A SUCCINCT INTEGRATION OF THE MOST MEMORABLE INTEGRAL FORMULA

$$\int \frac{dz}{(3 \pm z^2)\sqrt[3]{1 \pm 3z^2}}.$$

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§1

Let us focus on the positive signs[1], and let

$$dV = \frac{dz}{(3 + z^2)\sqrt[3]{1 + 3z^2}};$$

*Translated by Hannah Scaer, Assisted by Karl-Dieter Crisman and Graeme D. Bird.

[1]Orig. Latin "signs above."

And if it is posited that $\sqrt[3]{1+3z^2} = v$ with the result that $1+3z^2 = v^3$, it will be the case that $zdz = \frac{1}{2}v^2dv$; therefore $dz = \frac{v^2dv}{2z}$, whence we get $dV = \frac{vdv}{2z(3+z^2)}$.

§2

Now let it be decided that $p = \frac{1+z}{v}$ and $q = \frac{1-z}{v}$, so it will be the case that $p^3+q^3 = 2$ and $p^3-q^3 = \frac{6z+2z^3}{v^3}$, from whence we get $dV = \frac{dv}{v^2(p^3-q^3)}$. Further, since it is the case that $p+q = \frac{2}{v}$, then $dp+dq = -\frac{2dv}{v^2}$, and therefore

$$dV = -\frac{(dp+dq)}{2(p^3-q^3)}\,.$$

§3

Now let this formula be split into two parts, by setting $\frac{dp}{p^3-q^3} = dP$ and $\frac{dq}{p^3-q^3} = dQ$, with the result that $dV = -\frac{1}{2}dP - \frac{1}{2}dQ$, and because $q^3 = 2-p^3$, it will be the case that $dP = -\frac{dp}{2(1-p^3)}$; then indeed because $p^3 = 2-q^3$, dQ will be equal to $+\frac{dq}{2(1-q^3)}$, and so we will have $4dV = +\frac{dp}{1-p^3} - \frac{dq}{1-q^3}$.

§4

Now since it is generally agreed that

$$\int \frac{dp}{1-p^3} = \frac{1}{3}\ln\frac{\sqrt{1+p+p^2}}{1-p} + \frac{1}{\sqrt{3}}\arctan\frac{p\sqrt{3}}{2+p}\,,$$

because of the fact that $1+p+p^2 = \frac{1-p^3}{1-p}$, [meaning that $\frac{1+p+p^2}{(1-p)^2} = \frac{1-p^3}{(1-p)^3}]^2$, it will be the case that

$$\int \frac{dp}{1-p^3} = \frac{1}{6}\ln\frac{1-p^3}{(1-p)^3} + \frac{1}{\sqrt{3}}\arctan\frac{p\sqrt{3}}{2+p}\,.$$

§5

Since therefore in a similar way:

$$\int \frac{dq}{1-q^3} = \frac{1}{6}\ln\frac{1-q^3}{(1-q)^3} + \frac{1}{\sqrt{3}}\arctan\frac{q\sqrt{3}}{2+q}\,,$$

four times the sought-for integral will be:

$$4V = \frac{1}{6}\ln\frac{1-p^3}{(1-p)^3} - \frac{1}{6}\ln\frac{1-q^3}{(1-q)^3} + \frac{1}{\sqrt{3}}\arctan\frac{p\sqrt{3}}{2+p} - \frac{1}{\sqrt{3}}\arctan\left(\frac{q\sqrt{3}}{2+q}\right)\,.$$

§6

But if now the logarithms should be pulled together in this way so that they become $\frac{1}{6}\ln\frac{1-p^3}{1-q^3} + \frac{1}{6}\ln\frac{(1-p)^3}{(1-q)^3}$, this expression, because $1-p^3 = -(1-q^3)$, becomes $\frac{1}{6}\ln(-1) + \frac{1}{2}\ln\frac{1-q}{1-p}$, where the part before, because it is a constant, can be omitted[3], so that having been joined together in this way, the logarithms which we are considering make $\frac{1}{2}\ln\frac{1-q}{1-p}$, and therefore it should be considered to be

$$4V = \frac{1}{2}\ln\frac{1-q}{1-p} + \frac{1}{\sqrt{3}}\arctan\frac{p\sqrt{3}}{2+p} - \frac{1}{\sqrt{3}}\arctan\frac{q\sqrt{3}}{2+q}\,.$$

[2] This line has been inserted because it appears to have been omitted by the printer.

[3] Note that Euler treats indeterminate complex quantities as constants.

Moreover the double circular arcs are united[4], namely

$$\frac{1}{\sqrt{3}}\arctan\frac{(p-q)\sqrt{3}}{2+p+q+2pq},$$

and thus the sought-for integral will be

$$V=\frac{1}{8}\ln\frac{1-q}{1-p}+\frac{1}{4\sqrt{3}}\arctan\frac{(p-q)\sqrt{3}}{2+p+q+2pq}.$$

§7

Since we have already appointed p as $\frac{1+z}{v}$ and q as $\frac{1-z}{v}$, the logarithmic piece will take on this form[5]:

$$\frac{1}{8}\ln\frac{v-1+z}{v-1-z}=\frac{1}{8}\ln\frac{1-v-z}{1-v+z}.$$

Moreover, to simplify the circular arc, $p-q=\frac{2z}{v}$, then truly because $p+q=\frac{2}{v}$ and[6] $pq=\frac{1-z^2}{v^2}$, the arc will become

$$\frac{1}{4\sqrt{3}}\arctan\frac{vz\sqrt{3}}{1+v+v^2-z^2}.$$

and thus we have reached this fully elegant integration:

$$\int\frac{dz}{(3+z^2)\sqrt[3]{1+3z^2}}=\frac{1}{8}\ln\frac{1-v-z}{1-v+z}+\frac{1}{4\sqrt{3}}\arctan\frac{vz\sqrt{3}}{1+v+v^2-z^2},$$

with v being equal to $\sqrt[3]{1+3z^2}$.

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[4]The relevant difference formula is $\arctan(x)-\arctan(y)=\arctan\left(\frac{x-y}{1+xy}\right)$; presumably in Euler's day this was commonly used, but it has now gone out of common use.

[5]Euler reached this form by clearing fractions.

[6]The following formula has no square in the denominator in the original.

§8

Now for the other case, in which the lower signs prevail, let us make $z = y\sqrt{-1}$, with the result that $v = \sqrt[3]{1-3y^2}$, from which the above integration becomes:

$$\int \frac{dy\sqrt{-1}}{(3-y^2)\sqrt[3]{1-3y^2}} = \frac{1}{8}\ln\frac{1-v-y\sqrt{-1}}{1-v+y\sqrt{-1}} + \frac{1}{4\sqrt{3}}\arctan\frac{vy\sqrt{3}\sqrt{-1}}{1+v+v^2+y^2}$$

where it is only needful to eliminate the imaginary numbers.

§9

Towards this end, since in general

$$\arctan(t\sqrt{-1}) = \int \frac{dt\sqrt{-1}}{1-t^2} = \frac{\sqrt{-1}}{2}\ln\frac{1+t}{1-t},$$

in our case, because $t = \frac{vy\sqrt{3}}{1+v+v^2+y^2}$, the latter part of the contrived formula will be

$$\frac{\sqrt{-1}}{8\sqrt{3}}\ln\frac{1+v+v^2+y^2+vy\sqrt{3}}{1+v+v^2+y^2-vy\sqrt{3}}.$$

Let $t = u\sqrt{-1}$ be put in place of the logarithmic part in the canonical formula, and it will become

$$-\arctan u = \frac{\sqrt{-1}}{2}\ln\frac{1+u\sqrt{-1}}{1-u\sqrt{-1}},$$

and therefore

$$\ln\frac{1+u\sqrt{-1}}{1-u\sqrt{-1}} = 2\sqrt{-1}\arctan u.$$

Now for our case $u = -\frac{y}{1-v}$, and therefore

$$\ln\frac{1-v-y\sqrt{-1}}{1-v+y\sqrt{-1}} = 2\sqrt{-1}\arctan\left(-\frac{y}{1-v}\right).$$

When these values have been substituted, the integral of the imaginary formula at hand will be

$$-\frac{\sqrt{-1}}{4}\arctan\frac{y}{1-v}+\frac{\sqrt{-1}}{8\sqrt{3}}\ln\frac{1+v+v^2+y^2+vy\sqrt{3}}{1+v+v^2+y^2-vy\sqrt{3}}.$$

§10

Here clearly everything[7] is divisible by $\sqrt{-1}$, and so once the imaginary numbers are eliminated, we obtain this alternate integration

$$\int\frac{dy}{(3-y^2)\sqrt[3]{1-3y^2}}=\frac{1}{8\sqrt{3}}\ln\frac{1+v+v^2+y^2+vy\sqrt{3}}{1+v+v^2+y^2-vy\sqrt{3}}-\frac{1}{4}\arctan\frac{y}{1-v},$$

where it should be noted, if we multiply the fraction joined to the logarithm above and below by $1-v$, because $1-v^3=3y^2$, it will be $\frac{y(4-v)+v(1-v)\sqrt{3}}{y(4-v)-v(1-v)\sqrt{3}}$. In this way our integral takes on this form:

$$\int\frac{dy}{(3-y^2)\sqrt[3]{1-3y^2}}=\frac{1}{8\sqrt{3}}\ln\frac{y(4-v)+v(1-v)\sqrt{3}}{y(4-v)-v(1-v)\sqrt{3}}-\frac{1}{4}\arctan\frac{y}{1-v},$$

with $v=\sqrt[3]{1-3y^2}$.

[7]This includes the integral itself, which last appeared in section 8.

A More Natural Resolution for the Proposed Differential Formula

§11

Although the solution above solves the whole business most beautifully, however, this is left to be desired in it – that no reason is evident which might have been able to advise the substitutions used here; because of which, it will be by no means unpleasant to add another solution, whose rationale should be able to be comprehended somewhat more clearly.

§12

However, to one considering the earlier formula

$$dV = \frac{dz}{(3+z^2)\sqrt[3]{1+3z^2}}$$

the expressions $1+3z^2$ and $3z+z^3$ are able to suggest that in this method a substitution of this kind, $z = \frac{1+x}{1-x}$, can be called into use not without success, since one of the above expressions is the sum of two cubes, and the other the difference. And hence, it becomes $dz = \frac{2dx}{(1-x)^2}$, and then indeed

$$3+z^2 = \frac{4-4x+4x^2}{(1-x)^2} = \frac{4(1+x^3)}{(1+x)(1-x)^2}$$

and then it will be

$$1+3z^2 = \frac{4+4x+4x^2}{(1-x)^2} = \frac{4(1-x^3)}{(1-x)^3}$$

from which we get

$$\sqrt[3]{1+3z^2} = \frac{\sqrt[3]{4(1-x^3)}}{1-x}.$$

With these substitutions having been made, it produces

$$dV = \frac{1}{2\sqrt[3]{4}} * \frac{(1-x^2)dx}{(1+x^3)\sqrt[3]{1-x^3}}.$$

§13

In this way the discovered formula is divided into two parts unaided, and the integration can be represented in this way:

$$2V\sqrt[3]{4} = \int \frac{dx}{(1+x^3)\sqrt[3]{1-x^3}} - \int \frac{x^2dx}{(1+x^3)\sqrt[3]{1-x^3}},$$

of which formulas the former can be brought through to completion by placing $\frac{x}{\sqrt[3]{1-x^3}} = t$. Thus the first part becomes $\int \frac{tdx}{x(1+x^3)}$; then, also, x^3 will be equal to $t^3 - t^3x^3$, therefore $x^3 = \frac{t^3}{1+t^3}$, from which it immediately becomes $1+x^3 = \frac{1+2t^3}{1+t^3}$. However, by differentiating by means of taking logs[8] we obtain $\frac{dx}{x} = \frac{dt}{t(1+t^3)}$ and thus that part before comes out $\int \frac{dt}{1+2t^3}$, of which the integration is obvious.

§14

The treatment of the latter part is even easier. For with u having been set equal to[9] $\sqrt[3]{1-x^3} = u$, which becomes $x^3 = 1 - u^3$, then certainly

[8] That is, logarithmic differentiation.

[9] The following formula has a square root instead of a cube root in the original.

$x^2dx = -u^2du$ and $1 + x^3 = 2 - u^3$; in this way, therefore, we will have

$$\int \frac{x^2dx}{(1+x^3)\sqrt[3]{1-x^3}} = -\int \frac{udu}{2-u^3},.$$

Therefore the whole sought-for integral will be

$$2V = \sqrt[3]{4} = \int \frac{dt}{1+2t^3} + \int \frac{udu}{2-u^3}.$$

§15

Therefore in this way we have also transformed the proposed formula into two other clearly reasoned formulas, of which, therefore, the integration is easily extracted through understood rules, from whence for that reason the same integral which the prior method supplied ought to result, if only the needed reductions are set up correctly. However, it is easily visible that the final formula is much more easily obtained by the prior method than if we wished to unravel these last two formulas[10], and because of this very same reason the method previously related should be judged to snatch away the victory from this one.

§16

If we wish to handle the other formula $\frac{dz}{(3-z^2)\sqrt[3]{1-3z^2}}$ in a similar way, it will be necessary that z be set up as $\frac{1+x}{1-x}\sqrt{-1}$, with the result that this resolution cannot be established other than by means of imaginary numbers. Hence the paradox, already introduced previously, is much more confirmed, by which

[10]Note Euler does not show these details!

I had affirmed that it is possible for differential formulas of this kind to be presented, whose integration cannot be finished unless by proceeding by means of imaginary numbers[11]. From this the highest use of the calculus of imaginary numbers in the analysis is observed much better.

[11] Euler discussed this in some of this other papers as well, notably E707, *De insigni usu calculi imaginariorum in calculo integrali*, 'On the outstanding use of the calculus of the imaginaries in the integral calculus'.