

The Solution of certain more difficult Questions in the Calculus of Probabilities*

Leonhard Euler[†]

E600

Opuscula analytica 2, 1785, pp. 331-346

1. In this inquiry a game commonly established everywhere furnishes the opportunity, where from a list of 90, signified by the numbers 1, 2, 3, 4, . . . 90, at an appointed time five are customarily drawn from the list by chance. Hence therefore questions of this kind arise: namely how much is the probability, that, after a given number of drawings is completed, either all ninety numbers are drawn or at least 89 or 88 or less. Therefore here I have decided to resolve these questions, inasmuch as they are the most difficult ones, from the principles of the calculus of probability, which have long since been accepted by practice. Nor do the objections of the illustrious d'Alembert¹ deter me, who attempted to restore this suspected calculus. For after the greatest geometer bade farewell to mathematical studies, he even seems to declare war on them, when he undertook to overturn very many solidly established fundamentals. For although these objections must be of the greatest weight in the presence of the unknowledgable, it is not at all to be feared from this that any damage will be inflicted on the science itself.

2. Everyone who has carefully labored in investigations of this sort, easily will observe the solution of these questions to require the most intricate calculations, but which by benefit of certain characters, which now I have used several times² to good

*See commentaries 338 and 812.

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¹J.L. D'Alembert (1717–1783), “Réflexions sur le calcul des probabilités.” *Opuscules mathém.*, Paris t. II (1761), p. 1–25; t. IV (1768), p. 73–79, 283–310; t. VII (1780), p. 39–60, 384–386. “Doutes et questions sur le calcul des probabilités.” *Mélanges de litt., d'hist. et de philos.*, Amsterdam, t. V (1773), p. 273–304. See also the articles *Croix et Pile* and *Gageure*, which are contained in the *Encyclopédie, ou Dictionnaire raisonné des sciences, des arts et des métiers*, published by M. Diderot and M. D'Alembert, Paris 1751–1769. Besides see M. Cantor (1829–1920), *Vorlesungen über Geschichte der Mathematik*, Leipsig, III, 2nd edition (1901), p. 639; IV (1908), p. 222–229. L.G.D.

²See commentaries 521, 575, 584: “Théorèmes analytiques. Extraits de différentes lettres de M. Euler à M. le marquis de Condorcet,” *Mém. de l'acad. d. sc. de Paris* (1778), 1781, p. 603–614, “De mirabilibus proprietatibus unciarum, quae in evolutione binomii ad potestatem quamcunque evecti occurrunt,” *Acta acad. sc. Petrop.* (1781: I), 1784, p. 74–111, “De insignibus proprietatibus unciarum binomii ad uncias quorumvis polynomiorum extensis,” *Acta acad. sc. Petrop.* (1781: II), 1785, p. 76–89; *Leonhard Euler Opera Omnia*, series I, vol. 18 (Comm. 521) and 15 (Comm. 575 and 584). Euler use the same character yet in the latter Commentaries 663, 726, 768. Other ways of designating combinations were used: in the other commentaries (575, 584, 600) brackets, in others (521, 663, 726, 768) parenthesis are employed. L.G.D.

success, it is permitted by me to have the upper hand. Of this kind namely the character

$$\left(\frac{p}{q}\right),$$

where the fraction is represented enclosed by parentheses³, denotes to me such product

$$\frac{p}{1} \cdot \frac{p-1}{2} \cdot \frac{p-2}{3} \cdot \frac{p-3}{4} \dots \frac{p-q+1}{q},$$

the value of which therefore in any case whatsoever can be presented easily. Moreover concerning this character it will help to note the following.

1. There is always

$$\left(\frac{p}{q}\right) = \left(\frac{p}{p-q}\right).$$

2. If $q = 0$, there is always

$$\left(\frac{p}{0}\right) = 1.$$

3. If q is either a negative number or greater than p , the value itself $\left(\frac{p}{q}\right)$ is always = 0.

4. Next if p is a negative number, then such formula

$$\left(\frac{-p}{q}\right)$$

it is permitted to reduce to this

$$\pm \left(\frac{p+q-1}{q}\right),$$

where the + sign prevails, if q is an even number, – indeed, if odd; whence it is evident such form furthermore to be able to change to this $\pm \left(\frac{p+q-1}{p-1}\right)$.

3. In these first fruits I will have treated the questions most generally arising out of the mentioned game. The letter m will denote namely the number of tickets, which I assume each one is signified by the separate letters a, b, c, d etc., the use of the variable will not produce confusion. Next in any drawing whatsoever from these tickets I will assume i drawn tickets, whence the total number of variations, which are able to occur in this drawing, will be = $\left(\frac{m}{i}\right)$. In addition, if the number of successive planned drawings was = n , from the principles of combinations it is well known that the total number of variations, which are able to occur, is $\left(\frac{m}{i}\right)^n$. Therefore in this manner I will run through the following problems.

³*Translators note:* Actually Euler says here uncinulis which I understand to mean “by little hooks.” According to the Editor, Euler originally denoted the binomial coefficient by $\left[\frac{p}{q}\right]$. For the sake of consistency with other memoirs the brackets were replaced by parentheses.

PROBLEM 1

If the number of tickets designated by letters a, b, c, d etc. is m and thenceforth in any drawing whatsoever i tickets are removed and now the number of completed drawings will be= n, it is asked, how much is the probability, that all m of the letters a, b, c, d etc. were to have come forth.

SOLUTION

4. This is observed first, because in n drawings the number of tickets drawn is in, all letters are not able to come forth, unless there is $in > m$ and therefore

$$n > \frac{m}{i}$$

or at least not less. Denote now by Δ the number of all variations, which are able to happen in these n drawings, and there will be, as now we indicate

$$\Delta = \left(\frac{m}{i}\right)^n ;$$

which since it is the number of all possible cases, hence for our question all cases ought to be excluded, which contain fewer than m letters.

First therefore, if the number of letters were only= $m - 1$, because it is able to be made in m ways, the number of cases, which contain only $m - 1$ or fewer, will be

$$m \left(\frac{m-1}{i}\right)^n ,$$

which number we put = A.

In like manner, if two letters be excluded, because it is able to be made in $\left(\frac{m}{2}\right)$ ways, the number of cases containing only $m - 2$ or fewer letters will be

$$\left(\frac{m}{2}\right) \cdot \left(\frac{m-2}{i}\right)^n ,$$

which number we indicate by the letter B.

Further let C be the number of all cases, which contain only $m - 3$ letters or fewer, and it will be

$$C = \left(\frac{m}{3}\right) \cdot \left(\frac{m-3}{i}\right)^n .$$

And to the same purpose in the manner there is

$$D = \left(\frac{m}{4}\right) \cdot \left(\frac{m-4}{i}\right)^n , E = \left(\frac{m}{5}\right) \cdot \left(\frac{m-5}{i}\right)^n \text{ etc.}$$

And by these established elements I discover the number of all cases, which contain all m letters, to be

$$\Delta - A + B - C + D - \text{etc.},$$

which number we indicate by the letter Σ .

5. It is evident this number Σ to be able to be determined through one theorem of combinations and therefore by absolutely no doubt to be answerable, thus even as geometric truth is able to be tested. Hence also according to the principles of probability the number of favorable cases divided by the number of all possible cases will answer the question; which therefore if it is put= Π , will be

$$\Pi = \frac{\Sigma}{\Delta}.$$

Therefore since there is

$$\Sigma = \Delta - A + B - C + D - \text{etc.},$$

the assigned values will be substituted for the letters Δ, A, B, C etc.

$$\Sigma = \binom{m}{i}^n - \binom{m}{1} \binom{m-1}{i}^n + \binom{m}{2} \cdot \binom{m-2}{i}^n - \binom{m}{3} \cdot \binom{m-3}{i}^n + \text{etc.}$$

This formula divided by $\binom{m}{i}^n$ will give the probability, that after n drawings all m letters were removed, whence by necessity this expression Σ always ought to be equal to zero, whenever there will be $n < \frac{m}{i}$, which also will reveal the calculation by establishing the fact to occur for the simpler cases. As for instance if there was $m = 7, n = 3, i = 2$, there will be

$$\begin{aligned} \binom{m}{i}^n &= 21^3 = 9261, \\ \binom{m}{1} \binom{m-1}{i}^n &= 7 \cdot 15^3 = 23625, \\ \binom{m}{2} \cdot \binom{m-2}{i}^n &= 21 \cdot 10^3 = 21000 \\ \binom{m}{3} \cdot \binom{m-3}{i}^n &= 35 \cdot 6^3 = 7560, \\ \binom{m}{4} \cdot \binom{m-4}{i}^n &= 35 \cdot 3^3 = 945, \\ \binom{m}{5} \cdot \binom{m-5}{i}^n &= 21 \cdot 1^3 = 21, \\ \binom{m}{6} \cdot \binom{m-6}{i}^n &= 0, \end{aligned}$$

whence it produces $\Sigma = 0$.

6. Therefore until n be not less than $\frac{m}{i}$, there will be always $\Sigma = 0$; on the other hand if there were $n = \frac{m}{i}$ or if

$$m = in,$$

this extreme case is noteworthy; for then our found formula for Σ is able to reduce to the fixed product of pure factors⁴. For there will be

$$\Sigma = \binom{m}{i} \binom{m-i}{i} \binom{m-2i}{i} \binom{m-3i}{i} \dots \binom{i}{i};$$

⁴*Translators note:* The derivation of this formula is elementary. Each ticket must be drawn exactly once. There are $\binom{m}{i}$ ways to select i tickets from among m tickets. Now, given that these i tickets must not be drawn again, the second set of i tickets must be drawn from the remaining $m - i$. This occurs in $\binom{m-i}{i}$ ways. We now have $2i$ tickets which must not be drawn again. Thus the third set of i tickets must be chosen from the remaining $m - 2i$. The pattern is clear.

which for instance we illustrate by an example, I suppose, as before $n = 3$ and $i = 2$, there is in fact $m = 6$; while by the prior formula for Σ the data gives $\Sigma = 90$, the second indeed gives $\Sigma = 15 \cdot 6 \cdot 1 = 90$.

7. But nevertheless these formulas, if greater numbers be taken for n , greatly increase in size, still by logarithms it will be by no means difficult to assign the value of the probability to any cases whatever. For when there be

$$\begin{aligned}\frac{A}{\Delta} &= m \cdot \frac{(m-i)^n}{m^n} \\ \frac{B}{A} &= \frac{m-1}{2} \cdot \frac{(m-1-i)^n}{(m-1)^n} \\ \frac{C}{B} &= \frac{m-2}{3} \cdot \frac{(m-2-i)^n}{(m-2)^n} \\ &\text{etc.};\end{aligned}$$

hence by obtaining logarithms there will be

$$\begin{aligned}l\frac{A}{\Delta} &= lm - nl \frac{m}{m-i}, \\ l\frac{B}{A} &= l\frac{m-1}{2} - nl \frac{m-1}{m-1-i}, \\ l\frac{C}{B} &= l\frac{m-2}{3} - nl \frac{m-2}{m-2-i}, \\ &\text{etc.};\end{aligned}$$

from which is obtained

$$\begin{aligned}l\frac{A}{\Delta} &= lm - nl \frac{m}{m-i}, \\ l\frac{B}{\Delta} &= l\frac{A}{\Delta} + l\frac{m-1}{2} - nl \frac{m-1}{m-1-i}, \\ l\frac{C}{\Delta} &= l\frac{B}{\Delta} + l\frac{m-2}{3} - nl \frac{m-2}{m-2-i}, \\ &\text{etc.};\end{aligned}$$

Therefore from which it is easy to discover the values $\frac{A}{\Delta}$, $\frac{B}{\Delta}$, $\frac{C}{\Delta}$ etc., by which discovery the probability in question will be

$$\Pi = 1 - \frac{A}{\Delta} + \frac{B}{\Delta} - \frac{C}{\Delta} + \frac{D}{\Delta} - \text{etc.}$$

8. We apply this to the case of the game mentioned at the beginning, which is $m = 90$ and $i = 5$, and there will be, as follows:

$$\begin{aligned}l\frac{A}{\Delta} &= l90 - nl\frac{90}{85} &= 1,9542425 - n \cdot 0,0248235, \\ l\frac{B}{\Delta} &= l\frac{A}{\Delta} + l\frac{89}{2} - nl\frac{89}{84} &= l\frac{A}{\Delta} + 1,6483600 - n \cdot 0,0251107,⁴ \\ l\frac{C}{\Delta} &= l\frac{B}{\Delta} + l\frac{88}{3} - nl\frac{88}{83} &= l\frac{B}{\Delta} + 1,673614 - n \cdot 0,0254046, \\ l\frac{D}{\Delta} &= l\frac{C}{\Delta} + l\frac{87}{4} - nl\frac{87}{82} &= l\frac{C}{\Delta} + 1,3374593 - n \cdot 0,0257054, \\ l\frac{E}{\Delta} &= l\frac{D}{\Delta} + l\frac{86}{5} - nl\frac{86}{81} &= l\frac{D}{\Delta} + 1,2355286 - n \cdot 0,0260135,⁴ \\ l\frac{F}{\Delta} &= l\frac{E}{\Delta} + l\frac{85}{6} - nl\frac{85}{80} &= l\frac{E}{\Delta} + 1,1512676 - n \cdot 0,0263289, \\ l\frac{G}{\Delta} &= l\frac{F}{\Delta} + l\frac{84}{7} - nl\frac{84}{79} &= l\frac{F}{\Delta} + 1,0791813 - n \cdot 0,0266522 \\ &&\text{etc.}\end{aligned}$$

⁴First edition: $l\frac{A}{\Delta} + 1,6483600 - n \cdot 0,0250107$ and $l\frac{D}{\Delta} + 1,2355283 - n \cdot 0,0260133$. Now too consequently the values in subsequent paragraphs were corrected. L. G. D.

9. This is clear, where the greater number n of drawings is admitted, on this account such progression to converge rapidly, so that, if n denotes a very great number, what will be produced is $\Pi = 1$ nearly always; namely then to the greatest extent the probability will be that all m numbers entirely come forth. But on the other hand, if the number n too little surpasses the minimum value $\frac{m}{i} = 18$, a most laborious unrolling of these terms will happen, with many terms there is work, until one comes to vanishing.

We suppose $n = 100$, so that we respond to this question, how much is the probability, so that all ninety numbers will come forth after one hundred drawings. Therefore this will be

$$\begin{aligned} l_{\Delta}^A &= 9,47188, & \text{therefore } l_{\Delta}^A &= 0,2964, \\ l_{\Delta}^B &= 8,60917, & \text{therefore } l_{\Delta}^B &= 0,0407, \\ l_{\Delta}^C &= 7,53607, & \text{therefore } l_{\Delta}^C &= 0,0034, \\ l_{\Delta}^D &= 6,30299, & \text{therefore } l_{\Delta}^D &= 0,0002, \\ l_{\Delta}^E &= 4,93719, & \text{therefore } l_{\Delta}^E &= 0,0000, \end{aligned}$$

therefore

$$\Pi = 0,7411.$$

10. Let $n = 200$ and for this case there will be

$$\begin{aligned} l_{\Delta}^A &= 6,98952, & \text{therefore } l_{\Delta}^A &= 0,00098,^5 \\ l_{\Delta}^B &= 3,61574, & \text{therefore } l_{\Delta}^B &= 0,00000, \end{aligned}$$

whence the probability is obtained, that after 200 drawings all numbers come forth,

$$\Pi = 0,99902,$$

which probability that all come forth is very near certitude.

PROBLEM 2

By setting, what is established in the previous problem, it is asked, how much probability will there be, that at least $m - 1$ of the letters come forth after n drawings.

SOLUTION

11. Here therefore the number of the drawings containing all m letters is not excluded, whence it is well-known the number of drawings in our present case will be greater. However by the computed reckoning, if the number of these cases is put Σ' , it will be found

$$\Sigma' = \Delta - B + 2C - 3D + 4E - 5F + \text{etc.},$$

whence the probability, that after n drawings at least $m - 1$ letters come forth, will be

$$\Pi' = \frac{\Sigma'}{\Delta}$$

⁵First edition: $l_{\Delta}^A = 0,0009$ and $\Pi = 0,99903$. Correction L. G. D.

and therefore

$$\Pi' = 1 - \frac{B}{\Delta} + 2\frac{C}{\Delta} - 3\frac{D}{\Delta} + 4\frac{E}{\Delta} - \text{etc.}$$

12. Therefore this case will be

$$\Sigma' = \left(\frac{m}{i}\right)^n - \left(\frac{m}{2}\right) \left(\frac{m-2}{i}\right)^n + 2 \left(\frac{m}{3}\right) \left(\frac{m-3}{i}\right)^n - 3 \left(\frac{m}{4}\right) \left(\frac{m-4}{i}\right)^n + 4 \left(\frac{m}{5}\right) \left(\frac{m-5}{i}\right)^n - \text{etc.},$$

whence, if application be made to the mentioned game, while the letters Δ, A, B, C, D etc. retain their values, the calculation by means of the logarithms which had been made from the discovered values $\frac{B}{\Delta}, \frac{C}{\Delta}, \frac{D}{\Delta}$ etc. will be completed easily. Thus, if after 100 drawings the probability is required, that at least 89 numbers come forth, on account of

$$\frac{B}{\Delta} = 0,0407, \frac{C}{\Delta} = 0,0034, \frac{D}{\Delta} = 0,0002$$

this probability will be

$$\Pi' = 0,9655.$$

Whence the following probability, that as many fewer numbers will come forth, will be 0,0345.

PROBLEM 3

Put the same, as hitherto, it is asked, how much be the probability, that at least $m - 2$ letters were drawn after n drawings.

SOLUTION

13. The number of all cases, in which at least $m - 2$ letters are contained, through the letters Δ, A, B, C, D etc. established before thus is defined, as being

$$\Sigma'' = \Delta - C + 3D - 6E + 10F - \text{etc.},$$

which expression has by the restored values of itself in this way

$$\Sigma'' = \left(\frac{m}{i}\right)^n - \left(\frac{2}{2}\right) \left(\frac{m}{3}\right) \left(\frac{m-3}{i}\right)^n + \left(\frac{3}{2}\right) \left(\frac{m}{4}\right) \left(\frac{m-4}{i}\right)^n - \left(\frac{4}{2}\right) \left(\frac{m}{5}\right) \left(\frac{m-5}{i}\right)^n + \text{etc.},$$

and this probability will be

$$\Pi = 1 - \frac{C}{\Delta} + 3\frac{D}{\Delta} - 6\frac{E}{\Delta} + 10\frac{F}{\Delta} - \text{etc.}$$

Therefore for the previously mentioned game if the probability in question is, that after 100 drawings at least 88 numbers will come out, it will be discovered

$$\Pi'' = 0,9972,$$

whence the probability, that the contrary occur, will be = 0,0028.

THE GENERAL PROBLEM

Put the same, as before, it is asked, how much is the probability, that after n drawings, at least $m - \lambda$ letters come forth.

SOLUTION

14. The number of cases containing at least as many letters is easily expressed through our characters thus, as it is

$$\begin{aligned} & \left(\frac{m}{i}\right)^n - \left(\frac{\lambda}{\lambda}\right) \left(\frac{m}{\lambda+1}\right) \left(\frac{m-\lambda-1}{i}\right)^n + \left(\frac{\lambda+1}{\lambda}\right) \left(\frac{m}{\lambda+2}\right) \left(\frac{m-\lambda-2}{i}\right)^n \\ & - \left(\frac{\lambda+2}{\lambda}\right) \left(\frac{m}{\lambda+3}\right) \left(\frac{m-\lambda-3}{i}\right)^n + \text{etc.}, \end{aligned}$$

which formula divided by the first term $\left(\frac{m}{i}\right)^n$ will present the sought probability.

15. In determining these probabilities it is certainly assumed that all letters are equally subject to being drawn; however the illustrious d'Alembert denies that this can be assumed. For it is thought that one ought to look back to all the drawings which have already been completed before; for if any letters will have been extracted exceedingly frequently, then those will come forth in the following drawings more rarely; but that the contrary occurs if any letters come forth exceedingly rarely. If this reasoning were valid, it would also be valid, if the following drawings at least after a year or even as far as a whole century, why even in any other place whatsoever, were provided; and for the same reason an accounting would also need to be had for all the drawings which henceforth hereafter will have been completed in whatsoever places of the earth, than which surely scarcely anything more absurd can be imagined.

DEMONSTRATION OF THE PRECEDING SOLUTION

16. When the number of all letters a, b, c, d etc., by which I assume single lists have been designated, is $= m$, this complex of the letters I will call the principal system, from which other derived systems, which contain fewer letters, it will be convenient to form, which therefore I separate into orders, as the primary order includes all systems, some contain only $m - 1$ letters, of which therefore the number will be

$$= m.$$

To the second order indeed I will refer all systems, in which the number of letters is $m - 2$, of which the number will be

$$\frac{m}{1} \cdot \frac{m-1}{2} = \binom{m}{2}.$$

Moreover the third order will have all systems, where the number of letters is $m - 3$, of which the number is

$$\frac{m}{1} \cdot \frac{m-1}{2} \cdot \frac{m-3}{3} = \binom{m}{3}.$$

In the same manner the number of the systems of the fourth in the order containing only $m - 4$ letters will be

$$\binom{m}{4};$$

moreover the fifth in the order, where only $m - 5$ letters are involved, the number of systems will be

$$\binom{m}{5}$$

and therefore further.

17. For the reason which makes clearer, we observe the principal system corresponding to these six letters

$a \quad b \quad c \quad d \quad e \quad f,$

from which therefore the derived sequences and of each order will result, which we exhibit in this table:

I.	II.	III.	IV.	V.
<i>abcde</i>	<i>abcd</i>	<i>abc</i>	<i>ab</i>	<i>a</i>
<i>abcdef</i>	<i>abce</i>	<i>abd</i>	<i>ac</i>	<i>b</i>
<i>abcef</i>	<i>abcf</i>	<i>abe</i>	<i>ad</i>	<i>c</i>
<i>abdef</i>	<i>abde</i>	<i>abf</i>	<i>ae</i>	<i>d</i>
<i>acdef</i>	<i>abdf</i>	<i>acd</i>	<i>af</i>	<i>e</i>
<i>bcdef</i>	<i>abef</i>	<i>ace</i>	<i>bc</i>	<i>f</i>
	<i>acde</i>	<i>acf</i>	<i>bd</i>	
	<i>acdf</i>	<i>ade</i>	<i>be</i>	
	<i>acef</i>	<i>adf</i>	<i>bf</i>	
	<i>adef</i>	<i>ae f</i>	<i>cd</i>	
	<i>bcde</i>	<i>bcd</i>	<i>ce</i>	
	<i>bcdf</i>	<i>bcd</i>	<i>cf</i>	
	<i>bcef</i>	<i>bcf</i>	<i>de</i>	
	<i>bdef</i>	<i>bde</i>	<i>df</i>	
	<i>cdef</i>	<i>bdf</i>	<i>ef</i>	
		<i>bef</i>		
		<i>cde</i>		
		<i>cdf</i>		
		<i>cef</i>		
		<i>def</i>		

where therefore the number of systems of first order is $6 = \binom{6}{1}$, of second order $15 = \binom{6}{2}$, of third order $20 = \binom{6}{3}$, of fourth = $15 = \binom{6}{4}$, of fifth = $6 = \binom{6}{5}$.

18. Now it is evident each system and of any lower order to be contained in all greater, it makes a great deal of difference to observe how often this happens. Therefore for the case $m = 6$ the system of the first order *abcde* occurs once in accordance with this order. But the system of the second order *abcd* twice in the first order, in the second it occurs once. Next the system of the third order *abc* three times in the first order, in the second three times, but in the third it is found once. The system of the fourth order

ab four times in the first order, in the second six times, in the third four times, in the fourth it is in there once. Finally the system of the fifth order occurs in the first order five times, in the second ten times, in the third ten times, in the fourth five times, in the fifth once. From which it is clear these numbers coincide with the coefficients of a binomial raised to a power, if in fact all systems are contained in that principal once.

19. Thenceforth therefore in general for any system whatsoever of some lesser order one is able to assign easily, as many ways in whatsoever higher order it occurs, the fact which the following table makes clear, where I denote the principal system by the letter O , but the systems of the first, second, third fourth etc. order by the Roman numerals I, II, III, IV, V, VI etc.

	O	I	II	III	IV	V	VI
m	1						
$m - 1$	1	1					
$m - 2$	1	2	1				
$m - 3$	1	3	3	1			
$m - 4$	1	4	6	4	1		
$m - 5$	1	5	10	10	5	1	
$m - 6$	1	6	15	20	15	6	1
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$m = \lambda$	$\binom{\lambda}{0}$	$\binom{\lambda}{1}$	$\binom{\lambda}{2}$	$\binom{\lambda}{3}$	$\binom{\lambda}{4}$	$\binom{\lambda}{5}$	$\binom{\lambda}{6}$

20. Now we consider the number list, whatever things are being extracted from the principal system by drawing any in a sequence, which can be imagined to be extracted from the derived system, at least any will be able to be extracted easily from the principal. But if a single letter is extracted by drawing any whatever, for the principal system the number of different drawings will be

$$= \binom{m}{1};$$

but if on the contrary two letters are simultaneously extracted, the number of all different drawings will be

$$\frac{m}{1} \cdot \frac{m-1}{2} = \binom{m}{2}.$$

If any three letters whatever be removed by drawing, the number of different drawings will be

$$\binom{m}{3}$$

and in general, if any i letters whatever are extracted by drawing, the number of all different drawings will be

$$\binom{m}{i}$$

But if however such drawings are imagined to happen from the derived systems besides, the number of different drawings for whatsoever system of the first order will be $\binom{m-1}{i}$, of second order $\binom{m-2}{i}$, of third order $\binom{m-3}{i}$ and thus further.

21. But if now these drawings be twice repeated, since for the principal system all are able to be followed whatever had been extracted not just of the remaining, but furthermore the same ones, the number of different cases will be $\left(\frac{m}{i}\right)^2$. If three successive drawings are made, the number of all possible cases will be $\left(\frac{m}{i}\right)^3$; and in general, if n drawings succeed one another, the number of all possible cases will be $\left(\frac{m}{i}\right)^n$, which number we designated by the letter Δ above, just as it is

$$\Delta = \left(\frac{m}{i}\right)^n .$$

22. In like manner the number of all cases, which in whatever system of first order they can take place, is $\left(\frac{m-1}{i}\right)^n$; hence when the number of these systems be $\left(\frac{m}{i}\right)$, the number of all cases, which the first order allows, will be $\left(\frac{m}{i}\right) \left(\frac{m-1}{i}\right)^n$ and which we designate by the letter A , just as it is

$$A = \left(\frac{m}{1}\right) \left(\frac{m-1}{i}\right)^n .$$

In the same manner it is easy to understand the number of all cases, which from a single system are able to arise, to be for the second order

$$B = \left(\frac{m}{2}\right) \left(\frac{m-2}{i}\right)^n ,$$

for the third order

$$C = \left(\frac{m}{3}\right) \left(\frac{m-1}{i}\right)^n ,$$

for the fourth order

$$D = \left(\frac{m}{4}\right) \left(\frac{m-1}{i}\right)^n$$

and thus further. Now by these first fruits it is permitted to obtain the solution of each preceding problem easily.

FOR THE FIRST PROBLEM

23. Since in this problem out of all possible cases, of which the number is Δ , they, which involve all m letters, must be enumerated thenceforth we exclude first all cases, which contain as many as $m-1$ letters or less, because it will happen, if we withdraw all possible cases of first order, of which the number is A . For in this manner the cases, which contain $m-1$ letters, will be removed from the midst. But indeed the cases, which contain $m-2$ letters, will be withdrawn twice in this manner; whence in the formula $\Delta - A$ they will be lacking once, just as of them the number $1-2 = -1$. But for the case containing $m-3$ letters the number, which occur in the formula $\Delta - A$, will be $1-3 = -2$. In like manner for $m-4$ we will have $1-4 = -3$ and thus further, therefore any deficient cases must be restored again.

24. Moreover the deficient cases of form $m-2$ will be restored once, if B is added to formula $\Delta - A$. But in this manner the terms of the form $m-3$ are added three

times, when nevertheless they had been lacking only twice; therefore now they are one abundant, or the index will be +1. But the form $m - 4$ is added six times, when only three times had been lacking, and therefore the index will be +3. In the same manner for the terms of the form $m - 5$ the index will be $10 - 4 = +6$ and thus further.

25. Therefore in order that now we remove those abundant cases again, we subtract all cases of third order, = C . For in this manner the terms of the form $m - 3$ will be removed thoroughly, but the remainder will be removed much too often, certainly for the order $m - 4$ the index will be -1 , for the order $m - 5$ the index will be -4 etc.

26. Because the form $m - 4$ is lacking one, restitution happens by adding the letter D . But now the last will be too numerous according to the indexes 1, 5, 15 etc., whence these are removed by subtracting E ; because too much is subtracted, it will be restored by addition of the letter F and thus further.

27. Hence now it is clear well enough all cases containing fewer than m letters to have been removed from the form Δ ; of which therefore the number remaining will be

$$\Delta - A + B - C + D - E + F - \text{etc.},$$

which we have indicated by Σ ; and thus the solution of the first problem is firmly demonstrated.

FOR THE SECOND PROBLEM

28. It is clear according to the same, that we have used here, the reasoning in the preceding can prepare the demonstration to the solution of the second problem. And the work will be not at all so prolix to set forth. For when from the number of possible cases Δ they, which contain as many as $m - 1$ letters, are enumerated, immediately from here it is manifest to be excluded all the cases containing $m - 2$ letters, which will happen, if the number B is subtracted from the number Δ . But the table given above §19 makes clear in this manner the terms of the form $m - 3$ will be removed three times, when nevertheless they ought have been subtracted only once, and thus concerning the remaining forms. To restore them the number $2C$ is added, by which the deficient numbers of the form $m - 3$ will be removed thoroughly; but the numbers of the form $m - 4$ will overflow by index 3, and besides they overflow sequentially to a greater extent. Where the previous is removed, again the number $3D$ must be subtracted, by which removal the terms of the form $m - 4$ will have been excluded. The deficient numbers of the form $m - 5$ and of the following again will be restored by addition of the numbers $4E$ and thus further; by which completed operation the number of the remaining will be

$$\Sigma' = \Delta - B + 2C - 3D + 4E - 5F + \text{etc.};$$

and thus the solution of the second problem is demonstrated.

FOR THE THIRD PROBLEM

29. Here the number C must be subtracted from the number Δ , by which the cases $m - 3$ of the list are excluded; and because by this manner the number D^6

⁴The first edition has in place of D : $m - 4$.

is subtracted four times, when it was overabundant only once, again the number $3D$ must be added, by which the former and the sequentially deficient are restored. What exceeds by subtraction of the number $6E$ will be restored, indeed the deficiencies will have been restored by the addition of the number $10F$ and thus further; whence the number of cases containing $m - 2$ letters will be

$$\Sigma'' = \Delta - C + 3D - 6E + 10F - \text{etc.};$$

just as I had asserted in the solution of the third problem. In this manner therefore this solution is firmly demonstrated as well.