Translation of

Leonhard Euler's

De fractionibus continuis Wallisii

E745

(On the continued fractions of Wallis)

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Translator's Introduction

Kristin Masters teaches Latin and Christopher Tippie teaches mathematics at Millville Senior High School in New Jersey. Christopher is also a graduate student of mathematics at Rowan University, where Tom Osler is professor of mathematics. In this collaboration, we have attempted to provide not only an accurate translation, but to also comprehend and communicate Euler's discoveries. To this end we kept careful notes as we pondered Euler's ideas and filled in the missing details. These ideas are contained in our companion paper "Reflections and Notes on E745". In addition we have written a shorter "Synopsis of E745" to assist the reader. All of these can be found on the Euler Archive.

We thank Dominic Klyve and his Euler Archive for providing a home on the internet for this and other translations.

(1) After William Brouncker found his memorable continual fraction for the quadrature of a circle, and after he shared it with John Wallace without an explanation, Wallis devoted more time to it so that he could uncover the source of Brouncker's noted formula. Wallis reckoned that there was a use for these extraordinary formulas, which he brought to light in his work *Arithmetic of Infinities*. Then he produced, through a little too abstruse calculations, not only Brouncker's continual fraction, but also even more countless other ones like it, both of which (certainly including Brouncker's expression) are deemed worthy to be rescued from oblivion.

(2) However, the things which pertain to this from Wallis' *Arithmetic of Infinities*, which were brought to light long before the *Analysis of Infinities* was discovered, are now able to be represented in such a way that when the limits of integral formulas are extended from x = 0 to x = 1, the following quadratures are produced:

$$\int \frac{x\partial x}{\sqrt{1-xx}} = 1 = 1$$

$$\int \frac{x^3\partial x}{\sqrt{1-xx}} = \frac{2}{3} = \frac{2\cdot 2}{2\cdot 3}$$

$$\int \frac{x^5\partial x}{\sqrt{1-xx}} = \frac{2\cdot 4}{3\cdot 5} = \frac{2\cdot 2\cdot 4\cdot 4}{2\cdot 3\cdot 4\cdot 5}$$

$$\int \frac{x^7\partial x}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$$

$$\int \frac{x^9\partial x}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}, \text{ etc…}$$

(3) I arranged these formulas in the third column in such a way that the denominators clearly allow interpolation. And so it only remains that the numerators are transformed in such a way that they allow interpolation equally. If such a series progresses according to the law of uniformity, (of course if it were investigated as A, B, C, D, E, F, etc.) it would become:

AB = 1.1; BC = 2.2; CD=3.3; DE=4.4; etc

This very thing Wallis revealed through the highest wisdom of his genius. I will clear up this investigation much more generally, through a much easier calculation.

(4) However, when this series of letters A, B, C, D, etc. is found, the whole business will have been completely settled. For when as the following table reveals:

$$\int \frac{x\partial x}{\sqrt{1-xx}} = 1 = \frac{1}{A}\cdot\frac{A}{1},$$

$$\int \frac{x^3\partial x}{\sqrt{1-xx}} = \frac{BC}{2\cdot 3} = \frac{1}{A}\cdot\frac{ABC}{1\cdot 2\cdot 3},$$

$$\int \frac{x^5\partial x}{\sqrt{1-xx}} = \frac{BCDE}{2\cdot 3\cdot 4\cdot 5} = \frac{1}{A}\cdot\frac{ABCDE}{1\cdot 2\cdot 3\cdot 4\cdot 5},$$

$$\int \frac{x^7\partial x}{\sqrt{1-xx}} = \frac{BCDEFG}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = \frac{1}{A}\cdot\frac{ABCDEFG}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}, \text{ etc…}$$

Interpolation provides for us the following quadratures:

$$\int \frac{\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot 1,$$

$$\int \frac{xx\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot\frac{AB}{1\cdot 2},$$

$$\int \frac{x^4\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot\frac{ABCD}{1\cdot 2\cdot 3\cdot 4},$$

$$\int \frac{x^6\partial x}{\sqrt{1-xx}} = \frac{1}{A}\cdot\frac{ABCDEF}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}, \text{ etc…}$$

(5) Since now $\int \frac{\partial x}{\sqrt{1-xx}} = \frac{\pi}{2}$,where π denotes the circumference of a circle whose diameter equals 1, whose value I write for the sake of brevity $q = \frac{\pi}{2}$, all values of the letters *A, B, C, D*, etc…for this quantity *q* will be expressed in the following manner:

Difference

$$A = \frac{1}{q} = 0,636620$$

0,934176

$$B = q = 1,570796$$

0,975683

$$C = \frac{4}{q} = 2,546479$$

0,987813

$$D = \frac{9q}{4} = 3,534292$$

0,992782

$$E = \frac{4 \cdot 16}{9q} = 4,527074$$

0,995257

$$F = \frac{9 \cdot 25}{4 \cdot 16} q = 5,522331$$

(6) Here I attached a third column, which exhibits the numeric values of these letters to show more clearly the extent that these numbers increase according to the law of uniformity. This does not happen if I take a false value in the place of *q*.

Now that these things have been explained, I will find, in a much easier manner, continued fractions for each individual letter. What is more, I will arrange this investigation in much more generality while I resolve the following problem:

Problem:

To find a series of letters *A, B, C, D,* etc…progressing by the law of uniformity in such a way that AB =ff; BC= $(f+a)^2$; CD= $(f+2a)^2$; etc..

(7) *Solution*:

It is immediately clear from this point that A was a function of that very *f*, as it ought to be that B is a function of that very *f*+*a*, then moreover C is a function of *f*+*2a*; D is a function of *f*+*3a*; and so on.

Once this rule has been observed, if we assume $A = f - \frac{1}{2}a + \frac{\frac{1}{2}s}{A'}$, it ought to also be assumed that $B = f + \frac{1}{2}a + \frac{\frac{1}{2}s}{B'}$; where the letters A' and B' ought to hold the same ratio between themselves. In this way B' should arise from A' if $f+a$ is written in the place of *f*.

Therefore, when these fractions are removed, since $2A = 2f - a + \frac{s}{A'}$ and $2B = 2f + a + \frac{s}{B'}$, the product of these formulas is equal to 4*ff*. Now this equation arises from the fractions:

$$aaA'B' - A's(2f - a) - B's(2f + a) - ss = 0.$$

Therefore let us assume that *s*=*aa*, and the equation, divided by *aa*, should be

$$A'B' - A'(2f - a) - B'(2f + a) = aa,$$

which is easily factored in the following way:

$$(A' - 2f - a)(B' - 2f + a) = 4ff.$$

(8) Now, if both letters A' and B' are equal, the left part should be $A' = B' = 4f$, following the above pattern, we should find that $A' = 4f - 2a + \frac{s'}{A''}$ and $B' = 4f + 2a + \frac{s'}{B''}$. When you substitute these, the final equation takes the form:

$$(2f - 3a + \frac{s'}{A''})(2f + 3a + \frac{s'}{B''}) = 4ff.$$

When expanded and the fractions removed, the following equation arises:

$9aaA''B'' - A''s'(2f - 3a) - B''s'(2f + 3a) - s's' = 0$.

Therefore it is assumed here that $s' = 9aa$, and it follows that :

$A''B'' - A''(2f - 3a) - B''(2f + 3a) = 9aa$,

which again is able to be represented by factors as:

$(A'' - 2f - 3a)(B'' - 2f + 3a) = 4ff$.

(9) Now once again the *middle value* between A'' and B'' is *4f*, furthermore let us assume that $A'' = 4f - 2a + \frac{s''}{A'''}$ and $B'' = 4f + 2a + \frac{s''}{B'''}$, and once this substitution is made, we will get: $(2f - 5a + \frac{s''}{A'''})(2f + 5a + \frac{s''}{B'''}) = 4ff$.

When expanded it becomes:

$25aaA'''B''' - A'''s''(2f - 5a) - B'''s''(2f + 5a) - s''s'' = 0$.

Assume that $s'' = 25aa$, and this equation will take on the form:

$A'''B''' - A'''(2f - 5a) - B'''(2f + 5a) = 25aa$,

which is factored as:

$(A''' - 2f - 5a)(B''' - 2f + 5a) = 4ff$.

(10) Again, as before, let it be assumed that $A''' = 4f - 2a + \frac{s'''}{A^{IV}}$ and $B''' = 4f + 2a + \frac{s'''}{B^{IV}}$. Once this substitution is made, we get

$(2f - 7a + \frac{s'''}{A^{IV}})(2f + 7a + \frac{s'''}{B^{IV}} = 4ff$.

When this equation is manipulated, as before, we will get

$A^{IV}B^{IV} = A^{IV}(2f - 7a) - B^{IV}(2f + 7a) = 49aa$, where of course we put $s''' = 49aa$. Then in factored form it will be

$(A^{IV} - 2f - 7a)(B^{IV} - 2f + 7a) = 4ff$.

From here it is clear how these operations are to be continued further.

(11) When these things are reasoned, that $s = aa, s' = 9aa, s'' = 25aa, s''' = 49aa$, etc…,

we obtain for *2a* the following continued fraction:

$$2A = 2f - a + \cfrac{aa}{4f - 2a + \cfrac{9aa}{4f - 2a + \cfrac{25aa}{4f - 2a + \cfrac{49aa}{4f - 2a + etc...}}}}$$

If we write in the place of *f* the series *f+a, f+2a, f+3a,* etc…, we will create similar

continued fractions for *2B, 2C, 2D,* etc… which are:

$$2B = 2f + a + \cfrac{aa}{4f + 2a + \cfrac{9aa}{4f + 2a + \cfrac{25aa}{4f + 2a + \cfrac{49aa}{4f + 2a + etc...}}}}$$

$$2C = 2f + 3a + \cfrac{aa}{4f + 6a + \cfrac{9aa}{4f + 6a + \cfrac{25aa}{4f + 6a + \cfrac{49aa}{4f + 6a + etc...}}}}$$

$$2D = 2f + 5a + \cfrac{aa}{4f + 10a + \cfrac{9aa}{4f + 10a + \cfrac{25aa}{4f + 10a + \cfrac{49aa}{4f + 10a + etc...}}}}$$

(12) If now we put *f=1* and *a=1*, we will produce the very case handled by Wallis. From this Wallis found his continued fractions, when their values are expressed through the quadrature of a circle, they will be the following:

WALLIS' CONTINUAL FRACTIONS

$$2A = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{1 + etc...}}}} = \frac{2}{q} = \frac{4}{\pi},$$

$$2B = 3 + \cfrac{1}{6 + \cfrac{9}{6 + \cfrac{25}{6 + \cfrac{49}{6 + etc...}}}} = 2q = \pi,$$

$$2C = 5 + \cfrac{1}{10 + \cfrac{9}{10 + \cfrac{25}{10 + \cfrac{49}{10 + etc...}}}} = \frac{8}{q} = \frac{16}{\pi},$$

$$2D = 7 + \cfrac{1}{14 + \cfrac{9}{14 + \cfrac{25}{14 + \cfrac{49}{14 + etc...}}}} = \frac{9q}{2} = \frac{9\pi}{4},$$

$$2E = 9 + \cfrac{1}{18 + \cfrac{9}{18 + \cfrac{25}{18 + \cfrac{49}{18 + etc...}}}} = \frac{128}{9q} = \frac{256}{9\pi}.$$

The first of these is the very same continued fraction discovered by Brouncker.

(13) However, this is by no means similar to the roundabout way Brouncker came to his formula. Instead I believe that he derived it from the consideration of his famous series:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - etc... = \frac{\pi}{4},$$

This series is commonly accustomed to be attributed to Leibniz; however it was discovered much earlier by Jacob Gregory. From Gregory's work Brouncker was able to derive it.

Of course it was possible to come about through sufficiently easy and accessible operations in the following way:

When you insert…	It becomes…
$\frac{\pi}{4} = 1 - \alpha$	$\frac{4}{\pi} = \frac{1}{1-\alpha} = 1 + \frac{\alpha}{1-\alpha} = 1 + \cfrac{1}{-1 + \cfrac{1}{\alpha}}$
$\alpha = \frac{1}{3} - \beta$	$\frac{1}{\alpha} = \frac{3}{1-3\beta} = 3 + \frac{9\beta}{1-3\beta} = 3 + \cfrac{9}{-3 + \cfrac{1}{\beta}}$
$\beta = \frac{1}{5} - \gamma$	$\frac{1}{\beta} = \frac{5}{1-5\gamma} = 5 + \frac{25\gamma}{1-5\gamma} = 5 + \cfrac{25}{-5 + \cfrac{1}{\gamma}}$

$\gamma = \frac{1}{7} - \delta$ $\qquad$ $\frac{1}{\gamma} = \frac{7}{1-7\delta} = 7 + \frac{49\delta}{1-7\delta} = 7 + \cfrac{49}{-7 + \cfrac{1}{\delta}}$

But if these recently found values are substituted in the place of $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, etc...$

Brouncker's continued fraction is obtained

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + etc...}}}}$$

(14) Now, this brings me to my general solution of the problem. One can even express the values of individual continued fractions through certain quadratures, which I will show in the following problem:

Problem.

To investigate the values of the individual letters, expressed first (1) through continued products, then (2) expressed through integral formulas, for the sequence *A, B, C, D, etc*...that continues according to the law of uniformity in such a way that

$AB = ff; BC = (f+a)^2; CD = (f+2a)^2; etc...$

(15) Solution.

Therefore let $A = \frac{ff}{B}; B = \frac{(f+a)^2}{C}; C = \frac{(f+2a)^2}{D}; etc...$

When these values are substituted continuously, one finds that

$$A = \frac{ff(f+2a)^2(f+4a)^2(f+6a)^2(etc...}{(f+a)^2(f+3a)^2(f+5a)^2(etc...}$$, infinitely.

Because a factor remains either in the denominator or in the numerator whenever (the series) is broken off, no limiting value arises this way. This inconvenience, however, will be removed if I arrange the simple factors in the following

way: $A = f \cdot \frac{f(f+aa)}{(f+a)(f+a)} \cdot \frac{(f+2a)(f+4a)}{(f+3a)(f+3a)} \cdot \frac{(f+4a)(f+6a)}{(f+5a)(f+5a)} \cdot etc...$

In this way factors will continuously approach one, and infinitely they will equal one; and thus this expression will certainly have a limiting value.

(16) However, when I will show how one ought to reduce the value to integral expressions, I will call upon this *lemma* for help:

When integrals are evaluated from *x=0* to *x=1*, we will get:

$$\int \frac{x^{m-1}\partial x}{\sqrt{(1-x^n)^{n-k}}} = \frac{m+k}{m} \cdot \frac{m+k+n}{m+n} \cdot \frac{m+k+2n}{m+2n} \cdot \frac{m+k+3n}{m+3n} \cdot \frac{m+k+4n}{m+4n} \cdots \int \frac{x^{\infty}\partial x}{\sqrt[n]{(1-x^n)^{n-k}}}.$$ [1]

When I apply this lemma to my problem, since our members individual factors take an increase of $=2a$, thus it ought to be assumed that $n=2a$. Then when it is assumed that *m=f* and *k=a*, we will have:

[1] **There is a typographical error on the left hand side. The radical in the denominator should be identical to the corresponding radical on the right hand side.**

$$\int \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}} = \frac{f+a}{f} \cdot \frac{f+3a}{f+2a} \cdot \frac{f+5a}{f+3a} \cdots \int \frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}},$$

This expression, once adjusted, will yield previous factors of individual members. For the next equations, let us assume that $m=f+a$, (and k remains $=a$), when this is done, we get:

$$\int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}} = \frac{f+2a}{f+a} \cdot \frac{f+4a}{f+3a} \cdot \frac{f+6a}{f+5a} \cdots \int \frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}}.$$

(17) It is now evident that the second formula, once it has been divided by the previous formula, will show my continual product. When infinite integrals are cancelled out on both sides, we get:

$$A = \int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}}.$$

Similarly, straightaway we find

$$B = \int \frac{x^{f+2a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}}$$

$$C = \int \frac{x^{f+3a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f+2a-1}\partial x}{\sqrt{1-x^{2a}}} \quad \text{etc…}$$ [2]

But this investigation can still be generalized, as the following problem will show.

A MORE GENERAL PROBLEM

To find a series *A, B, C, D,* etc. that proceeds uniformly in such a way that

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[2] **There is an error in each of the above three equations**. **The right hand sides should be multiplied by f,** $f+a$**, and** $f+2a$ **respectively. See section 27.**

$$AB = ff + c; BC = (f + a)^2 + c; CD = (f + 2a)^2 + c; DE = (f + 3a)^2 + c;$$

Where the letter f is increased by a quantity a.

PREVIOUS SOLUTION FOR CONTINUED FRACTIONS

(18) Again, here it is evident that A is a function of f and B ought to be a function of $f + a$; C is a function of $f + 2a$; D is a function of $f + 3a$, and so on. Therefore, when $AB = ff + c$, if A and B are equal, when c is omitted, $A = B = f$. If A is considered less than f , B ought to be considered greater than f. From here, when $A = f - x$, B will equal $f + x$. Because A comes from B, if $f + a$ is written in the place of f , it ought to be that $B = f + a - x$. From here we conclude that $x = ½ a$; and so the principal parts for A and B will be

$$A = f - ½ a \text{ and } B = f + ½ a ,$$

or

$$2A = 2f - a \text{ and } 2B = 2f + a.$$

Therefore for the following letters, we find

$$2C = 2f + 3a;\ 2D = 2f + 5a;\ 2E = 2f + 7a, \text{ etc.}$$

(19) When these principal values have been found, we will consider that

$$2A = 2f - a + \frac{s}{A'}; 2B = 2f + a + \frac{s}{B'};$$

But a suitable value will emerge for s . Therefore from here, we get:

$$4AB = 4ff - aa + \frac{s}{A'}(2f + a) + \frac{s}{B'}(2f - a) + \frac{ss}{A'B'} = 4ff + 4c .$$

This equation, once the fractions are removed, will take on the form:

$$A'B'(aa + 4c) - A's(2f - a) - B's(2f + a) - ss = 0.$$

Now let us assume that $s = aa + 4c$, and once division is made we get:

$$A'B' - A'(2f - a) - B(2f + a) = aa + 4c.$$

This equation can be expressed in factored form as:

$$(A' - 2f - a)(B' - 2f + a) = 4ff + ac.$$

(20) Similarly, as it can be recognized through deduction, if *A*' and *B*' are equal, the member on the left would be

$$A'A' - 4fA' = 0, \text{ and therefore } A' = B' = 4f.$$

Because *B'* ought to arise from *A'*, if $f + a$ is written in the place of *f* , it is evident that the principal parts are

$$A' = 4f - 2a \quad \text{and} \quad B' = 4f + 2a.$$

Therefore let us assume that

$$A' = 4f - 2a + \frac{s'}{A''} \text{ and } B' = 4f + 2a + \frac{s'}{B''}.$$

From here, if these values are substituted, the preceding equation will take on this form in factors:

$$(2f - 3a + \frac{s'}{A''})(2f + 3a + \frac{s'}{B''}) = 4ff + 4c$$

This equation, once the expansion has been made, will bring us to this equation:

$$(4ff - 9aa) + \frac{s'}{A''}(2f + 3a) + \frac{s'}{B''}(2f - 3a) + \frac{s's'}{A''B''} = 4ff + 4c\,.$$

Once the fractions have been removed, we get:

$$A''B''(9aa + 4c) - A''s'(2f - 3a) - B''s'(2f + 3a) - s's' = 0.$$

Therefore, once it is assumed that $s' = 9aa + 4c$, and the division has been made, this equation arises:

$$A''B'' - A''(2f - 3a) - B''(2f + 3a) = 9aa + 4c.$$

This can be represented by factors in the following way:

$$(A'' - 2f - 3a)(B'' - 2f + 3a) = 4ff + 4c.$$

(21) Because this equation is similar to the previous one, again in this case if *A'' = B''*, we will get *4f* . Furthermore, it is assumed that

$$A'' = 4f - 2a + \frac{s''}{A'''} \text{ and } B'' = 4f + 2a + \frac{s''}{B'''}.$$

From here the final equation will be in factor form:

$$(2f - 5a + \frac{s''}{A'''})(2f + 5a + \frac{s''}{B'''}) = 4ff + 4c.$$

But once the expansion has been made, and the fractions have been removed, it will produce:

$$A'''B'''(25aa + 4c) - A'''s''(2f - 5a) - B'''s''(2f + 5a) - s''s'' = 0.$$

Therefore, by assuming that *s'' = 25aa + 4c* and by dividing by *s''*, it will become:

$$A'''B''' - A'''(2f - 5a) - B'''(2f + 5a) = 25aa + 4c$$

or, in product form:

$$(A''' - 2f - 5a)(B''' - 2f + 5a) = 4ff + 4c.$$

(22) Furthermore it will be established that

$$A''' = 4f - 2a + \frac{s'''}{A^{(4)}} \text{ and } B''' = 4f + 2a + \frac{s'''}{B^{(4)}} \quad ,$$

and when these values are substituted, the preceding equation will be in product form:

$$(2f - 7a + \frac{s'''}{A^{(4)}})(2f + 7a + \frac{s'''}{B^{(4)}}) = 4ff + 4c.$$

When these same operations have been repeated, and when it has been assumed that $s''' = 49aa + 4c$, this equation is reduced to the following:

$A^{(4)}B^{(4)} - A^{(4)}(2f - 7a) - B^{(4)}(2f + 7a) = 49aa + 4c$.

Or, in factored form, it will be:

$$(A^{(4)} - 2f - 7a)(B^{(4)} - 2f + 7a) = 4ff + 4c\,.$$

From this it is now abundantly clear how the final calculation ought to proceed.

(23) Therefore once these values are successively substituted, because $s = aa + 4c; s' = 9aa + 4c; s'' = 25aa + 4c; s''' = 49aa + 4c;$ etc., we obtain the following continued fraction for A:

$$2A = 2f - a + \cfrac{aa + 4c}{4f - 2a + \cfrac{9aa + 4c}{4f - 2a + \cfrac{25aa + 4c}{4f - 2a + \cfrac{49aa + 4c}{4f - 2a + etc...}}}}$$

Similarly,

$$2B = 2f + a + \cfrac{aa + 4c}{4f + 2a + \cfrac{9aa + 4c}{4f + 2a + \cfrac{25aa + 4c}{4f + 2a + \cfrac{49aa + 4c}{4f + 2a + etc...}}}}$$

$$2C = 2f + 3a + \cfrac{aa + 4c}{4f + 6a + \cfrac{9aa + 4c}{4f + 6a + \cfrac{25aa + 4c}{4f + 6a + \cfrac{49aa + 4c}{4f + 6a + etc...}}}}$$

$$2D = 2f + 5a + \cfrac{aa + 4c}{4f + 10a + \cfrac{9aa + 4c}{4f + 10a + \cfrac{25aa + 4c}{4f + 10a + \cfrac{49aa + 4c}{4f + 10a + etc...}}}}$$

ANOTHER SOLUTION FOR CONTINUED PRODUCTS

(24) When

$AB = ff + c; BC = (f + a)^2 + c; CD = (f + 2a)^2 + c; DE = (f + 3a)^2 + c; etc...$,we get:

$$A = \frac{(ff + c)((f + 2a)^2 + c)((f + 4a)^2 + c)((f + 6a)^2 + c)(etc...)}{((f + a)^2 + c)((f + 3a)^2 + c)((f + 5a)^2 + c)(etc...)}.$$

But in this expression, whenever you stop, there will be a factor left either in the numerator or in the denominator. This will become clearer if we substitute in the letter F as follows $A = \frac{ff + c}{(f + a)^2 + c} \cdot \frac{(f + 2a)^2 + c}{(f + 3a)^2 + c} \cdot (f + 4a)^2 + c \cdot \frac{1}{F}$.

However, when we substitute in the following letter G, it will become:

$$A = \frac{ff + c}{(f + a)^2 + c} \cdot \frac{(f + 2a)^2 + c}{(f + 3a)^2 + c} \cdot \frac{(f + 4a)^2 + c}{(f + 5a)^2 + c}, G.$$

(25) But if therefore these two expressions are continued infinitely and are calculated into themselves, the final letter factor, which here is $\frac{G}{F}$, will equal one. Since in this case the number leaves a factor in the numerator of one, I will write its first factor apart in the front part, and the product will be expressed in the following way:

$$A^2 = (ff + c) \cdot \frac{(ff + c)((f + 2a)^2 + c)}{((f + a)^2 + c)((f + a)^2 + c)} \cdot \frac{((f + 2a)^2 + c)((f + 4a)^2 + c)}{((f + 3a)^2 + c)((f + 3a)^2 + c)} \cdot etc...$$

Now the factors near infinity approach one, and thus this expression proceeds according to the law of uniformity. Here, however, it is fitting to distinguish between two cases: if *c* is a positive or a negative number.

Case # 1: Where *c* = - *bb*

(26) In the first case any factor allows itself to be resolved in two ways. Therefore if we first establish that $c = -bb$, the continued fraction can be expressed in the following way:

$$2A = 2f - a + \cfrac{(a + 2b)(a - 2b)}{4f - 2a + \cfrac{(3a + 2b)(3a - 2b)}{4f - 2a + \cfrac{(5a + 2b)(5a - 2b)}{4f - 2a + \cfrac{(7a + 2b)(7a - 2b)}{4f - 2a + etc...}}}}$$

and in the place of this expression, we now have a sequence in continued factors; for the simple letter *A*, it is:

$$A = (f - b) \cdot \frac{(f + b)(f + 2a - b)}{(f + a + b)(f + a - b)} \cdot \frac{(f + 2a - b)(f + 4a - b)}{(f + 3a + b)(f + 3a - b)} \cdot etc...$$ [3]

Anywhere in the main factors of this expression, the sum of the factors in the numerator is equal to the sum factor in the denominator. Because of this, these factors are able to be expressed in an integral formula.

[3] There is a misprint in the last factor. It should be $\frac{(f + 2a + b)(f + 4a - b)}{(f + 3a + b)(f + 3a - b)}$.

(27) It is agreed that, if this integral formula: $\int \frac{x^{m-1}\partial x}{\sqrt[n]{(1-x^n)^{n-k}}}$, is evaluated from $x = 0$ to $x = 1$, the value is reduced to the following infinite product:

$$\frac{m+k}{m} \cdot \frac{m+k+n}{m+n} \cdot \frac{m+k+2n}{m+2n} \cdots\cdots \int \frac{x^{\infty}\partial x}{\sqrt[n]{(1-x^n)^{n-k}}}.$$

Therefore, when we apply this formula to our expression, since single factors are increased in the following factor by the quantity $2a$, one can assume that $n = 2a$. However, when it is assumed that $m = f + b$ and $k = a$,, it is found to be:

$$\frac{f+a+b}{f+b} \cdot \frac{f+3a+b}{f+2a+b} \cdot \frac{f+5a+b}{f+4a+b} \cdots\cdots \int \frac{x^{\infty}\partial x}{\sqrt{(1-x^{2a})}} = \int \frac{x^{f+b-1}\partial x}{\sqrt{1-x^{2a}}}.$$

This expression, once inverted, contains the previous factors of its members. For the latter ones, however, if it remains that $n = 2a$, it is assumed that $m = f + a - b$ and $k = a$.

Once this is done, it will create this equation:

$$\frac{f+2a-b}{f+a-b} \cdot \frac{f+4a-b}{f+3a-b} \cdot \frac{f+6a-b}{f+5a-b} \cdots\cdots \int \frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}} = \int \frac{x^{f+a-b-1}\partial x}{\sqrt{1-x^{2a}}}.$$

If, therefore, this equation is divided by the preceding one, later integral factors cancel each other out, and it will produce an infinite product, giving the value A, that is expressed by two integrals as:

$$A=(f-b)\cdot\int\frac{x^{f+a-b-1}\partial x}{\sqrt{1-x^{2a}}}:\int\frac{x^{f+b-1}\partial x}{\sqrt{1-x^{2a}}}.$$[4]

(28) Let me show this by the following example. If I assume that $f = 2$, $a = 1$, $b = 1$, we get these values: $AB = 3$, $BC = 8$, $CD = 15$, $DE = 24$, *etc…* In this case our continued fraction is: $2A=3-\cfrac{3}{6+\cfrac{5}{6+\cfrac{21}{6+\cfrac{45}{6+\cfrac{77}{6+etc...}}}}}$

But it will be this in continued products: $A=\frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot etc...$

Then it will be this way in integral formulas: $A=\int\frac{x\partial x}{\sqrt{1-xx}}:\int\frac{xx\partial x}{\sqrt{1-xx}}.$

It is agreed that for my limits of integration (from $x = 0$ to $x = 1$), it is $\int\frac{x\partial x}{\sqrt{1-xx}}=1$, and $\int\frac{xx\partial x}{\sqrt{1-xx}}=\frac{\pi}{4}$. From here A is seen to be $\frac{\pi}{4}$. This fits splendidly with Wallis' product, where $\frac{\pi}{2}=\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot etc...$

CASE # 2: Where *c* = + *bb*

[4] **Note that this time he has the correct factor $(f-b)$ which he neglected in section 17 when $b=0$.**

(29) Let us solve now another case, $c = +bb$, whose continual fraction takes on this

form: $$2A = 2f - a + \cfrac{aa+4bb}{4f-2a+\cfrac{9aa+4bb}{4f-2a+\cfrac{25aa+4bb}{4f-2a+\cfrac{49aa+4bb}{4f-2a+etc...}}}}$$

However, when $b\sqrt{-1}$ is written in the place of *b*, in the previous continued product the new form is expressed imaginarily:

$$A = (f-b\sqrt{-1})\cdot\frac{(f+b\sqrt{-1})(f+2a-b\sqrt{-1})}{(f+a+b\sqrt{-1})(f+a-b\sqrt{-1})}\cdot\frac{(f+2a+b\sqrt{-1})(f+4a-b\sqrt{-1})}{(f+3a+b\sqrt{-1})(f+3a-b\sqrt{-1})}\cdot etc...$$

However it is evident that when (paragraph 26) is applied to the same expression that $-b\sqrt{-1}$ can be written in the place of *b*, from which we get:

$$A = (f+b\sqrt{-1})\cdot\frac{(f-b\sqrt{-1})(f+2a+b\sqrt{-1})}{(f+a-b\sqrt{-1})(f+a+b\sqrt{-1})}\cdot\frac{(f+2a-b\sqrt{-1})(f+4a+b\sqrt{-1})}{(f+3a-b\sqrt{-1})(f+3a+b\sqrt{-1})}\cdot etc...$$

Therefore the product of these two expressions becomes real, it will be:

$$A^2 = (ff+bb)\frac{(ff+bb)((f+2a)^2+bb)}{((f+a)^2+bb)((f+a)^2+bb}\cdot\frac{((f+2a)^2+bb)((f+4a)^2+bb)}{((f+3a)^2+bb)((f+3a)^2+bb)}\cdot etc...$$

This expression is congruous with the previous one found in paragraph 25.

(30) But this expression becomes imaginary through integral formulas. For if $b\sqrt{-1}$ is written in the place of b in the formula in paragraph 27 , the following expression arises:

$$A = (f-b\sqrt{-1})\int\frac{x^{f+a-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} : \int\frac{x^{f-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}.$$

However, once the sign of the imaginary numbers have been changed, it becomes:

$$A = (f + b\sqrt{-1})\int \frac{x^{f+a-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}.$$

Here there is no doubt that in both expressions the imaginary numbers cancel each other out, although there is no apparent method to show the mutual cancellation of imaginary numbers.

(31) However, if both expressions are brought together, then this cancellation can be shown easily. Then the product is $A^2 = (ff + bb)\dfrac{\int \frac{x^{f+a-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} \cdot \int \frac{x^{f+a-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}}{\int \frac{x^{f-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} \cdot \int \frac{x^{f-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}}$

It can be demonstrated that imaginary numbers cancel out in both the numerator and the denominator. Because it is sufficient to have shown this for the denominator, since the numerator arises from it by writing $f + a$ in the place of f.

(32) So that this demonstration can be dealt with more succinctly, let us assume (for the sake of brevity) that $\frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}} = \partial V$. When this has been done and once the imaginary numbers have been dealt with, the denominator of my expression will be

$$\int x^{+b\sqrt{-1}}\partial V \cdot \int x^{-b\sqrt{-1}}\partial V.$$

Now let us assume the factors:

The sum: $\int (x^{b\sqrt{-1}} + x^{-b\sqrt{-1}}\partial V = p$

Difference: $\int (x^{b\sqrt{-1}} - x^{-b\sqrt{-1}}\partial V = q$

And it is noted that the proposed product will be: $\int (x^{b\sqrt{-1}} \partial V \cdot \int x^{-b\sqrt{-1}} \partial V = \frac{pp-qq}{4}$

Now I will show how both pp and qq can be reduced to real quantities.

(33) In this evaluation in the place of x in possible imaginary numbers, let us write e^{ix},

so it becomes

$$p = \int (e^{bix\sqrt{-1}} + e^{-bix\sqrt{-1}}) \partial V,\ q = \int (e^{bix\sqrt{-1}} - e^{-bix\sqrt{-1}}) \partial V,$$

(Here Euler writes *lx* for our $\log x$.)

Since we know that

$$e^{\phi\sqrt{-1}} + e^{-\phi\sqrt{-1}} = 2\cos\phi \text{ and } e^{\phi\sqrt{-1}} - e^{-\phi\sqrt{-1}} = 2\sqrt{-1}\sin\phi .$$

Assume, for the sake of clarity, that $bix = \phi$, and get

$$p = 2\int \partial V \cos\phi \text{ and } q = 2\sqrt{-1}\int \partial V \sin\phi .$$

From here the denominator becomes:

$$\frac{pp-qq}{4} = (\int \partial V \cos\phi)^2 + (\int \partial V \sin\phi)^2$$

which clearly is a real expression.

(34) Here the value of the numerator easily is brought together to become:

$$(\int x^a \partial V \cos\phi)^2 + (\int x^a \partial V \sin\phi)^2 ,$$

In such a way that our expression, where the imaginary numbers have been removed, A^2 is represented in real numbers in the following way:

$$A^2 = (ff + bb)\frac{(\int x^a \partial V \cos\phi)^2 + (\int x^a \partial V \sin\phi)^2}{(\int \partial V \cos\phi)^2 + (\int \partial V \sin\phi)^2}$$

We get : $\partial V = \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}} and \phi = blx.$

(35) In Analysis, however, a method is still missing for dealing with integrating formulas of this kind: $\int \frac{x^{f-1}\partial x \cos blx}{\sqrt{1-x^{2a}}}$ and $\int \frac{x^{f-1}\partial x \sin blx}{\sqrt{1-x^{2a}}}$.[5]

Meanwhile, however, if the denominator is absent, both formulas are able to be integrated, which a jewel of my work shows in the following way.

(36) For this can be exhibited with the help of a famous reduction, $\int P\partial Q = PQ - \int Q\partial P$.

If $P = \cos blx$ and $\partial Q = x^{f-1}\partial x$ is assumed for the previous formula, we get:

$$\int x^{f-1}\partial x \cos blx = \frac{x^f}{f}\cos blx + \frac{b}{f}\int x^{f-1}\partial x \sin blx.$$

However, for the other formula, if $P = \sin bix$ and $\partial Q = x^{f-1}\partial x$ is assumed, it will be

$$\int x^{f-1}\partial x \sin blx = \frac{x^f}{f}\sin blx - \frac{b}{f}\int x^{f-1}\partial x \cos blx.$$

From here we get by substituting

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[5] **Recall that Euler uses $lx = \log x$.**

$$\int x^{f-1} \partial x \cos blx = \frac{x^f}{ff + bb}(f \cos blx + b \sin blx);$$

$$\int x^{f-1} \partial x \sin blx = \frac{x^f}{ff + bb}(f \sin blx - b \cos blx).$$

However, when including the denominator, nothing else is known except to express the integral as a kind of transcendental quantity that is still unknown.