A specimen of a singular transformation of series*

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§1 I considered this series

$$s = 1 + \frac{ab}{1 \cdot c}x + \prod \frac{(a+1)(b+1)}{2 \cdot (c+1)}x^2 + \prod \frac{(a+2)(b+2)}{3 \cdot (c+2)}x^3 + \text{etc.}$$

where in usual manner Π denotes the coefficient of the preceding term. This series has the property, that its sum seems to be impossible to exhibit in general, although in all cases, in which either a or b is a whole negative number, it truncates and its sum is expressed in finite terms.

§2 Hence if we put

$$s = z(1-x)^{c-a-b}$$

and further let be

$$c - a = \alpha \quad \text{and} \quad c - b = \beta$$

the letter z will express the sum of the following series, quite similar to the preceding one,

$$z = 1 + \frac{\alpha\beta}{1 \cdot c}x + \prod \frac{(\alpha+1)(\beta+1)}{2(\alpha+1)}x^2 + \prod \frac{(\alpha+2)(\beta+2)}{3(c+2)}x^3 + \text{etc.},$$

which now truncates in all cases, in which either α or β is a positive integer, and hence as often as either $a - c$ or $b - c$ is a positive integer.

*Original titele: „Specimen transformationis singularis serierum“, first published in „*Nova Acta Academiae Scientarum Imperialis Petropolitinae* 12, 1801, pp. 58-70“, reprinted in „*Opera Omnia*: Series 1, Volume 16,2, pp. 41 - 55 “, Eneström-Number E710, translated by: Alexander Aycock, for the project „Euler-Kreis Mainz“

§3 This transformation is to be considered to be of even higher importance, because it seems that it cannot be found in a straight-forward way and even only by means of a differential equation of second order. Hence it will be worth the effort, to have explained the whole analysis, on which this transformation is based.

§4 Because it is

$$s = 1 + \frac{ab}{1 \cdot c}x + \frac{ab}{1 \cdot c} \cdot \frac{(a+1)(b+1)}{2(c+1)} \cdot xx + \text{etc.},$$

and hence in each following term the numerator as the denominator gets two new factors, let us eliminate the two last factors from each term by differentiation, which is achieved by these operations

$$\frac{\partial s}{\partial x} = \frac{ab}{1 \cdot c} + \frac{ab}{1 \cdot c}\frac{(a+1)(b+1)}{c+1}x + \text{etc.},$$

which multiplied x^c and differentiated again yields

$$\partial \cdot x^c \partial s = abx^{c-1} + \frac{ab}{1 \cdot c}(a+1)(b+1)x^c + \text{etc.},$$

where we, seeking for brevity, omitted the element ∂x, which can be easily be remembered in the following passages.

§5 Now in the same way let us add two new factors to the single numerators as follows:

1. Our series, multiplied by x^a and differentiated, will give

$$\partial \cdot x^a s = ax^{a-1} + \frac{ab}{1 \cdot c}(a+1)x^a + \text{etc.},$$

which

2. multiplied by x^{b+1-a} and differentiated again yields

$$\partial \cdot x^{b+1-a}\partial x^a s = abx^{b-1} + \frac{ab}{1 \cdot c}(a+1)(b+1)x^b + \text{etc.},$$

which form arises from the preceding, if that one is muliplied by x^{b-c}.

§6 Hence we obtain this equation

$$\partial \cdot x^{b-a+1}\partial \cdot x^a s = x^{b-c}\partial \cdot x^c \partial s,$$

which equation expanded is reduced to this form

$$x^{b+1}\partial\partial s + (a+b+1)x^b \partial s + abx^{b-1}s = x^b \partial\partial s + cx^{b-1}\partial s.$$

This equation, after having divided by x^{b-1} and having translated all terms to the right-hand side, will attain this form

$$0 = x(1-x)\partial\partial s + \left[c - (a+b+1)x\right]\partial s - abs$$

so that the summation of the propounded series depends on the resolution of this differential equation of second order. But this equation seems to be conditioned in such a way, that it in general admits no integration.

§7 But although this differential equation hardly seems to be of any use for us, one can perform an extraordinary trasformation, that completes our whole task. For let us use this general substitution

$$s = (1-x)^n z,$$

whence it is

$$\log s = n \log (1-x) + \log z,$$

and by differentiation it will be

$$\frac{\partial s}{s} = \frac{\partial z}{z} - \frac{n\partial x}{1-x},$$

which equation, differentiated again, yields

$$\frac{\partial\partial s}{s} - \frac{\partial s^2}{ss} = \frac{\partial\partial z}{z} - \frac{\partial z^2}{zz} - \frac{n\partial x^2}{(1-x)^2}.$$

To this one let us add this equation

$$\frac{\partial s^2}{ss} = \frac{\partial z^2}{zz} - \frac{2n\partial x \partial z}{z(1-x)} + \frac{nn\partial x^2}{(1-x)^2}$$

and it will arise

$$\frac{\partial\partial s}{s} = \frac{\partial\partial z}{z} - \frac{2n\partial x\partial z}{z(1-x)} + \frac{n(n-1)\partial x^2}{(1-x)^2}.$$

§8 Hence if the propounded equation, divided by s, is represented as follows

$$0 = x(1-x)\frac{\partial\partial s}{s} + [c-(a+b+1)x]\,\frac{\partial s}{s} - abs$$

we will, after having done the substitution, get to a differential equation of second order between z and x, which will be

$$\begin{aligned} x(1-x)\frac{\partial\partial z}{z} - \frac{2nx\partial x\partial z}{z} + [c-(a+b+1)x]\,\frac{\partial z}{z} \\ + \frac{n(n-1)x\partial x^2}{1-x} - \frac{n[c-(a+b+1)x]\partial x}{1-x} - ab = 0. \end{aligned}$$

§9 But here it is evident, that the number n can be assumed in such a way, that the last members having the denominator $1-x$, can be divided by it, what happens in the case $n = -a-b+c$, after having introduced which value, that it is $s = (1-x)^{c-a-b}z$, the equation between z and x will attain this form

$$x(1-x)\partial\partial z + [c+(a+b-2c-1)x]\,\partial z - (c-a)(c-b)z = 0$$

§10 If we put $c-a=\alpha$ and $c-b=\beta$ in this equation, the equation between z and x will appear in this form:

$$x(1-x)\partial\partial z + [c-(\alpha+\beta+1)x]\,\partial z - \alpha\beta z = 0,$$

which differs from the first one only in that aspect, that instead of the letters a and b we here have α and β. Because the first difference-differential equation arose from this series

$$s = 1 + \frac{ab}{1\cdot c}x + \prod\frac{(a+1)(b+1)}{2(c+1)}xx + \prod\frac{(a+2)(b+2)}{3(c+2)}x^3 + \text{etc.},$$

vice versa from the last equation this series will arise

$$z = 1 + \frac{\alpha\beta}{1\cdot c}x + \prod\frac{(\alpha+1)(\beta+1)}{2(c+1)}xx + \prod\frac{(\alpha+1)(\beta+1)}{3(c+2)}x^3 + \text{etc.},$$

while $\alpha = c-a$ and $\beta = c-b$; and these two series s and z depend on each other in such a way, that it is $s = (1-x)^{c-a-b}z$ and $\frac{s}{z} = (1-x)^{c-a-b}$.

§11 But from the last difference-differential equation by a direct method one can find the same series for z. Hence because from the first series for $x = 0$ it is $s = 1$, but we now put $z = (1-x)^{a+b-c}$, in the same case $x = 0$ it will be $z = s = 1$. Having noted this, let us assume a series for z:

$$z = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{etc.},$$

whence it is

$$\partial z = A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + \text{etc.}$$

and

$$\partial\partial z = 2B + 6Cx + 12Dx^2 + 20Ex^3 + 30Fx^4 + \text{etc.}$$

after having substitued which values this becomes

$$\begin{array}{rlllll} x(1-x)\partial\partial z & = & 2Bx & +6Cx^2 & +12Dx^3 & +\text{etc.} \\ & & & -2Bx^2 & -6Cx^3 & -\text{etc.} \\ c\partial z & = Ac & +Bcx & +3Ccx^2 & +4Dcx^3 & +\text{etc.} \\ -(\alpha+\beta+1)x\partial z & = & -(\alpha+\beta+1)Ax & -2(\alpha+\beta+1)Bx^2 & -3(\alpha+\beta+1)Cx^3 & -\text{etc.} \\ -\alpha\beta z & = -\alpha\beta & -A\alpha\beta x & -B\alpha\beta x^2 & -C\alpha\beta x^3 & -\text{etc.} \\ \hline \end{array}$$

$$x(1-x)\partial\partial z + c\partial z - (\alpha+\beta+1)x\partial z - \alpha\beta z = 0$$

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§12 Having put the single terms equal to zero we will reach the following equations:

I. $Ac - \alpha\beta = 0$
II. $2B(c+1) - (\alpha+1)(\beta+1)A = 0$
III. $3C(c+2) - (\alpha+2)(\beta+2)B = 0$
IV. $4D(c+3) - (\alpha+3)(\beta+3)C = 0$
V. $5E(c+4) - (\alpha+4)(\beta+4)D = 0$
etc.

§13 From there therefore the same coefficients, that we alread had, are found, of course

$$A = \frac{\alpha\beta}{1\cdot c}$$
$$B = \frac{A(\alpha+1)(\beta+1)}{2(c+1)}$$
$$C = \frac{B(\alpha+2)(\beta+2)}{3(c+2)}$$
$$D = \frac{C(\alpha+3)(\beta+3)}{4(c+3)}$$
etc.

But because the method, by which we obtained the extraordinary transformation, is quite strange and proceeds very non straight-forward, it is very much desired, that another more direct and more natural method is discovered, by which a increment not to be contemned would be brought into analysis.

§14 Because in these series the number of factors grows continuously, that we can use the characters, taken from the power of a binom, here more comfortable, let us attribute negative values to the letters a and b, and in the same way to the letters α and β, by putting

$$a=-f, \quad b=-g, \quad \alpha=-\zeta \quad \text{and} \quad \beta=-\eta$$

so that it is

$$\zeta=-c-f \quad \text{and} \quad \eta=-c-g,$$

and now our two series s and z will depend on each other in such a way, that it is

$$s=(1-x)^{c+f+g}z.$$

Now let us at first expand the series s in terms of these values, and it will be

$$s=1+\frac{fg}{1\cdot c}x+\prod\frac{(f-1)(g-1)}{2(c+1)}x^2+\prod\frac{(f-2)(g-2)}{3(c+2)}x^3+\text{etc.}$$

and in the same way the second series will be

$$z=1+\frac{\zeta\eta}{1\cdot c}x+\prod\frac{(\zeta-1)(\eta-1)}{2(c+1)}x^2+\prod\frac{(\zeta-2)(\eta-2)}{3(c+2)}x^3+\text{etc..}$$

§15 Here we can already use the mentioned characters in a comfortable way. So let $\left(\frac{m}{n}\right)$ denote the coefficient of the term v^n, which corresponds to the same in the expansion of the power of the binom, $(1+v)^m$, so that in this manner we have

$$(1+v)^m = 1 + \left(\frac{m}{1}\right) v + \left(\frac{m}{2}\right) v^2 + \left(\frac{m}{3}\right) v^3 + \text{etc..}$$

Hence for the first of our series it will be $\frac{f}{1} = \left(\frac{f}{1}\right)$; further $\frac{f(f-1)}{1\cdot 2} = \left(\frac{f}{2}\right)$; $\frac{f(f-1)(f-2)}{1\cdot 2\cdot 3} = \left(\frac{f}{3}\right)$ etc. and so this series is shorter represented as follows:

$$s = 1 + \frac{g}{c}\left(\frac{f}{1}\right) x + \frac{g}{c}\cdot\frac{g-1}{c+1}\cdot\left(\frac{f}{2}\right) x^2 + \frac{g}{c}\cdot\frac{g-1}{c+1}\cdot\frac{g-2}{c+2}\cdot\left(\frac{f}{3}\right) x^3 + \text{etc.}$$

To contract the terms, containing the letter g, in the same way, let us multiply the series by the character $\left(\frac{g+c-1}{c-1}\right)$ on both sides; for it will be

$$\left(\frac{g+c-1}{c-1}\right)\frac{g}{c} = \left(\frac{g+c-1}{c}\right); \left(\frac{g+c-1}{c-1}\right)\frac{g}{c}\cdot\frac{g-1}{c+1} = \left(\frac{g+c-1}{c+1}\right),$$

which expression further multiplied by $\frac{g-2}{c+2}$ will give this character $\left(\frac{g+c-1}{c+2}\right)$. Having noted all this, we obtain this series:

$$s\left(\frac{g+c-1}{c-1}\right) = \left(\frac{g+c-1}{c-1}\right) + \left(\frac{f}{1}\right)\left(\frac{g+c-1}{c}\right) x + \left(\frac{f}{2}\right)\left(\frac{g+c-1}{c+1}\right) x^2 + \left(\frac{f}{3}\right)\left(\frac{g+c-1}{c+2}\right) x^3 + \text{etc.}$$

§16 In the same way it will be possible to transform the other series; but there it is to be noted, that this transformation can be done in two ways, depending on whether the factors of the denominators 1, 2, 3, 4 etc. are either combined with the letter ζ or with η. At first we will therefore from the first series, if we write ζ instead of f and η instead of g, obtain this series:

$$z\left(\frac{\eta+c-1}{c-1}\right) = \left(\frac{\eta+c-1}{c-1}\right) + \left(\frac{\zeta}{1}\right)\left(\frac{\eta+c-1}{c}\right) x + \left(\frac{\zeta}{2}\right)\left(\frac{\eta+c-1}{c+1}\right) x^2 + \left(\frac{\zeta}{3}\right)\left(\frac{\eta+c-1}{c+2}\right) x^3 + \text{etc.}$$

But if we instead of f and g in inverse order write η and ζ, it arises

$$z\left(\frac{\zeta+c-1}{c-1}\right)=\left(\frac{\zeta+c-1}{c-1}\right)+\left(\frac{\eta}{1}\right)\left(\frac{\zeta+c-1}{c}\right)x+\left(\frac{\eta}{2}\right)\left(\frac{\zeta+c-1}{c+1}\right)x^2$$
$$+\left(\frac{\eta}{3}\right)\left(\frac{\zeta+c-1}{c+2}\right)x^3+\text{etc.}$$

But for each of both the relation remains the same, of course

$$s=(1-x)^{c+f+g}z$$

§17 That it becomes clearer, how much these two series found for z differ from each other, let us write the assumed values instead of ζ and η, of course

$$\zeta=-c-f \quad\text{and}\quad \eta=-c-g$$

and the two last series for the letter z will be:

$$z\left(\frac{-g-1}{c-1}\right)=\left(\frac{-g-1}{c-1}\right)+\left(\frac{-c-f}{1}\right)\left(\frac{-g-1}{c}\right)x+\left(\frac{-c-f}{2}\right)\left(\frac{-g-1}{c+1}\right)x^2+\text{etc.}$$
$$z\left(\frac{-f-1}{c-1}\right)=\left(\frac{-f-1}{c-1}\right)+\left(\frac{-c-g}{1}\right)\left(\frac{-f-1}{c}\right)x+\left(\frac{-c-g}{2}\right)\left(\frac{-f-1}{c+1}\right)x^2+\text{etc.}$$

§18 To reduce these series to a more convenient from, let us put

$$g+c-1=h \quad\text{and}\quad c-1=e,$$

so that it is

$$c=e+1 \quad\text{and}\quad g=h-e;$$

hence our principal series will be

$$s\left(\frac{h}{e}\right)=\left(\frac{h}{e}\right)+\left(\frac{f}{1}\right)\left(\frac{h}{e+1}\right)x+\left(\frac{f}{2}\right)\left(\frac{h}{e+2}\right)x^2+\left(\frac{f}{3}\right)\left(\frac{h}{e+3}\right)x^3+\text{etc.}$$

But the two following series, formed from the letter z, will be:
the first

$$z\left(\frac{e-h-1}{e}\right)=\left(\frac{e-h-1}{e}\right)+\left(\frac{-e-f-1}{1}\right)\left(\frac{e-h-1}{e+1}\right)x$$
$$+\left(\frac{-e-f-1}{2}\right)\left(\frac{e-h-1}{e+2}\right)x^2+\text{etc.}$$

the second

$$z\left(\frac{-f-1}{e}\right)=\left(\frac{-f-1}{e}\right)+\left(\frac{-1-h}{1}\right)\left(\frac{-f-1}{e+1}\right)x$$
$$+\left(\frac{-1-h}{2}\right)\left(\frac{-f-1}{e+2}\right)x^2+\text{etc.};$$

but both quantities s and z depend on each other in such a way, that it is

$$s=(1-x)^{f+b+1}z.$$

§19 Now let us show the great use of this transformation in a most memorable case, taken from the integral formula

$$\int\frac{\partial\varphi\cos i\varphi}{(1+aa-2a\cos\varphi)^{n+1}},$$

whose integral from the boundary $\varphi=0$ to the boundary $\varphi=180°$ I, based on conjecture alone, concluded to be

$$=\frac{\pi a^i}{(1-aa)^{2n+1}}V,$$

with

$$V=\left(\frac{n-1}{0}\right)\left(\frac{n+i}{i}\right)+\left(\frac{n-i}{1}\right)\left(\frac{n+i}{i+1}\right)aa+\left(\frac{n-i}{2}\right)\left(\frac{n+i}{i+2}\right)a^4+\text{etc.};$$

this series, if it is compared to our principal one, that it is

$$V=s\left(\frac{h}{e}\right),$$

will yield

$$h=n+i\ \text{ and }\ e=i,$$

but then

$$f=n-i\ \text{ and }\ x=aa;$$

therefore the other series, formed from there, will be:
the first

$$z\left(\frac{-n-1}{i}\right)=\left(\frac{-n-1}{i}\right)+\left(\frac{-n-1}{i}\right)\left(\frac{-n-1}{i+1}\right)a^2$$

$$+\left(\frac{-n-1}{2}\right)\left(\frac{-n-1}{i+2}\right)a^4 + \text{etc.}$$

the second

$$z\left(\frac{i-n-1}{i}\right) = \left(\frac{i-n-1}{i}\right) + \left(\frac{-n-i-1}{1}\right)\left(\frac{i-n-1}{i+1}\right)a^2$$

$$+\left(\frac{-n-i-1}{2}\right)\left(\frac{i-n-1}{i+2}\right)a^4 + \text{etc.},$$

which series arises from the series V itself by writing $-n-1$ instead of n. But this relation between s and z will be

$$s = (1-aa)^{2n+1}z;$$

but then it is

$$V = s\left(\frac{n+i}{i}\right).$$

§20 Because therefore it is

$$\int \frac{\partial\varphi \cos i\varphi}{(1+aa-2a\cos\varphi)^{n+1}} = \frac{\pi a^i}{(1-aa)^{2n+1}}V = \frac{\pi a^i}{(1-aa)^{2n+1}}\left(\frac{n+i}{i}\right)s,$$

let us in this form write $-n-1$ instead of n and let be

$$\int \frac{\partial\phi \cos i\phi}{(1+aa-2a\cos\phi)^{-n}} \begin{bmatrix} \text{from} & \phi & = & 0^\circ \\ \text{to} & \phi & = & 180^\circ \end{bmatrix} = \frac{\pi a^i}{(1-aa)^{-2n-1}}U,$$

it will be

$$U = \left(\frac{-n-1-i}{0}\right)\left(\frac{-n-1+i}{i}\right) + \left(\frac{-n-1-i}{1}\right)\left(\frac{-n-1+i}{i+1}\right)aa + \text{etc.}$$

and therefore

$$U = z\left(\frac{i-n-1}{i}\right) = \left(\frac{i-n-1}{i}\right)(1-aa)^{-2n-1}s.$$

§21 Now let us put

$$1 + aa - 2a\cos\phi = \Delta$$

and let us consider these two values of the integrals, that we just obtained:

$$\text{I.} \quad \int \frac{\partial\phi \cos i\phi}{\Delta^{n+1}} = \frac{\pi a^i}{(1-aa)^{2n+1}} \left(\frac{n+i}{i}\right) s$$

$$\text{II.} \quad \int \Delta^n \partial\phi \cos i\phi = \frac{\pi a^i}{(1-aa)^{-2n-1}} \left(\frac{i-n-1}{i}\right) (1-aa)^{-2n-1} s = \pi a^i \left(\frac{i-n-1}{i}\right) s;$$

as a consequence we obtain this most memorable relation between these to integral formulas, extended from the boundary $\varphi = 0$ to the boundary $\varphi = 180^\circ$:

$$\int \frac{\partial\varphi \cos i\phi}{\Delta^{n+1}} : \int \Delta^n \partial \cos i\varphi = \left(\frac{n+i}{i}\right) : \left(\frac{i-n-1}{i}\right) (1-aa)^{2n+1}$$

or it will be

$$\left(\frac{n+i}{i}\right) (1-aa)^{-n} \int \Delta^n \partial\varphi \cos i\varphi = \left(\frac{-n-1+i}{i}\right) (1-aa)^{n+1} \int \Delta^{-n-1} \partial\phi \cos i\varphi.$$

§22 I had already found this last theorem some time ago by induction alone, and I almost despaired of its proof, which now follows directly from the mentioned transformation of the series; hence the very great use of this transformation, which is validly to be considered to be of largest profoundity, is seen even more.

§23 But after having propounded the same theorem some time ago, the ratio between the two integral formulas given there seems to differ from the one found here quite a bit; but nevertheless they are detected to agree perfectly, if one just uses the following proportion, according to which it is in general

$$\left(\frac{n}{i}\right) : \left(\frac{-n-1}{i}\right) = \left(\frac{-n-1+i}{i}\right) : \left(\frac{n+i}{i}\right),$$

the reason for which is obvious from this, that is in general

$$\left(\frac{-a}{i}\right) = \pm \left(\frac{a+i-1}{i}\right)$$

and therefore also

$$\left(\frac{b}{i}\right) = \pm\left(\frac{b-1+i}{i}\right),$$

where the superior signs hold, if i was a even number, the inferior ones on the other hand, if i was an odd number. Hence it will be

$$\left(\frac{n+i}{i}\right) = \pm\left(\frac{-n-1}{i}\right) \quad \text{und} \quad \left(\frac{-n-1+i}{i}\right) = \pm\left(\frac{n}{i}\right).$$

§24 Hence our theorem can be enunciated even more comfortable. If we put for the sake of brevity

$$\frac{1+aa-2a\cos\phi}{1-aa} = \Theta,$$

so that it is

$$\Delta = (1-aa)\Theta$$

then this proportion will arise:

$$\int \Theta^n \partial\varphi \cos i\varphi : \int \frac{\partial\varphi \cos i\varphi}{\Theta^{n+1}} = \int \frac{\Delta^n \partial \cos i\varphi}{(1-aa)^n} : \int \frac{\partial\phi \cos i\varphi (1-aa)^{n+1}}{\Delta^{n+1}}$$

$$= \left(\frac{-n-1+i}{i}\right) : \left(\frac{n+i}{i}\right) = \left(\frac{n}{i}\right) : \left(\frac{-n-1}{i}\right)$$

and so it will be

$$\left(\frac{n}{i}\right) \int \frac{\partial\varphi \cos i\varphi}{\Theta^{n+1}} = \left(\frac{-n-1}{i}\right) \int \Theta^n \partial\varphi \cos i\varphi.$$

Theorem

§25 *If the sum of this series was known:*

$$\frac{h}{e} + \left(\frac{f}{1}\right)\left(\frac{h}{e+1}\right)x + \left(\frac{f}{2}\right)\left(\frac{h}{e+2}\right)x^2 + \left(\frac{f}{3}\right)\left(\frac{h}{e+3}\right)x^3 + \text{etc.},$$

which sum we want to put $= S$*, then also the sums of the following two series can be exhibited, the first of which is this:*

$$\left(\frac{e-h-1}{e}\right) + \left(\frac{-e-f-1}{1}\right)\left(\frac{e-h-1}{e+1}\right)x + \left(\frac{-e-f-1}{2}\right)\left(\frac{e-h-1}{e+2}\right)x^2 + \text{etc.},$$

whose sum will be

$$\left(\frac{e-h-1}{e}\right)\frac{s}{\left(\frac{h}{e}\right)(1-x)^{f+h+1}},$$

where it is to be noted, that it is

$$\left(\frac{e-h-1}{e}\right)=\pm\left(\frac{h}{e}\right),$$

where the superior sign holds, if i is an even number, the inferior, if an odd one; hence the sum of the series is

$$\frac{\pm s}{(1-x)^{f+h+1}}$$

The other series, whose sum can be defined from there, will be

$$\left(\frac{-f-1}{e}\right)+\left(\frac{-h-1}{1}\right)\left(\frac{-f-1}{e+1}\right)x+\left(\frac{-h-1}{2}\right)\left(\frac{-f-1}{e+2}\right)x^2+\text{etc.},$$

whose sum will be

$$\left(\frac{-f-1}{e}\right)\frac{s}{\left(\frac{h}{e}\right)(1-x)^{f+h+1}},$$

which can also be expressed in this way:

$$\pm\left(\frac{f+e}{e}\right)\frac{s}{\left(\frac{h}{e}\right)(1-x)^{f+h+1}}.$$

§26 If the sums of these three series are stated as follows

$$\mathfrak{A}=\left(\frac{h}{e}\right)+\left(\frac{f}{1}\right)\left(\frac{h}{e+1}\right)x+\left(\frac{f}{2}\right)\left(\frac{h}{e+2}\right)x^2+\left(\frac{f}{3}\right)\left(\frac{h}{e+3}\right)x^3+\text{etc.};$$

$$\mathfrak{B}=\left(\frac{e-h-1}{e}\right)+\left(\frac{-e-f-1}{1}\right)\left(\frac{e-h-1}{e+1}\right)x+\left(\frac{-e-f-1}{2}\right)\left(\frac{e-h-1}{e+2}\right)x^2+\text{etc.};$$

$$\mathfrak{C}=\left(\frac{-f-1}{e}\right)+\left(\frac{-h-1}{1}\right)\left(\frac{-f-1}{e+1}\right)x+\left(\frac{-h-1}{2}\right)\left(\frac{-f-1}{e+2}\right)x^2+\text{etc.},$$

they are related to each other, that it is

$$\left(\frac{e-h-1}{e}\right)\mathfrak{A}=\left(\frac{h}{e}\right)(1-x)^{f+h+1}\mathfrak{B}$$

$$\left(\frac{-f-1}{e}\right)\mathfrak{A}=\left(\frac{h}{e}\right)(1-x)^{f+h+1}\mathfrak{C}$$

$$\left(\frac{-f-1}{e}\right)\mathfrak{B}=\left(\frac{e-h-1}{e}\right)\mathfrak{C}.$$