A Problem of Geometry Resolved by Diophantine Analysis[1]

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§1.

The subject of the problem which appears in this paper is drawn from Rational Trigonometry. One asks for the three sides x, y, z, of a triangle of which the lines drawn from the angles through the triangle's center are all expressible as rational numbers; that is to say: one asks for three numbers, x, y, z, such that

$$\begin{aligned} 2x^2 + 2y^2 - z^2 &= \square \\ 2y^2 + 2z^2 - x^2 &= \square \\ 2z^2 + 2x^2 - y^2 &= \square \end{aligned}$$

I have already given, on different occasions, some solutions to this problem, which are not entirely satisfactory to me. The beautiful solution that I present here is of the greatest generality. But before entering into this matter it will be good to facilitate the solution with the following lemma:

§2.

Lemma. Two numbers of the form $A^2 + 2PAB + B^2$ and $A^2 + 2QAB + B^2$ will always be squares when $A = 4(P + Q)$ and $B = (P - Q)^2 - 4$.

Proof. Multiplying the first expression by the second yields:

$$A^4 + 2(P + Q)A^3B + 2(2PQ + 1)A^2B^2 + 2(P + Q)AB^3 + B^4.$$

Let this be the square of $A^2 + (P + Q)AB + B^2$ and as the square of the latter is

$$A^4 + 2(P + Q)A^3B + [(P + Q)^2 - 2]A^2B^2 - 2(P + Q)AB^3 + B^4,$$

we compare the former with the latter and see that equality requires that $((P - Q)^2 - 4)A = 4(P + Q)B$. Thus $A = 4(P + Q)$ and $B = (P - Q)^2 - 4$.

Substituting these values into either of the two equations of this lemma yields a square. The first, for example, becomes

$$16(P + Q)^2 + 2P[4(P + Q)(P - Q)^2 - 16(P + Q)] + (P - Q)^4 - 8(P - Q)^2 + 16.$$

It is necessary to note that

$$(P - Q)^4 + 8P(P + Q)(P - Q)^2 = (P - Q)^2(3P + Q)^2$$

[1] Translation from original French by Benjamin Linowitz, Dartmouth College

and that

$$16(P+Q)^2 - 32P(P+Q) - 8(P-Q)^2 = -8(P-Q)(3P+Q).$$

In this fashion the equation reduces to $((P-Q)(3P+Q)-4)^2$. Now that both the first of the two equations in the lemma and the product of the equations were shown to be squares, it is clearly necessary that the other equation be a square as well. One finds, by a similar process, the root to be $(Q-P)(sQ+P)-4$.

Corollary.

§3.

Regarding the values of A and B it is necessary to remark that:

1. By symmetry one could also set $A = (P-Q)^2 - 4$ and $B = 4(P+Q)$;

2. These values can be simplified in certain cases. As $(P-Q)^2 = (P+Q)^2 - 4PQ$, we can put this value into the expression for B and have $B = (P+Q)^2 - 4(PQ+1)$, so that now, whenever $PQ+1 = n(P+Q)$, we can divide both A and B by the same number $(P+Q)$ and have $A = 4$ and $B = P+Q-4n$. As for the roots of the two proposed equations (recall that they were $(P-Q)(3P+Q)-4$ and $(Q-P)(3Q+P)-4$), as the first can be expressed as

$$(P+Q)(P-Q) + 2P(P-Q) - 4,$$

and as $2P(P-Q)-4 = 2P(P+Q)-4(PQ+1)$, because $(PQ+1) = n(P+Q)$, one can divide by $P+Q$ so that the root of the first equation equals $3P-Q-4n$, and by the symmetry between P and Q the root of the other equation is $3Q-P-4n$.

Solution to the Proposed Problem

§4.

Let

$$2x^2 + 2y^2 - z^2 = p^2$$
$$2y^2 + 2z^2 - x^2 = q^2$$
$$2z^2 + 2x^2 - y^2 = r^2$$

and put $x^2 + y^2 + z^2 = s$. Then we have $p^2 + 3z^2 = q^2 + 3y^2 = r^2 + 3x^2 = 2s$. One also finds that

$$\begin{aligned} 2p^2 + 2q^2 - r^2 &= 9x^2 \\ 2p^2 + 2r^2 - q^2 &= 9y^2 \\ 2q^2 + 2r^2 - p^2 &= 9z^2. \end{aligned}$$

As these properties don't contribute in any manner to the solution of this problem, they are mentioned only in passing. As for the solution of the problem, it is deduced from the following operations:

§5.

We begin by taking the difference between the first and the second of our three fundamental equations, which is $p^2 - q^2 = 3(y^2 - z^2)$. This factors as

$$(p+q)(p-q) = 3(y+z)(y-z).$$

Let $(p+q) = 3ab(y-z)$ and $(p-q) = \frac{b}{a}(y+z)$. Then the sum of their squares will be

$$(p+q)^2 + (p-q)^2 = 2p^2 + 2q^2 = \frac{9a^2}{b^2}(y-z)^2 + \frac{b^2}{a^2}(y+z)^2.$$

Now the fundamental equations give

$$2p^2 + 2q^2 = 8x^2 + 2y^2 + 2z^2$$

or rather,

$$2p^2 + 2q^2 = 8x^2 + (y+z)^2 + (y-z)^2,$$

from which we get the following equation in x, y, z:

$$\frac{9a^2}{b^2}(y-z)^2 + \frac{b^2}{a^2}(y+z)^2 = 8x^2 + (y+z)^2 + (y-z)^2,$$

which can also be expressed as:

$$8x^2 = \frac{9a^2 - b^2}{b^2}(y-z)^2 + \frac{b^2 - a^2}{a^2}(y+z)^2.$$

§6.

The third fundamental equation is easily transformed to:

$$(y+z)^2 + (y-z)^2 - x^2 = r^2,$$

which yields the following after being multiplied by 8:

$$8r^2 = 8(y+z)^2 + 8(y-z)^2 - 8x^2.$$

If we put in the place of $8x^2$ the value which precedes the § in §6., it will be

$$8r^2 = \frac{9(b^2-a^2)}{b^2}(y-z)^2 + \frac{9a^2-b^2}{a^2}(y+z)^2.$$

§7.

If we now let $(y+z) = a(c+d)$ and let $(y-z) = b(c-d)$, then the two expressions found for $8x^2$ and $8r^2$ take on the following forms:

$$2x^2 = 2a^2(c^2+d^2) + cd(b^2-5a^2);$$
$$2\mathrm{r}^2 = 2b^2(c^2+d^2) + cd(9a^2-5b^2);$$

which, divided by $2aa^2$ and $2b^2$ respectively, give:

$$\frac{x^2}{a^2} = c^2 + d^2 + \frac{b^2-5a^2}{2a^2} \cdot cd;$$
$$\frac{r^2}{b^2} = c^2 + d^2 + \frac{9a^2-5b^2}{2b^2} \cdot cd;$$

§8.

Comparing these two expressions with those found in the Lemma, we see that $A = c$ and $B = d$. Furthermore,

$$P = \frac{b^2-5a^2}{4a^a} \text{ and } Q = \frac{9a^2-5b^2}{4b^2}.$$

From this we easily deduce that:

$$n(P+Q) = \frac{n(b^4-10a^2b^3+9a^4)}{4a^2b^2}$$

$$PQ+1 = -\frac{5}{4}\frac{b^4-10a^2b^3+9a^4}{4a^2b^2}$$

Hence, $n = -\frac{5}{4}$.

§9.

By virtue of section 3's corollary, we have that $A = 4$ and that $B = P+Q-4n$, hence $c = 4$ and $d = \frac{(9a^2+b^2)(a^2+b^2)}{4a^2b^2}$, yielding:

$$y+z = \frac{a(16a^2b^2+(9a^2+b^2)(a^2+b^2))}{4a^2b^2}$$

$$y-z = \frac{b(16a^2b^2-(9a^2+b^2)(a^2+b^2)))}{4a^2b^2}$$

Thus, by virtue of the same corollary,

$$\frac{x}{a} = 3P - Q - 4n \text{ and } \frac{r}{b} = 3Q - P - 4n.$$

We also have

$$x = \frac{a((9a^2+b^2)(a^2+b^2)-2(9a^4-b^4))}{4a^2b^2};$$

$$r = \frac{b((9a^2+b^2)(a^2+b^2)+2(9a^4-b^4))}{4a^2b^2};$$

Finally we have

$$p + q = \frac{3a}{b}(y - z);$$

$$p - q = \frac{b}{a}(y + z).$$

§10.

By putting

$C = 16a^2b^2$;
$D = (9a^2 + b^2)(a^2 + b^2)$;
$F = 2(9a^4 - b^4)$;

and eliminating the common divisor $4a^2b^2$, we will have

$x = a(D - F)$	$r = b(D + F)$
$y + z = a(C + D)$	$p + q = 3a(C - D)$
$y - z = b(C - D)$	$p - q = b(C + D)$.

Example 1.

§11.

Let $a = 1$ and $b = 2$ and we will have $C = 64$, $D = 65$ $F = -14$; therefore

$x = 79$	$r = 102$
$y + z = 129$	$p + q = -3$
$y - z = -2$	$p - q = 258$.

and consequently we have

$x = 79$	$p = \frac{255}{12}$
$y = \frac{127}{2}$	$q = \frac{261}{2}$
$z = \frac{131}{2}$	$r = 102$.

Example 2.

§12.

Let $a = 2$ and $b = 1$ and we will have $C = 64$, $D = 185$ $F = 286$; therefore

$x = -202$	$r = 471$
$y + z = 498$	$p + q = -726$
$y - z = -121$	$p - q = 249.$

and consequently we have

$x = 202$	$p = \frac{471}{2}$
$y = \frac{377}{2}$	$q = \frac{975}{2}$
$z = \frac{619}{2}$	$r = 471.$

§13.

If one wishes to have integral solutions, it is clear that he need only multiply by 2 each of the six numbers in the preceding examples. We therefore have some more solutions:

68	87	85
158	127	131
159	325	314
619	377	404
477	277	446
569	881	649.