De Centro Similitudinis
(On the Center of Similitude)
Nova acta academiae scientiarum Petropolitaneae **9** (1791)
Submitted October 23, 1777

1. If two similar figures (see Fig.1) are described in the same plane, and if we let the larger one be called AB and the corresponding smaller one be called ab, then there always will be, in that very plane, a certain point Γ that relates similarly to either segment, so that the triangles ΓAB and Γab are similar. Let the point Γ be called the *center of similitude* of the two proposed similar figures, and let us investigate how it may be found in any case whatsoever. First of all, it is plain that the point Γ should be situated so that, when the lines ΓA and Γa are drawn, angles ΓAB and Γab are equal and also so that $\Gamma A : \Gamma a = AB : ab$, that is, in the same ratio as the homologous sides. We indicate this ratio as $A : a$.

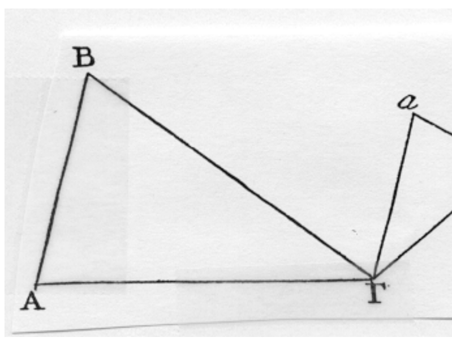


Fig. 1

2. In fact the following is immediately clear: if the homologous sides AB and ab (see Figure 2) are parallel, then the center of similitude is quite easily assigned, since it is always found at the intersection of the lines Aa and Bb. For since ab is parallel to AB, if the lines Aa and Bb are extended to a meeting point Γ, the triangles $AB\Gamma$ and $ab\Gamma$ obviously are similar. The outcome is the same when the homologous sides are laid out in opposite directions, as in Figure 3, where the center of similitude falls between the segments.

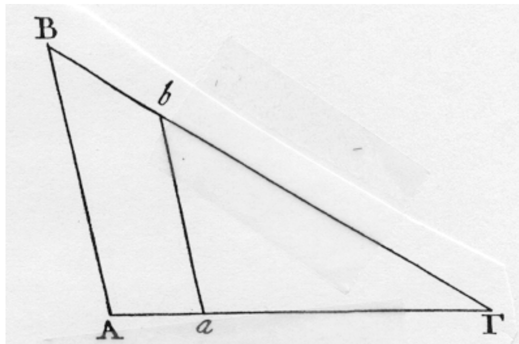


Fig. 2

3. Moreover (see Figure 4), if sides AB and ab should be non-parallel, let them be extended to an intersection at O, making between them the angle AOa, and clearly both of the homologous sides are inclined together at this same angle, whence it follows that if the circle OAa is drawn through the points O, A, and a, then all points O ought to lie on the boundary of this circle, since all angles that subtend the arc Aa are equal.

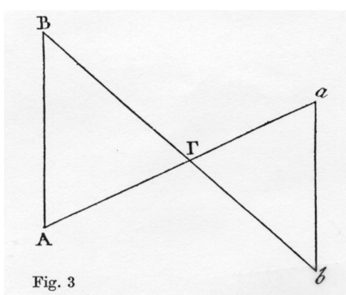


Fig. 3

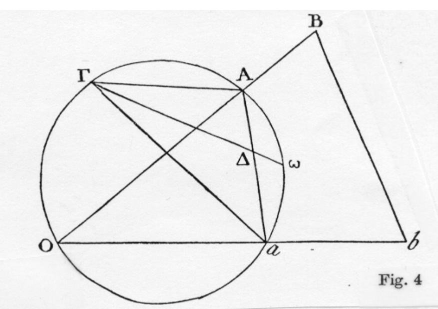


Fig. 4

4. Thereupon it may also be ascertained that the center of similitude should be located on the boundary of this very circle. Indeed, since ΓA and Γa can be seen as homologous sides, they would also make between them an angle $A\Gamma a$ that is equal to angle AOa itself. For this reason, the whole business reduces to the following: that one find, on the boundary, the point Γ so that, when lines ΓA and Γa are drawn, $\Gamma A : \Gamma a = A : a$. To this end, let the segment Aa be cut in a point Δ so that $A\Delta : aA = A : a$; and since it is required that $\Gamma A : \Gamma a = A\Delta : a\Delta$, it follows that the line $\Gamma\Delta$ must bisect angle $A\Gamma a$, and therefore when it is extended it should bisect the arc Aa in a point ω.

5. Having considered these matters, we deduce the following construction: first, we understand the arc Aa to be bisected at a point ω, and then the chord Aa is cut so that $A\Delta : a\Delta = A : a$. When this is done. let there be produced, through the points ω and Δ, a line $\omega\Delta\Gamma$, cutting the boundary of the circle in the point Γ, which will be the center of similitude that we seek. It is automatically

clear from this construction that $\Gamma A : \Gamma a = A\Delta : a\Delta = A : a$. It is also evident that the lines ΓA and Γa incline equally to the sides AB and ab respectively, since the angles $OA\Gamma$ and $Oa\Gamma$, subtending the same arc $O\Gamma$, are equal.

6. Just as we have determined the center of similitude Γ from the homologous points A and a, their place could have been taken by the homologous points B and b, which are related in the same way to the point of intersection O. This would have allowed a circle OBb to be described, and because the center Γ ought also to be located on the boundary of this circle, this circle must necessarily intersect the previous circle in the point Γ itself. Therefore, since these two circles pass through the same point O, their other intersection necessarily falls at the point Γ: this is the center of similitude that was sought.

7. Also, the center of similitude will exist even if the two given figures are not planar, but have similar projections over the same plane, provided that the bases of such similar bodies are set up on the same plane. From this it is understood that, if a plane is produced through the center of similitude and any two homologous points of such bodies, and both bodies are cut along this plane, then both of their sections will also be similar to each other. From here it is possible to see that, no matter how the two similar figures are given, it is always possible to find the center of similitude, which of course is related in like manner to both bodies. This can be demonstrated through the following calculation.

8. Let A and a be any pair of homologous points of the two similar bodies (see Figure 5), through which the plane of the table is conceived to pass; in this plane let some side AB of the larger body be considered to lie, and let the corresponding side ab in the smaller body be located in some other plane, which intersects the plane of the table along line ai, at some inclination θ. Now let perpendicular $b\beta$ be dropped from the point b to the plane of the table, then let the normal line $\beta\mathtt{b}$ be drawn from β to ai. If the points $\mathtt{b}$ and b are joined by the line $b\mathtt{b}$, then the angle $b\mathtt{b}\beta$ has inclination θ to the plane. Then indeed it will be noted that $AB : ab = A : a = \lambda : 1$, where we set $\lambda = \frac{A}{a}$. Also, in the same plane of the table, let the angle BAI be set equal to angle bai, and from B let the normal $B\mathtt{B}$ to AI be produced, and the points $\mathtt{B}$ and $\mathtt{b}$ will be homologous, just as β and b are.

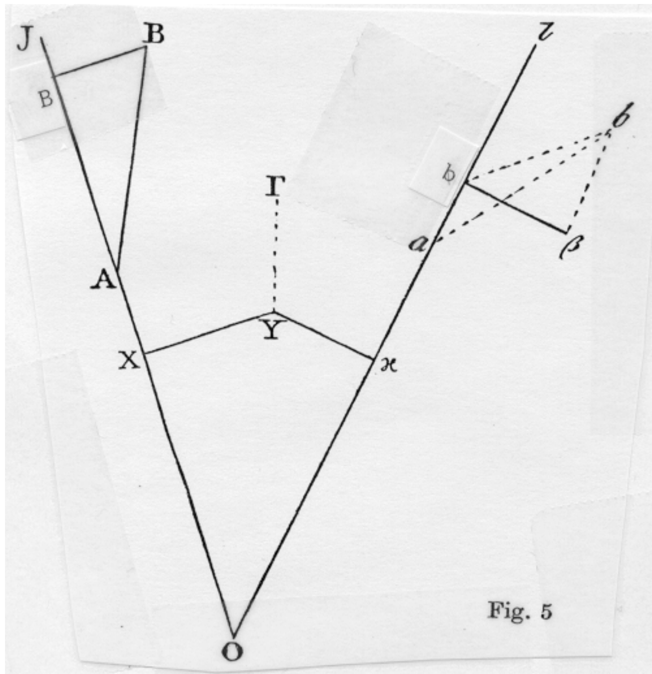


Fig. 5

9. Now, for the larger figure let us set $A\mathtt{B} = A$, and $\mathtt{B}B = B$; for the smaller figure, set $a\mathtt{b} = a$ and $\mathtt{b}b = b$; on account of similarity, $A = \lambda a$ and $B = \lambda b$; then indeed in the triangle $b\mathtt{b}\beta$ we will have $b\beta = \sin\theta$, and $\mathtt{b}\beta = \mathtt{b}\cos\theta$. These things being established, let the lines ia and IA be extended to their intersection at O, and, since the desired center of similitude Γ is located above, let the perpendicular ΓY be dropped from it to the plane of the table. From the point Y let normal lines YX and Yx be drawn to AO and aO respectively, and ley us set $AX = X$ and $XY = Y$, and then set $ax = x$ and $xY = y$. Let the perpendicular $Y\Gamma$ itself be set equal to z. These things being established, if we set the distance $AO = f$ and $aO = g$, and the angle $AOa = \omega$, then since $Ox = g - x$ it will be easily apparent that

$$\begin{aligned} AX &= X = f - (g-x)\cos\omega - y\sin\omega, \\ XY &= Y = (g-x)\sin\omega - y\cos\omega,, \end{aligned}$$

whence it is understood how the terms X and Y may be determined through x and y.

10. Now, therefore, we ought to show that a point Γ of this kind can be given—namely, one which is related in like fashion to A, B and $\mathtt{B}$ and to a, b and $\mathtt{b}$ respectively, so that

$$\Gamma A = \lambda\Gamma a, \quad \Gamma\mathtt{B} = \lambda\Gamma\mathtt{b}, \quad \text{and } \Gamma B = \lambda\Gamma b,$$

whence we obtain three equations and from these three the unknowns x, y and z can be determined. These equations are expressed in the following way:

$$\begin{aligned}
&\textbf{I.} && \Gamma A^2 = X^2 + Y^2 + z^2 = \lambda^2 \left(x^2 + y^2 + z^2\right), \\
&\textbf{II.} && \Gamma \mathsf{B}^2 = (X + \lambda a)^2 + Y^2 + z^2 = \lambda^2 \left((x+a)^2 + yy + zz\right), \\
&\textbf{III.} && \Gamma B^2 = (X + \lambda a)^2 + (Y - \lambda b)^2 + z^2. \\
& && \quad = \lambda^2 \left((x+a)^2 + (y + b\cos\theta)^2 + (z - b\sin\theta)^2\right)
\end{aligned}$$

11. Now then, the first of these three equations gives

$$X^2 + Y^2 + Z^2 = \lambda\lambda xx + \lambda\lambda yy + \lambda\lambda zz,$$

which when subtracted from the second equation leaves

$$2\lambda a X + \lambda\lambda aa = 2\lambda\lambda ax + \lambda\lambda aa,$$

and therefore $X = \lambda x$. Then the second equation, subtracted from the third, leaves:

$$-2\lambda bY + \lambda\lambda b = 2\lambda\lambda by\cos\theta - 2\lambda\lambda bz\sin\theta + \lambda\lambda bb,$$

from which we gather that

$$Y = \lambda\left(z\sin\theta - y\cos\theta\right).$$

In this way the entire investigation has been reduced to three equations involving three variables x, y and z, from which the terms a and b themselves have vanished—just as the nature of the matter requires, since the center of similitude Γ ought to relate in the same way to all homologous sides. However, these three equations are to complicated to permit the terms x, y and z to be determined from them in practice.

12. Rather, we have brought this calculation to attention in order to demonstrate that no matter how two similar bodies are arranged, there always exists a point Γ that relates to both equally. This having been shown, it is fitting to consider another method – henceforth to be sought – of determining the center of similitude, wherein all sections of the two bodies, formed through the two homologous points and the center of similitude, are always similar figures.

13. Therefore let us imagine (see Fig. 6) an arbitrary plane passing through homologous points A and a; there will be another plane passing through the same points A and a, in which the center of similitude Γ will be located, so that the intersection would be the line Aa. In order to investigate this point we should seek out two other homologous points, located outside of the [original] plane, which are located in the same plane as the line Aa, so that the four points A, a, B, and b lie on this plane. However, considerable complications would be entailed in the general resolution of such an investigation.

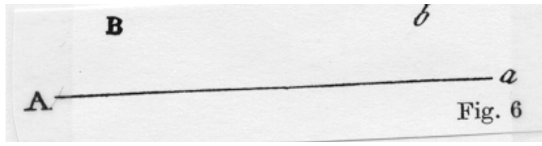


Fig. 6

14. Nevertheless, we may avoid such complications if we take (see Figure 7) the point B so that it lies on the line Aa itself. Then let the homologous point lie at b, from which point let perpendicular $b\beta$ be dropped to the plane of the table, and from β to the line Aa (extended if need be) let the normal line $\beta\mathsf{b}$ be dropped. In this way the four points A, B and a, b certainly will be located on the same plane, since this plane intersects the plane of the table along the line Aa and is inclined to it at the angle $b\mathsf{b}\beta$. Therefore the desired center of similitude must necessarily be located in the same plane, and this center will be found most promptly, without any difficulty, by application of the method described in the beginning.

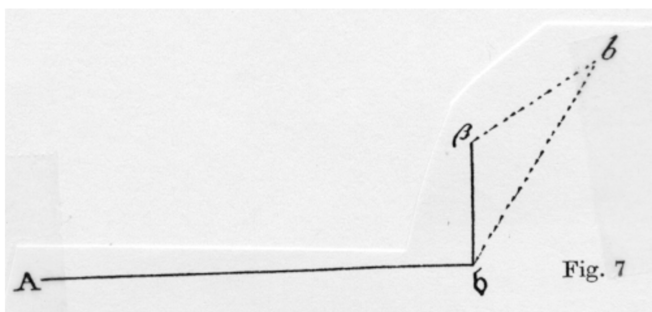
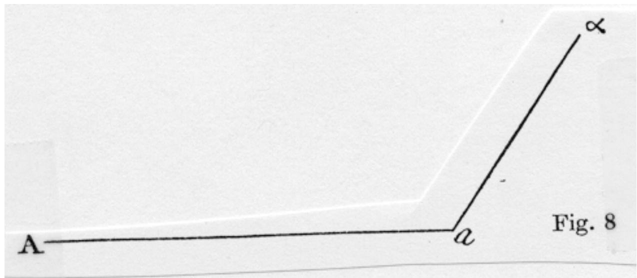


Fig. 7

15. The problem we have dealt with up to now may be seen as concerning the science of Perspective, since, if a copy of some object is drawn accurately, it will be quite pertinent to determine that location from which, if both the object and the copy were viewed, all homologous parts would appear as contained in equal angles. It is understood that this location will be found in that point which we have called the center of similitude.

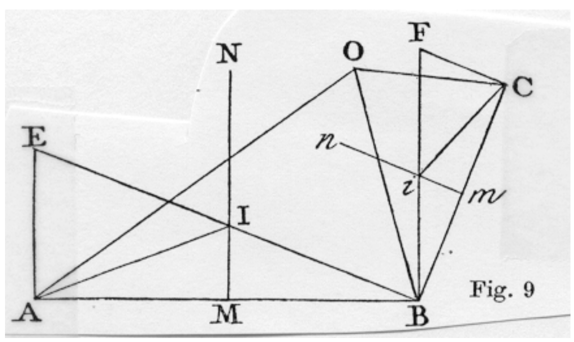
16. The center of similitude can be located easily, from the considerations adduced above, no matter how the object and its image are situated. For let us now consider (see Figure 8) two homologous points A and a, of which the first point A is taken to be in the object it self, and the other is taken to be in the image; then, just as a segment Aa may be seen extending to the object itself, so also in the image a similar segment $a\alpha$ may be drawn from the point a. This segment is understood to relate to the image in the same way in which the line Aa relates to the object itself. From what was done in the beginning, it should be maintained that the center of similitude is located in the plane that is determined by these two segments Aa and $a\alpha$ (that is, by the three points A, a, and α), and its location ought to be so defined that it relates in like fashion to

either segment Aa or $a\alpha$. The means of finding this point is to be sought in the following problem:



Geometrical Problem

(Figure 9) *Given three points A, B and C located arbitrarily in the plane of the table: to find in that same plane a point O so that, when segments AB, BC, OC, and OA are drawn, the two triangles OAB and OBC are similar, that is: so that it tuns out that angle $AOB = BOC$, $OAB = OBC$, and $OBA = OCB$.*



Solution

17. Having extended segment AB to some point D, let is set angle $CBD = \theta$: this is now given. Then likewise let angle OAB be denoted ϕ: this will be an unknown. Since angle OBC is supposed to equal angle OAB, angle OBD will be $\phi + \theta$, and since OBD is external to triangle AOB, the angle AOB will be θ, so it is given, too. Also, the angle BOC will be equal to it: from this condition the point O itself will be determined with hardly any difficulty.

18. Now since the triangle AOB is arranged so that its angle $AOB = \theta$, it is known from basic principles that over the base AB infinitely many triangles

of the same sort may be erected, where apex angles AOB are all of that same magnitude θ, so long as these angles are located on the boundary of a particular circle described over the base AB. Therefore the center of this circle will be somewhere on the line MN, erected perpendicularly from the midpoint M of segment AB: whence, if the center is to be at I it is necessary that the central angle AIB equal twice the angle θ, and therefore its half angle $BIM = \theta$, and hence angle $MBI = 90^\circ - \theta$; from this it is clear that CBI is a right angle – in other words, that the line BI must be drawn so that it is normal to CB, and in this way the desired center of similitude O will be lie somewhere on the boundary of this circle.

19. If the other segment BC is bisected in the same way at m and normal mn is erected upon it, then over this segment BC one can also draw a circle so that all of the angles standing on the base BC and extending to the boundary of the circle are also equal to θ. The center of this circle will be at a point i, so that the angle $Bim = \theta$; whence it is apparent that the line Bi is normal to BA. Therefore, since the desired point O is likewise located on the boundary of the circle drawn by means of its center i and passing through points C and B, it is evident that this point should lie at the intersection of the two aforesaid circles.

20. Moreover, these matters can be settled much more easily in the following way: to the line BA let the perpendicular AE be erected from A, meeting line BI at E, and it will be the case that $IE = BI = AI$, hence the point E will be on the circle and therefore the segment BE will be the diameter of this circle. In this way, if in the other part of the diagram the perpendicular CF to BC is raised from C, meeting the extended line Bi at F, it will also be the case that $iF = Bi =$ Ci, and hence BF will be the diameter of this other circle. Wherefore, if on the diameters BE and BF two circles are constructed, their intersection O will give the desired center of similitude O.

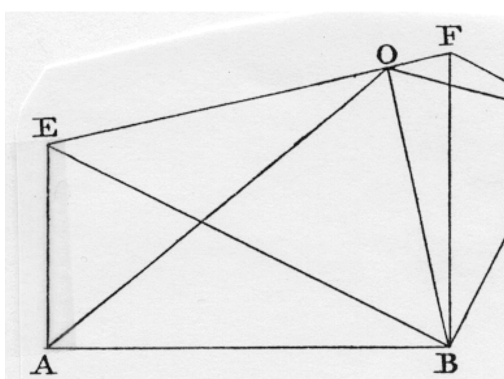


Fig. 10

21. Let us make a new diagram (see Figure 10) with superfluous lines omitted, and first let us erect, from points A and C, perpendiculars AE and

CF to segments BA and BC, which meet segments BE and BF (themselves joined at a right angle) in points E and F; then indeed, over the segments BE and BF considered as diameters let the two circles, meeting at the point O, be understood to be constructed. That point O will lie on the first semicircle built up over the segment BE and therefore the angle BOE is a right angle; then likewise the same point O will lie on the semicircle built up over the segment BF, whence the angle BOF will also be a right angle. From this it is obvious that the two segments EO and FO are situated on the same line. Therefore, if one draws segment EF, the point O can be found on EF if the perpendicular BO is dropped to it from the point B. From this the following construction is quite easily derived:

Construction of the Proposed Problem

22. From the three given points A, B and C let segments be drawn that are normal to AB and BC and whose intersections give two points E and F; then draw EF and let the perpendicular BO be dropped to it from B, and the point O will be the desired center of similitude, so that, when segments OA and OC are drawn, the two triangles AOB and BOC will be similar to one another.

Demonstration of this Construction

23. To begin with, it should be noted that quadrilateral $AEOB$ is inscribed in a circle; from the nature of such a quadrilateral it follows that angles

$$ABE = AOE, \quad EAO = EBO, \quad BAO = BEO, \quad AEB = AOB.$$

Next, in the same fashion quadrilateral $BOFC$ is inscribed in a circle, whence the following equalities of angles ensue:

$$BOC = BFC, \quad CBF = COF, \quad OBF = OCF, \quad BCO = BFO.$$

24. Once these things have been observed, it can be shown that triangle AOB is similar to triangle EBF. For, in the first place, angle $BAO = FEB$ (from the preceding paragraph); next, angle $AOB = AEB$, which angle is the complement of angle ABE, but the complement of angle EBF is the same angle ABE, whence it follows that angle $AOB = EBF$.

25. In the same way it can be shown that $\triangle BOC \sim \triangle EBF$. For, in the first place, angle $BOC = BFC$. Since BE is parallel to CF, the alternate angle EBF is equal to BFC, and thus angle $BOC = EBF$, whence also the third angles OBC and BEF are likewise equal. Therefore, since triangles AOB and

BOC are both similar to the same triangle EBF, it is necessary that they are also similar to one another. Q.E.D.

26. It will also help to have considered the case in which the point O falls outside of the segment, as in Figure 11. Here, as before, quadrilateral $ABOE$ is inscribed in a circle; next, it is evident that quadrilateral $BFOC$ resides on a semicircle described over the diameter BF, from which it will be possible for the demonstration of the construction to be given as before, in which it is shown that both triangles AOB and BOC are similar to triangle EBF and therefore are similar to each other as well. In addition the case, in which the two segments AB and BC are normal to one another, deserves to be pointed out (see Figure 12); for then it is plain that the points E and F fall at the terminal points A and C, whence, if the hypotenuse EF (or AC) is joined, and perpendicular BO is dropped to it from B, the point O will be the desired center of similitude, since triangles AOB and BOC plainly are similar both to one another and to the third triangle ABC.

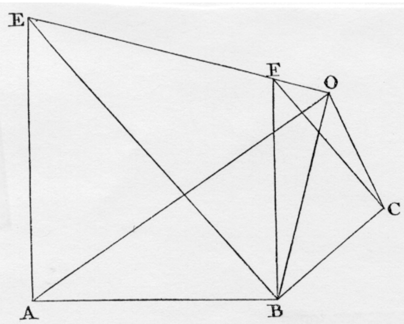


Fig. 11

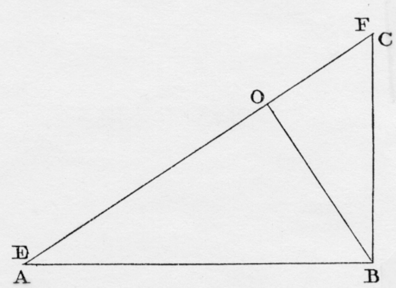


Fig. 12

27. Finally, it is fitting that the case (see Figure 13), in which BC and AB are inclined at an acute angle, should be considered, namely: the case in which the pair of perpendiculars AE and BF fall on opposite sides, as can be seen in the adjacent figure, where the point O falls within angle ABC so that triangle AOB and triangle BOC are similar to one another—for they will always be similar as well to triangle EBF.

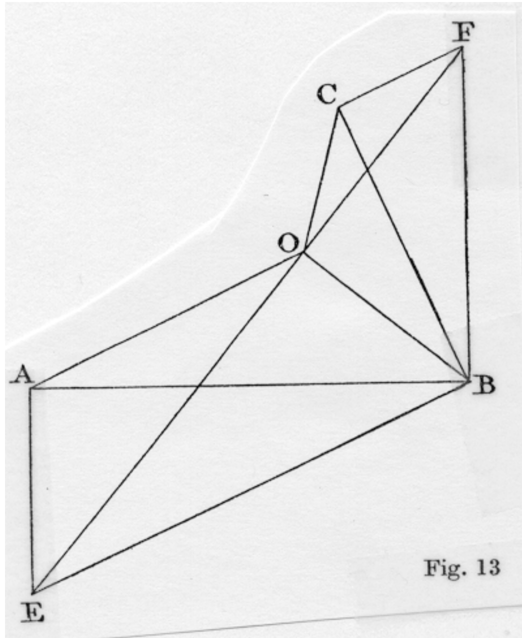


Fig. 13

28. However, one more case occurs, to which the previous solution seems not to apply. The case obtains when the segments AB and BC lie on one line; in this case, the two points E and F are removed to infinity, so that the preceding construction clearly cannot apply. Moreover, it is clear at once that in this case the center of similitude O must necessarily fall on the extended line ABC, so that $AO : BO = BO : CO$. For the sake of finding this point, therefore, let us set $AB = a$, $BC = c$, and $BO = z$; therefore we can get

$$a + z : z = z : z - c$$

from which we have

$$z = \frac{ac}{a - c} \text{ and therefore } AO = \frac{aa}{a - c},$$

and this location is conveniently constructed in the following manner: at point A of segment AB, at an arbitrary angle to AB, let segment Ab be drawn so that $Ab = AB = a$; let this segment be cut at point c so that $bc = BC = c$, and, segment cB having been drawn, let the segment BO be made parallel to it. The intersection of BO with the given segment AB will present the desired point O.