentily simple form:

$$\delta \int V dx = \int dx\left(N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + etc.\right)$$

of which thus this value, through the whole curve AYC, is extended until $x = a$; it should be equal to zero.

§. 14. Let be denoted that $\int P'd\omega = P'\omega - \int \omega dP'$ holds, by which this formula is further reduced; thereafter $\int Q'dd\omega = Q'd\omega - \int d\omega dQ'$. Yet it verily holds that $\int d\omega dQ' = \omega dQ' - \int \omega ddQ'$, and for that reason $\int Q'dd\omega = Q'd\omega - \omega dQ' + \int \omega ddQ'$. In the same way it will hold that

$$\int R'd^3\omega = R'dd\omega - d\omega dR' + \omega ddR' - \int \omega d^3 R'$$

and so on in this way; where, because in the final end C no variation ω is applied, it is allowed to onwards neglect absolute boundary terms, and for that reason we will have

$$\delta\int Vdx = \int \omega dx \left(N' - \frac{dP'}{dx} + \frac{ddQ'}{dx^2} - \frac{d^3R'}{dx^3} + etc\right)$$

of which thus the value through the whole curve, extended from A to C, should be equal to zero, whenever the variations ω are accepted.

§. 15. It is however evident, that this cannot happen otherwise, if it won't hold that $N' = N - \frac{\Pi\mathfrak{N}}{\Lambda}; P' = P - \frac{\Pi\mathfrak{P}}{\Lambda}; etc.$ Then it will verily hold that $\Lambda = e^{\int \mathfrak{L}dx}$ and $\Pi = \int L\Lambda dx$, which integral should in this way be taken, such that it vanishes, having posed $x = a$. Besides, it is necessary that all proceeding constants are verily defined by integration in this way, such that it is satisfied by all circumstances, that is, such that, having assumed that $x = 0$, y, too, becomes 0; thereafter, having assumed that $x = a$, y becomes $BC = b$. Moreover, a certain given value for the quantity v, for the case $x = 0$, should be granted.

An Application
For Brachistochrones in a Resistant Medium

§. 16. Because the time of descent through the arc AY is $\int \frac{ds}{v}$, on account of $ds = dx\sqrt{1+pp}$, the integral formula, extended from the end A, where $x = 0$, until the end C, where $x = a$ and $y = b$, and to be reduced to a minimum, will be $\int \frac{dx\sqrt{1+pp}}{v}$ and for that reason $V = \frac{\sqrt{1+pp}}{v}$, which formula, because it contains only two variables v and p, will be $L = -\frac{\sqrt{1+pp}}{v}, M = 0, N = 0$, but $P = \frac{p}{v\sqrt{1+pp}}$. Next, because $dv = \frac{gdx - hv^{n+1}dx\sqrt{1+pp}}{v}$, $\mathfrak{V}$ will be $\frac{g}{v} - hv^n\sqrt{1+pp}$; whence further $\mathfrak{L} = -\frac{g}{vv} - nhv^{n-1}\sqrt{1+pp}; \mathfrak{M} = 0; \mathfrak{N} = 0$; but $\mathfrak{P} = -\frac{hv^n p}{\sqrt{1+pp}}$. From these values $\frac{d\Lambda}{\Lambda}$ will already first be $\mathfrak{L}dx$; then verily $\Pi = \int L\Lambda dx$

§. 17. Having invented this, firstly N' will be 0; $P' = P - \frac{\Pi\mathfrak{P}}{\Lambda}$; wherefore the equation for the sought curve will be $N' - \frac{dP'}{dx} = 0$, or $\frac{dP'}{dx} = 0$, whence immediately by integrating $P' = C$ is obtained; having thus substituted the values for P and $\mathfrak{P}$, this equation for the curve appears:

$$\frac{p}{v\sqrt{1+pp}} + \frac{h\Pi v^n p}{\Lambda\sqrt{1+pp}} = C$$

From this equation we then elicit the value of Π, for which we surely gave an integral formula, and it will hold that:

$$\Pi = \frac{C\Lambda v\sqrt{1+pp} - \Lambda p}{hpv^{n+1}}$$

Let us here pose for the grace of brevity $\frac{C}{v^n} \cdot \frac{\sqrt{1+pp}}{p} - \frac{1}{v^{n+1}} = \Theta$, such that $\Pi = \frac{\Lambda\Theta}{h}$, and on account of $d\Lambda = \Lambda\mathfrak{L}dx$, it will hold that:

$$d\Pi = L\Lambda dx = \frac{\Theta\Lambda\mathfrak{L}dx}{h} + \frac{\Lambda d\Theta}{h}$$

Which equation, divided by Λ, will be $hLdx = \Theta\mathfrak{L}dx + d\Theta$. It verily holds that:

$$d\Theta = -\frac{nCdv}{v^{n+1}} \cdot \frac{\sqrt{1+pp}}{p} + \frac{C}{v^n} d \cdot \frac{\sqrt{1+pp}}{p} + \frac{(n+1)\,dv}{v^{n+2}}$$

Whence our equation will be:

$$-\frac{hdx\sqrt{1+pp}}{vv} = \frac{C\mathfrak{L}dx}{v^n} \cdot \frac{\sqrt{1+pp}}{p} - \frac{\mathfrak{L}dx}{v^{n+1}} - \frac{nCdv}{v^{n+1}} \cdot \frac{\sqrt{1+pp}}{p} + \frac{(n+1)\,dv}{v^{n+2}} + \frac{C}{v^n} \cdot d \cdot \frac{\sqrt{1+pp}}{p}$$

where $\mathfrak{L} = -\frac{g}{vv} - nhv^{n-1}dx\sqrt{1+pp}$ enters.

§. 18. This equation, already freed from the integral formula, moreover contains three variables, of course p and v with the differential dx, and from this the segment dx can be easily removed. Because $vdv = gdx - hv^{n+1}dx\sqrt{1+pp}$, dx will be $\frac{vdv}{g - hv^{n+1}\sqrt{1+pp}}$, which value is substituted, the equation, only containing two variables, is obtained. To this end, let us resolve in our equation all terms containing the segment dx to the same side, and it will hold that:

$$\frac{\mathfrak{L}dx}{v^{n+1}} - \frac{dx\sqrt{1+pp}}{vv}\left(h + \frac{C\mathfrak{L}}{pv^{n-2}}\right) = \frac{(n+1)\,dv}{v^{n+2}} - \frac{nCdv}{v^{n+1}} \cdot \frac{\sqrt{1+pp}}{p} + \frac{C}{v^n} \cdot d \cdot \frac{\sqrt{1+pp}}{p}$$

Because, if we want to substitute their values instead of dx and $\mathfrak{L}$, a very complicated equation arises, as appointing this will be superfluous. It is meanwhile yet evident that the future differential equation between p and v is first degree, whence we can rightly postulate its, so to speak, allowed resolution in such a laborious matter.

§. **19.** Because then consequently the quantity p is given by v in this equation, and on account of integration, a new constant quantity is engaged in, all remaining things, which pertain to the solution, will be allowed to easily be procured. Because firstly $\sqrt{1+pp}$ is a certain function of that v, the quantity x will also be allowed to be defined by v with help of the equation $dx = \frac{vdv}{q-hv^{n+1}\sqrt{1+pp}}$, whence again a new constant is introduced, it is allowed to define which in this way, such that, having assumed that $v = 0$, x becomes 0. Thereafter verily also $\int \mathfrak{L}dx$ will be determined by only v, and hence further that value of the letter Π from the equation $\Pi = \frac{C\Lambda v\sqrt{1+pp}-\Lambda p}{hpv^{n+1}}$; where the constant C should be determined in this way, such that, having posed $x = a$, that value vanishes, because thus, if we assume that in the case that $x = a$, v becomes C, it happens in this way; and just like that, having defined all constants duly, that construction of the curve toils moreover without trouble. Because x and p are already given by v, the attached y will also be possible to be ascribed to v on account of $y = \int pdx$, and in these determinations we should acquiesce in the so sublime investigation, to what extent of course a general solution, that extends to all values of the exponent n, is desired

Supplement
in which the nature of the brachistochrones in a resistant medium is determined more accurately.

§. **20.** Although the last differential equation between both variables p and v, to which our method of Maxima and Minima leads, will in this way be regarded complex, such that barely anything about knowing the nature of those curves is thenceforth seen to be able to be concluded: the following plenily convenient equation appears, having built the calculus duly:

$$0 = \frac{(n+2)\,dv}{vv} - \frac{(n+1)\,Cdv\sqrt{1+pp}}{pv} + C\left(1 - \frac{h}{g}v^{n+1}\sqrt{1+pp}\right)d\cdot\frac{\sqrt{1+pp}}{p}$$

which only consists of four terms, and by no means difficultly can it be reduced to a simpler form.

§. **21.** Let us truly first state $C = \frac{1}{c}$ and $\frac{\sqrt{1+pp}}{p} = t$, whence it holds that $p = \frac{1}{\sqrt{tt-1}}$ and $\sqrt{1+pp} = \frac{t}{\sqrt{tt-1}}$, having substituted which values, this equation appears:

$$\frac{(n+2)\,cdv}{vv} - \frac{(n+1)\,tdv}{v} + dt - \frac{h}{g}\cdot\frac{v^{n+1}tdt}{\sqrt{tt-1}} = 0$$

Where it is immediately exposed that both middle terms $dt - \frac{(n+1)tdv}{v}$ are rendered integrable, if they are divided by v^{n+1}, because the integral surely advances $= \frac{t}{v^{n+1}}$. Further on, however, the first and last term spontaneously admit integration, such that the complete integral of this equation becomes in this way:

$$\frac{t}{v^{n+1}} - \frac{c}{v^{n+2}} - \frac{h}{g}\sqrt{tt-1} = \Delta$$

which equation, having restored the values $t = \frac{\sqrt{1+pp}}{p}$ and $\sqrt{tt-1} = \frac{1}{p}$, multiplying by v^{n+1}, assumes this form:

$$\frac{\sqrt{1+pp}}{p} - \frac{c}{v} - \frac{h}{g} \cdot \frac{v^{n+1}}{p} = \Delta v^{n+1}$$

whence thus the value of that p is defined by v with only extraction of the square root.

§. 22. It will moreover before all help here to have noted that the constant Δ is defined from the locus of the last point C, where the descent is terminated. Because truly in this end $\Pi = 0$ should hold, and the method of Maxima and Minima immediately had satisfied this equation: $P - \frac{\Pi\mathfrak{P}}{\Lambda} = C$, it is evident that the quantity Π can not vanish, if not in that place, where $P = C$ holds. Moreover $P = \frac{p}{v\sqrt{1+pp}}$ held, and because we now posed $C = \frac{1}{c}$, this will happen, where c is $\frac{v\sqrt{1+pp}}{p}$. In this case however our invented equation will provide the value $\Delta = -\frac{h}{gp}$, where p expresses the tangent of the angle, with which the curve declines from the present vertical; on which account, if we want, that this incline in the point C is equal to a given angle α, of which the tangent is θ, Δ will be $-\frac{h}{g\theta}$, having thus substituted which value, our equation will be thoroughly determined, and becomes

$$\frac{\sqrt{1+pp}}{p} - \frac{c}{v} + \frac{h}{g}v^{n+1}\left(\frac{1}{\theta} - \frac{1}{p}\right) = 0$$

or

$$\sqrt{1+pp} - \frac{cp}{v} + \frac{h}{g}v^{n+1}\left(\frac{p}{\theta} - 1\right) = 0$$

This determination, however, of the edge point C by a given incline of the curve to the present vertical is regarded much more accommodated to a matter of nature, than if we want to define that point by the abscissa $x = a$ and $y = b$.

§. **23.** Since then consequently the quantity p is equal to a so much algebraic function of that v, hence the construction of the curve will be able to be established sufficiently conveniently. Because $dx = \frac{vdv}{g-hv^{n+1}\sqrt{1+pp}}$ holds, dy will be $\frac{pvdv}{g-hv^{n+1}\sqrt{1+pp}}$ and each formula should in this way be integrated, such that, having posed $v = 0$, that which occurs in that beginning A, the integrals vanish, and in this way both coordinates x and y are obtained for that point of this curve, where the speed of the body is v. Of course it will hold that $x = \int \frac{vdv}{g-hv^{n+1}\sqrt{1+pp}}$ and $y = \int \frac{pvdv}{g-hv^{n+1}\sqrt{1+pp}}$, and this curve, continued until there, where p is θ, will be a true brachistochrone, over which the body descends in the shortest time from A to C.

Evolution of the case, in which $h = 0$,
or the resistance vanishes.

§. **24.** In this case our equation then consequently shortens to this most simple form: $\sqrt{1+pp} - \frac{cp}{v} = 0$, to which the equation $P - C = 0$ corresponds; whence it is exposed, that it is allowed for the curve that the point Y can be assumed for the final end, such that in this way all portions of this curve, beginning from the beginning A, rejoice with the brachistochronism property, which, as is agreed upon, is the distinguished property of the brachistochrone, already long ago invented for vacuum.

§. **25.** Because therefore here h is 0, p will be $\frac{v}{\sqrt{cc-vv}}$ and both coordinates are expressed in this way: $x = \int \frac{vdv}{g}$ and $y = \int \frac{vvdv}{g\sqrt{cc-vv}}$. Thence consequently x will be $\frac{vv}{2g}$, whence in turn $v = \sqrt{2gx}$, which value, substituted in the other formula, yields $y = \int \frac{dx\sqrt{2gx}}{\sqrt{cc-2gx}}$, which equation is manifest for the Cycloid, of which the cusp happens in that beginning A and describes a revolution of a circle over the horizontal straight.

Evolution of the case, in which $n = -1$,
or the resistance is the same everywhere.

§. **26.** In this case our equation between p and v assumes thus this form:

$$\sqrt{1+pp} - \frac{cp}{v} + \frac{h}{g}\left(\frac{p}{\theta} - 1\right) = 0$$

from which equation $v = \frac{cp}{\sqrt{1+pp}+\frac{b}{g}\left(\frac{p}{\theta}-1\right)}$ is derived. Whence, having assumed $p = \theta$, the speed in the last end C will be $v = \frac{c\theta}{\sqrt{1+\theta\theta}}$. Moreover, the coordinates are now expressed by v in this way, such that $x = \int \frac{vdv}{g-h\sqrt{1+pp}}$ holds and $y = \int \frac{pvdv}{g-h\sqrt{1+pp}}$, which, if instead of v the discovered value is substituted, are found to be expressed by p. It will be however superfluous to establish this operation.

§. 27. This curve will thus be a brachistochrone in a medium, of which the friction is constant, and does not depend on the speed, or, how *Newton* describes such friction, she is proportional to the momentum of times.

Conclusion

§. 28. If we carefully examine the equation between p and v invented here more accurately, we discover, that she can be extended much more widely, such that not only a certain friction is proportional to a power of the velocity v, but it so much follows a ratio of whichever function of that v such that, having assumed V for that function of the speed v, we have this equation for the motion of the body in this way:

$$vdv = gdx - hVdx\sqrt{1+pp}$$

Because truly in our integral equation the exponent n does not occur, if not in the exponent of that v, it is hence not allowed to conclude safely, that there is need of nothing else, except as in our formulae instead of v^{n+1} V is writtenw. In this way the equation between p and v will therefore now have itself in this way:

$$\sqrt{1+pp} - \frac{cp}{v} + \frac{h}{g}V\left(\frac{p}{\theta} - 1\right) = 0$$

Whence, because $dx = \frac{vdv}{g-hV\sqrt{1+pp}}$ holds, dy will be $\frac{pvdv}{g-hV\sqrt{1+pp}}$, all remaining things are determined in the same way, as before.

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