CALCULATIONS ON AEROSTATIC BALLOONS MADE BY THE LATE MR. LEONHARD EULER, AS THEY WERE FOUND ON HIS BLACKBOARD, AFTER HIS DEATH ON SEPTEMBER 7, 1783

LEONHARD EULER

FOREWORD

The experiment performed in Annonay on June 5, 1783, by Messrs. Montgolfier,[1] to demonstrate the possibility of raising bodies of large capacity relative to their weight in air, by filling them with an expandable fluid that is lighter than the air in the atmosphere and whose elasticity nevertheless is in equilibrium with that of air. No sooner was this known than Europe's scientists hastened to attend to a matter that presented new questions to resolve to almost all of the sciences, giving hope to some of obtaining a new pathway to discoveries, and arousing curiosity through the host of real and fanciful applications that the means of navigating through an element, which up until now had been closed, presented at first glance.

Mr. Euler had only little time before his death to learn about Mr. Montgolfier's[2] discovery. The first thing that came to his mind was the idea of searching for the

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[1]Translator's note: These are the brothers Montgolfier, Joseph-Michel and Jacques-Étienne, inventors of the Montgolfière style hot-air balloon.

[2]Translator's note: It is unclear why there is a switch to a single "Mr. Montgolfier" at this point; this could simply be an error made by the author of the foreword or the printer. The version that appears in the *Opera Omnia* retains this reference to one person.

laws of vertical motion of a sphere rising in still air by virtue of an upward force due to its lightness. He immediately tried to apply computation to this question. And when he was taken unexpectedly by death, the blackboard on which he wrote with chalk since being almost deprived of his sight, was laden with these calculations, the last ones made by this great man, perhaps as singular for the incredible number of his works as for the depth and the force of his genius.

Mr. Euler's son,[3] his successor to the position of Foreign Associate of the Academy of Sciences[4] that he held among us, was pleased to send a copy of these calculations to the Academy, which then hastened to publish them as a precious monument bearing the final thoughts of one of the men who bestowed the most honor to the sciences, as a singular proof of the mind's vigor, that can still exist on a few hours before the moment where an unknown cause destroys the secret essence of intelligence and of life, and finally as an honor rendered to the author of the new discovery, since this very attempt at calculation exhibits the interest it has stirred in one of these men whose approbation is the most worthy reward that genius can strive after.

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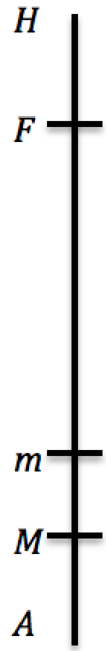
[3]Translator's note: Johann-Albrecht
[4]Translator's note: Paris



Let the aerostatic sphere have radius $= a$ and weight $= M$, and therefore its volume $= \frac{4}{3}\pi a^3$, with π denoting the circumference of a circle whose diameter $= 1$. Let the height of the air column $= k =$ 24000 feet approximately; and if we suppose that the sphere reaches an altitude of $AM = x$, then the air pressure would be $= e^{-\frac{x}{k}}$.

Let the velocity of the sphere at point $M = v$ and the weight of the airborne sphere be $= N$. Since the hemisphere has surface area $= \frac{\pi aa}{2}$, the resistance at this point M will be

$$= \frac{vv}{4g} \cdot \frac{\pi aa}{2} \cdot \frac{N}{\frac{4}{3}\pi a^3} = \frac{3N}{8a} \cdot \frac{vv}{4g},$$

where g denotes the altitude fallen due to gravity in one second.

The principles of mechanics supply this equation:

$$2v\partial v = \frac{4g\partial x}{M} \cdot P,$$

where ∂x is the element of altitude Mm and P the force acting upon it, which is composed of the expressions for air, weight of the sphere and resitance, so that

$$P = Ne^{\frac{x}{k}} - M - \frac{3N}{8a} \cdot \frac{vv}{4g} \cdot e^{-\frac{x}{k}},$$

whence

$$2v\partial v = \frac{4g\partial x}{M}\left(Ne^{-\frac{x}{k}} - M - \frac{3N}{8a} \cdot \frac{vv}{4g} \cdot e^{-\frac{x}{k}}\right),$$

or

$$2v\partial v = 4g\partial x\frac{N}{M}e^{-\frac{x}{k}} - 4g\partial x - \frac{3}{8a}\cdot\frac{N}{M}\cdot vv\partial xe^{-\frac{x}{k}}.$$

Let $\frac{N}{M} = \lambda$; then

$$2v\partial v + \frac{3\lambda}{8a}\cdot vve^{-\frac{x}{k}}\partial x = 4g\partial x\left(\lambda e^{-\frac{x}{k}} - 1\right),$$

whose integral, setting $\frac{8a}{3\lambda} = b$, will be

$$vve^{\frac{x}{b}} = \int 4g\partial x\left(\lambda - 1 - \frac{\lambda x}{k}\right)e^{\frac{x}{b}},$$

so that it would lead to:

$$vve^{\frac{x}{b}} = 4\lambda g\int e^{\frac{x}{b}}\partial x\left(\frac{\lambda-1}{\lambda} - \frac{x}{k}\right) = \frac{4\lambda g}{k}\int e^{\frac{x}{b}}x\left(f - x\right),$$

where

$$f = \frac{\left(\lambda-1\right)k}{\lambda}.$$

However

$$\int e^{\frac{x}{b}}\partial x\left(f - x\right) = b\left(b + f\right)\left(e^{\frac{x}{b}} - 1\right) - be^{\frac{x}{b}}x,$$

therefore

$$vve^{\frac{x}{b}} = \frac{4\lambda gb}{k}\left[\left(b+f\right)\left(e^{\frac{x}{b}} - 1\right) - e^{\frac{x}{b}}x\right],$$

whence

$$vv = \frac{4\lambda gb}{k}\left[\left(b+f\right)\left(1 - e^{-\frac{x}{b}}\right) - x\right],$$

an expression which determines the speed of the sphere at any altitude.

To determine the maximum altitude that the sphere can reach, it may be established that the speed v, or rather its square vv, vanishes at point H, and setting the elevation $AH = h$, which is then determined by the equation

$$(b+f)\left(1-e^{-\frac{x}{b}}\right)-h=0,$$

from which

$$b+f=\frac{h}{1-e^{-\frac{h}{b}}}=\frac{he^{\frac{h}{b}}}{e^{\frac{h}{b}}-1}.$$

Let $f = nb$, then

$$b+f=(n+1)b,$$

and since h is much greater than b, then we can, without percentible error, set

$$e^{\frac{h}{b}}-1=e^{\frac{h}{b}},$$

from which we obtain

$$b+f=b\left(n+1\right),$$

and therefore the maximum altitude $AH = b\left(n+1\right)$, where $b = \frac{8a}{3\lambda}$, and

$$n=\frac{3(\lambda-1)k}{8a}\ \left[\text{for}\left(\frac{\lambda-1}{\lambda}\right)k=f=nb=\frac{8an}{3\lambda}\right].$$

For the time of ascent the equation $v\partial t = \partial x$ leads to

$$\partial x=\partial t\sqrt{\frac{4\lambda gb}{k}}\sqrt{\left[h\left(1-e^{-\frac{x}{b}}\right)-x\right]},$$

with ∂t denoting the time element. Therefore

$$\partial t\sqrt{\frac{4\lambda gb}{k}} = \frac{\partial x}{\sqrt{\left[h\left(1-e^{-\frac{x}{b}}\right)-x\right]}}.$$

Setting

$$t = \sqrt{\frac{k}{4\lambda gb}}\int\frac{\partial x}{\sqrt{h-x}}$$

and the integral will be[5]

$$t = \sqrt{\frac{k}{4\lambda gb}}\left[C-2\sqrt{h-x}\right] = \sqrt{\frac{k}{4\lambda gb}}\left[2\sqrt{h}-2\sqrt{h-x}\right]$$

from which we obtain that the ascent time through the interval AM is

$$t = \sqrt{\frac{kh}{\lambda gb}}\left(1-\sqrt{\frac{h-x}{h}}\right)$$

and the total ascent time is

$$\sqrt{\frac{kh}{\lambda gb}}.$$

To determine the altitude F, where the speed is a maximum, let

$$\partial\cdot\left[(b+f)\left(1-e^{-\frac{x}{b}}\right)-x\right] = 0$$

and thus

$$\frac{\partial x}{b}(b+f)\,e^{-\frac{x}{b}} = \partial x,$$

[5]Translator's note: In the original publication, ∂t appears in place of t in the previous expression. This was corrected and noted in the *Opera Omnia*.

whence

$$\frac{b+f}{b} = e^{\frac{x}{b}}.$$

Consequently,

$$x = bl\,(b+f) - blb = bl\,(n+1)\,,$$

and therefore $AF = bl\,(n+1)$. Substituting this value into the expression for speed we obtain

$$vv = \frac{4\lambda gbb}{k}\left[(n+1)\left(1-e^{-l(n+1)}\right) - l\,(n+1)\right].$$

However

$$e^{-l(n+1)} = \frac{1}{n+1},$$

and thus

$$vv = \frac{4\lambda gbb}{k}\left[n - l\,(n+1)\right].$$

Therefore, the maximum speed at F will be

$$= 2b\sqrt{\frac{\lambda g}{k}\left[n - l\,(n+1)\right]},\ \text{or}\ 2b\sqrt{\frac{\lambda n g}{k}},$$

for very large n.

Example

Let $a = 30$ ft., $\lambda = 5$, it will be that $b = 16$ and $n = 1200$, whence the maximum altitude $AH = 19200$ feet, the altitude corresponding to the maximum velocity is $AF = 112$ feet, the maximum speed 64 feet per second, and the time of ascent 10 minutes and 32 seconds.