On a new type of rational and highly convergent series, by which the ratio of the circumference to the diameter is able to be expressed*

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1. The principle, from which these series are deduced, rests in this binomial formula: $4+x^4$, which is evidently composed of these two rational factors: $2+2x+xx$ and $2-2x+xx$. Then indeed it at once follows for this integral formula: $\int \frac{\partial x(2+2x+xx)}{4+x^4}$, which we shall indicate with the sign $\odot$, to be reduced to this: $\odot = \int \frac{\partial x}{2-2x+xx}$, whose integral, having been obtained so that it vanishes when it is put $x=0$, is Atang. $\frac{x}{2-x}$. Whereby it may be observed in the case $x=1$ to be $\odot = \frac{\pi}{4}$; while indeed in the case $x=\frac{1}{2}$ it will be $\odot = \text{Atang.}\,\frac{1}{3}$; then indeed in the case $x=\frac{1}{4}$ it will be $\odot = \text{Atang.}\,\frac{1}{7}$. It is noted moreover for it to be

$$2\,\text{Atang.}\,\frac{1}{3} + \text{Atang.}\,\frac{1}{7} = \text{Atang.}\,1 = \frac{\pi}{4}.$$

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2. With therefore this integral formula indicated by the sign $\odot$ which is comprised by three parts, each of which we shall unfold separately, which by the grace of brevity we shall indicate by the following characters:

$$\text{I.} \int \frac{\partial x}{4+x^4} = \text{♄}; \quad \text{II.} \int \frac{x\partial x}{4+x^4} = \text{♃}; \quad \text{III.} \int \frac{xx\partial x}{4+x^4} = \text{♂};$$

so that it will thus be

$$\odot = 2\text{♄} + 2\text{♃} + \text{♂} = \text{Atang.}\, \frac{x}{2-x}.$$

Now therefore we may unfold these three integral formulas in the usual manner into infinite series, which are thereupon to be formed, insofar as it will be

$$\frac{1}{4+x^4} = \frac{1}{4}\Big(1 - \frac{x^4}{4} + \frac{x^8}{4^2} - \frac{x^{12}}{4^3} + \frac{x^{16}}{4^4} - \text{ etc.}\Big).$$

3. But if now first we adjoin this series with ∂x and we then integrate, the first formula ♄ will be expressed by the following series:

$$\text{♄} = \frac{x}{4}[1 - \frac{1}{5}\cdot\frac{x^4}{4} + \frac{1}{9}\Big(\frac{x^4}{4}\Big)^2 - \frac{1}{3}\Big(\frac{x^4}{4}\Big)^3 + \text{ etc.}].$$

While indeed adjoining the former series by $x\partial x$ and integrating gives

$$\text{♃} = \frac{xx}{8}[1 - \frac{1}{3}\cdot\frac{x^4}{4} + \frac{1}{5}\Big(\frac{x^4}{4}\Big)^2 - \frac{1}{7}\Big(\frac{x^4}{4}\Big)^3 + \text{ etc.}].$$

Then adjoining the very same series with $xx\partial x$ and integrating produces

$$\text{♂} = \frac{x^3}{4}[\frac{1}{3} - \frac{1}{7}\frac{x^4}{4} + \frac{1}{11}\Big(\frac{x^4}{4}\Big)^2 - \frac{1}{15}\Big(\frac{x^4}{4}\Big)^3 + \text{ etc.}].$$

4. With therefore it being $\odot = 2\text{♄} + 2\text{♃} + \text{♂}$, we shall unfold some particular cases recalled from before, in which it is $x = 1$, $x = \frac{1}{2}$ and $x = \frac{1}{4}$, of which the first is $\frac{x^4}{4} = \frac{1}{4}$; for the second indeed it is $\frac{x^4}{4} = \frac{1}{64}$; for the third indeed $\frac{x^4}{4} = \frac{1}{1024}$; from which it stands open for the two last cases to converge most greatly, but that the first, whose terms decrease by a ratio of four, indeed converges more so than the series of Leibnitz, by taking an arc whose tangent is $\frac{1}{\sqrt{3}}$, seeing that this calculation is perturbed by no irrational.

The expansion of the first case, where $x = 1$ and $\odot = \text{Atang.}\ \frac{\pi}{4}$.

5. Seeing therefore here that it is $\frac{x^4}{4} = \frac{1}{4}$, our three principal series for $♄, ♃, \odot$ [sic] proceed in the following way:

$$
\begin{aligned}
♄ &= \tfrac{1}{4}[1 - \tfrac{1}{5}\cdot\tfrac{1}{4} + \tfrac{1}{9}(\tfrac{1}{4})^2 - \tfrac{1}{13}(\tfrac{1}{4})^3 + \tfrac{1}{17}(\tfrac{1}{4})^4 - \text{ etc.}]\\
♃ &= \tfrac{1}{8}[1 - \tfrac{1}{3}\cdot\tfrac{1}{4} + \tfrac{1}{5}(\tfrac{1}{4})^2 - \tfrac{1}{7}(\tfrac{1}{4})^3 + \tfrac{1}{9}(\tfrac{1}{4})^4 - \text{ etc.}]\\
♂ &= \tfrac{1}{4}[\tfrac{1}{3} - \tfrac{1}{7}\cdot\tfrac{1}{4} + \tfrac{1}{11}(\tfrac{1}{4})^2 - \tfrac{1}{13}(\tfrac{1}{4})^3 + \tfrac{1}{19}(\tfrac{1}{4})^4 - \text{ etc.}]
\end{aligned}
$$

6. Seeing therefore that it is $\odot = 2♄ + 2♃ + ♂ = \frac{\pi}{4}$, by multiplying the value of π by 4, the following three series are expressed

$$
\pi = \begin{cases}
2(1 - \frac{1}{5}\cdot\frac{1}{4} + \frac{1}{9}\cdot\frac{1}{4^2} - \frac{1}{13}\cdot\frac{1}{4^3} + \frac{1}{17}\cdot\frac{1}{4^4} - \text{ etc.})\\
1(1 - \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4^2} - \frac{1}{7}\cdot\frac{1}{4^3} + \frac{1}{9}\cdot\frac{1}{4^4} - \text{ etc.})\\
1(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{4} + \frac{1}{11}\cdot\frac{1}{4^2} - \frac{1}{15}\cdot\frac{1}{4^3} + \frac{1}{19}\cdot\frac{1}{4^4} - \text{ etc.})
\end{cases}
$$

7. From these particular three series the ratio of the circumference to the diameter is able to be calculated with much less work than by the series of Leibnitz, which method the most meritous Authors Sharp, Machin and de Lagny have used, of whom the first has determined π in a decimal fraction to 72 figures, the second to 100, and the last indeed to 128. And truly they were able to lift up the following cases with much more effort.

The expansion of the second case, where $x = \frac{1}{2}$.

8. In this case it will therefore be $\frac{x^4}{4} = \frac{1}{64}$, from which the three series are drawn forth in the following way:

$$
\begin{aligned}
♄ &= \tfrac{1}{8}(1 - \tfrac{1}{5}\cdot\tfrac{1}{64} + \tfrac{1}{9}\cdot\tfrac{1}{64^2} - \tfrac{1}{13}\cdot\tfrac{1}{64^3} + \text{ etc.})\\
♃ &= \tfrac{1}{32}(1 - \tfrac{1}{3}\cdot\tfrac{1}{64} + \tfrac{1}{5}\cdot\tfrac{1}{64^2} - \tfrac{1}{7}\cdot\tfrac{1}{64^3} + \text{ etc.})\\
♂ &= \tfrac{1}{32}(\tfrac{1}{3} - \tfrac{1}{7}\cdot\tfrac{1}{64} + \tfrac{1}{11}\cdot\tfrac{1}{64^2} - \tfrac{1}{15}\cdot\tfrac{1}{64^3} + \text{ etc.})
\end{aligned}
$$

9. Therefore with it $2♄ + 2♃ + ♂ = \text{Atang.}\,\frac{1}{3}$, it will be

$$\text{Atang.}\,\frac{1}{3} = \begin{cases} \frac{1}{4}(1 - \frac{1}{5}\cdot\frac{1}{64} + \frac{1}{9}\cdot\frac{1}{64^2} - \frac{1}{13}\cdot\frac{1}{64^3} + \text{ etc.}) \\ \frac{1}{16}(1 - \frac{1}{3}\cdot\frac{1}{64} + \frac{1}{5}\cdot\frac{1}{64^2} - \frac{1}{7}\cdot\frac{1}{64^3} + \text{ etc.}) \\ \frac{1}{32}(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{64} + \frac{1}{11}\cdot\frac{1}{64^2} - \frac{1}{15}\cdot\frac{1}{64^3} + \text{ etc.}) \end{cases}$$

Even though here these three series are to be computed, however, because each successively decreases by the same ratio $1 : 64$, this labor will be able to be shortened in a wonderful way.

The expansion of the third case, where $x = \frac{1}{4}$.

10. Seeing therefore here that it is $\frac{x^4}{4} = \frac{1}{1024}$, our three principal series will be had as follows:

$$\begin{aligned} ♄ &= \tfrac{1}{16}(1 - \tfrac{1}{5}\cdot\tfrac{1}{1024} + \tfrac{1}{9}\cdot\tfrac{1}{1024^2} - \tfrac{1}{13}\cdot\tfrac{1}{1024^3} + \text{ etc.}) \\ ♃ &= \tfrac{1}{128}(1 - \tfrac{1}{3}\cdot\tfrac{1}{1024} + \tfrac{1}{5}\cdot\tfrac{1}{1024^2} - \tfrac{1}{7}\cdot\tfrac{1}{1024^3} + \text{ etc.}) \\ ♂ &= \tfrac{1}{256}(\tfrac{1}{3} - \tfrac{1}{7}\cdot\tfrac{1}{1024} + \tfrac{1}{11}\cdot\tfrac{1}{1024^2} - \tfrac{1}{15}\cdot\tfrac{1}{1024^3} + \text{ etc.}) \end{aligned}$$

11. Therefore with $2♄ + 2♃ + ♂ = \text{Atang.}\,\frac{1}{7}$, it will properly be by joining these series:

$$\text{Atang.}\,\frac{1}{7} = \begin{cases} \frac{1}{8}(1 - \frac{1}{5}\cdot\frac{1}{1024} + \frac{1}{9}\cdot\frac{1}{1024^2} - \frac{1}{13}\cdot\frac{1}{1024^3} + \text{ etc.}) \\ \frac{1}{64}(1 - \frac{1}{3}\cdot\frac{1}{1024} + \frac{1}{5}\cdot\frac{1}{1024^2} - \frac{1}{7}\cdot\frac{1}{1024^3} + \text{ etc.}) \\ \frac{1}{256}(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{1024} + \frac{1}{11}\cdot\frac{1}{1024^2} - \frac{1}{15}\cdot\frac{1}{1024^3} + \text{ etc.}) \end{cases}$$

An application of the last two cases for expressing a circumference by highly convergent series.

12. With, as we have already observed, $\frac{\pi}{4} = 2\,\text{Atang.}\,\frac{1}{3} + \text{Atang.}\,\frac{1}{7}$ it will be $\pi = 8\,\text{Atang.}\,\frac{1}{3} + 4\,\text{Atang.}\,\frac{1}{7}$, by substituting the value of π into the series

found above, the following six series may be expressed together:

$$\pi = \begin{cases} 2(1 - \frac{1}{5} \cdot \frac{1}{64} + \frac{1}{9} \cdot \frac{1}{64^2} - \frac{1}{13} \cdot \frac{1}{64^3} + \text{ etc.}) \\ \frac{1}{2}(1 - \frac{1}{3} \cdot \frac{1}{64} + \frac{1}{5} \cdot \frac{1}{64^2} - \frac{1}{7} \cdot \frac{1}{64^3} + \text{ etc.}) \\ \frac{1}{4}(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{64} + \frac{1}{11} \cdot \frac{1}{64^2} - \frac{1}{15} \cdot \frac{1}{64^3} + \text{ etc.}) \\ \frac{1}{2}(1 - \frac{1}{5} \cdot \frac{1}{1024} + \frac{1}{9} \cdot \frac{1}{1024^2} - \frac{1}{13} \cdot \frac{1}{1024^3} + \text{ etc.}) \\ \frac{1}{16}(1 - \frac{1}{3} \cdot \frac{1}{1024} + \frac{1}{5} \cdot \frac{1}{1024^2} - \frac{1}{7} \cdot \frac{1}{1024^3} + \text{ etc.}) \\ \frac{1}{64}(1 - \frac{1}{7} \cdot \frac{1}{1024} + \frac{1}{11} \cdot \frac{1}{1024^2} - \frac{1}{15} \cdot \frac{1}{1024^3} + \text{ etc.}) \end{cases}$$

This occurs as most noteworthy, because all these series proceed only by powers of two.