THE CONSTRUCTION OF A CERTAIN PROBLEM OF

PAPPUS OF ALEXANDRIA[*]

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(Translated by Cameron Friend, Research Fellow, Quest University, and Dr. Cynthia J. Huffman, Pittsburg State University, figures are courtesy of the Linda Hall Library, www.lindahall.org)

Theorem.

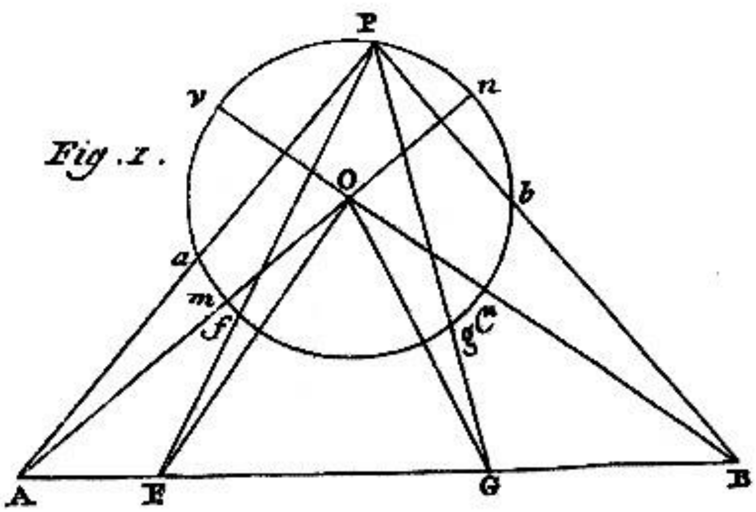
If lines[†] AP and BP from the ends of any line AB to any point P of any circle are drawn, cutting the circle at a and b,[‡] then moreover [if] points F and G are taken, so that it is

$$AF = \frac{AP \cdot Aa}{AB} \text{ and } BG = \frac{BP \cdot Bb}{AB},$$

then it will always be [the case that]

$$FP \cdot Ff = GP \cdot Gg = AF \cdot BG.$$

Fig. 1.



[*] Proposition 117 of Book 7 of Pappus' *Collection*. The problem of Pappus is, given a circle and 3 collinear points outside the circle, inscribe a triangle on the circle, such that when the sides are extended, they pass through the three collinear points. In this paper, Euclid generalizes the problem to the situation where the three points are not necessarily collinear.

[†] While "rectae" could be translated as either "line," or "line segment," it is here translated as "line" to maintain historical accuracy, though mathematical context dictates that this is truly a line segment.

[‡] In the original text, this "a and b" portion has capital letters for the points, but it is clear from the diagram that they are meant to be lowercase.

Demonstration.

Let the position of line segment AB with [those of] points F and G be shown with respect to the center O of that circle [see Fig.1], and let it be set forth that $AO = a, BO = b$,[§] the radius of the circle $Om = On = r$, and $AB = c$, then moreover let $FO = f, GO = g$, and it will be[**] $AP \cdot Aa = An \cdot Am$. Truly it is $An = a + r$ and $Am = a - r$, and therefore

$$AP \cdot Aa = aa - rr.$$

In a similar manner, it will be that

$$BP \cdot Bb = B\nu \cdot B\mu = (b + r)(b - r),$$

or

$$BP \cdot Bb = bb - rr.$$

In the same way, it is inferred that it will be the case that

$$FP \cdot Ff = ff - rr \text{ and } GP \cdot Gg = gg - rr.$$

With these [equations] having been assumed, consequently

$$AF = \frac{aa - rr}{c} \text{ and } BG = \frac{bb - rr}{c},$$

it must demonstrated that it will be

$$ff - rr = gg - rr = \frac{(aa - rr)(bb - rr)}{cc},$$

to which conclusion the following Lemma must be invoked in support.

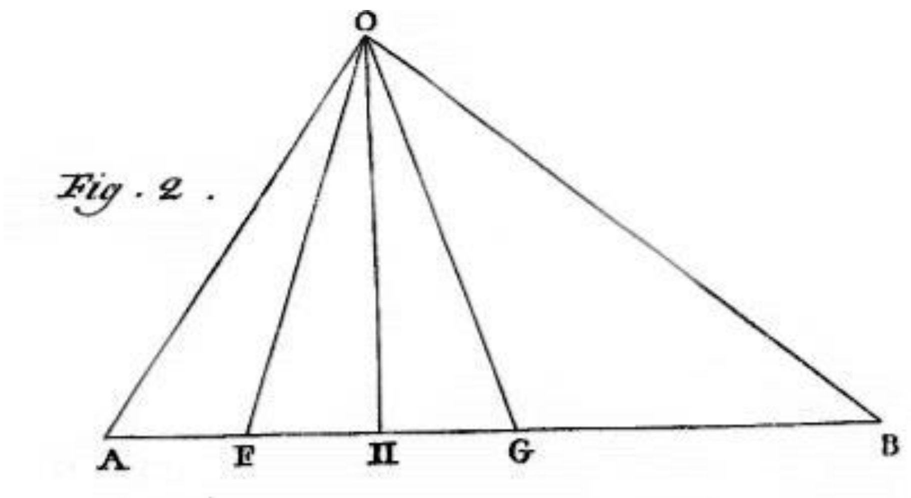
Lemma.

If the line segment OF is drawn from the point O of triangle AOB to a given point F of the opposite side AB, it will be

$$FO^2 = \frac{AO^2 \cdot BF + BO^2 \cdot AF}{AB} - AF \cdot BF.$$

[§] Note that these are different from the a and b in Fig.1, which are points rather than lengths.

[**] By the Intersecting Secants Theorem (http://www.mathopenref.com/secantsintersecting.html) which is proven using similar triangles and related to the Intersecting Chords Theorem, Euclid III.36



Demonstration.

Having dropped a perpendicular $O\Pi$ from O [††] to AB, it will be

$$AO^2 = A\Pi^2 + \Pi O^2 = (AF + F\Pi)^2 + FO^2 - F\Pi^2,$$

or

$$AO^2 = AF^2 + FO^2 + 2AF \cdot F\Pi;$$

and in the same manner, it will be

$$BO^2 = BF^2 + FO^2 - 2BF \cdot F\Pi \ .$$

If the former of these equations multiplied by BF is added to the other multiplied by AF, it will produce

$$AO^2 \cdot BF + BO^2 \cdot AF = BF\left(AF^2 + FO^2\right) + AF\left(BF^2 + FO^2\right)$$

or rather

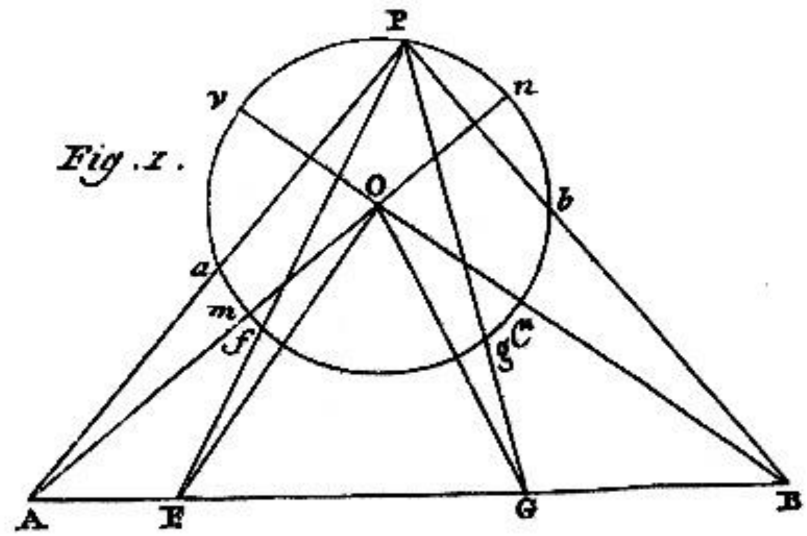
$$AO^2 \cdot BF + BO^2 \cdot AF = FO^2 \cdot AB + BF \cdot AF \cdot AB,$$

whence

$$FO^2 = \frac{AO^2 \cdot BF + BO^2 \cdot AF}{AB} - AF \cdot BF \ .$$ Q.E.D.

[††] The original text has “ E ” here.

Continuation of the previous demonstration.



Let it be set forth that $AF = \frac{aa - rr}{c} = \alpha$, $BG = \frac{bb - rr}{c} = \beta$, and it will be

$$aa = \alpha c + rr \text{ and } bb = \beta c + rr .$$

Now, from the Lemma it will be

$$cff = aa(c - \alpha) + bb\alpha - \alpha c(c - \alpha) ,$$

and if the values given now are substituted in the place of aa and bb, it will be had

$$cff = crr + \alpha\beta c, \text{ or } ff - rr = \alpha\beta .$$

Using the former, let it be

$$GO^2 = \frac{BO^2 \cdot AG + AO^2 \cdot BG}{AB} - AG \cdot BG ,^{‡‡}$$

in the same way, it is shown that

$$cgg = crr + \alpha\beta c, \text{ resulting in } gg - rr = \alpha\beta$$

hence

$$ff - rr = gg - rr = \frac{(aa - rr)(bb - rr)}{cc} . \text{ Q. E. D.}$$

‡‡ The original uses F on the right side instead of G, but Euler is applying the previous lemma to GO^2 not FO^2.

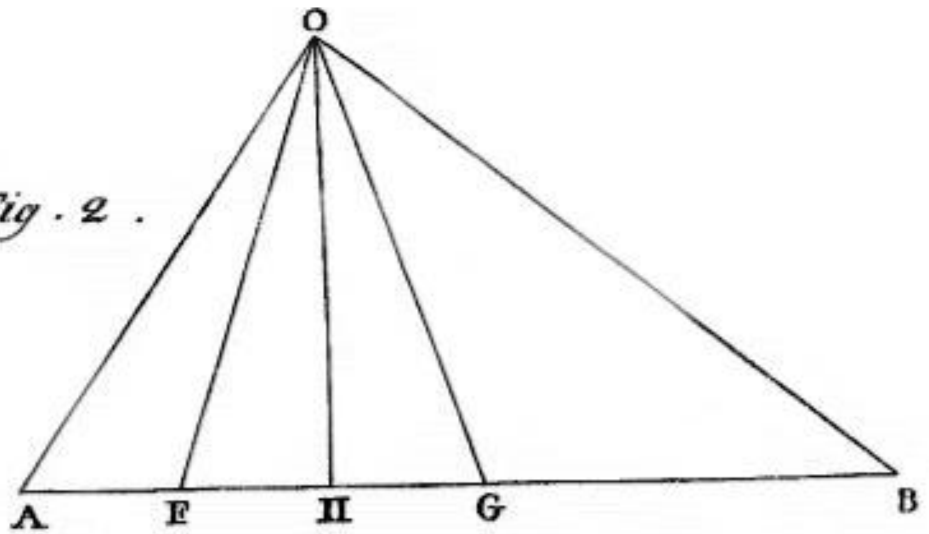
From this can be formed the following

Theorem.

If from the point O of triangle ABO [§§], two lines OF and OG are drawn to the base AB [which] are equal to each other, it will be

$$AO^2 - AB \cdot AF = BO^2 - AB \cdot BG .$$

Fig. 2.



Demonstration.

With a perpendicular $O\Pi$ having been dropped from vertex O, it will be

$$F\Pi = G\Pi = \frac{1}{2} FG .$$

And therefore it would be

$$AO^2 = AF^2 + FO^2 + 2AF \cdot F\Pi , \text{ making}$$
$$AO^2 = AF^2 + FO^2 + AF \cdot FG , \text{ it will be}$$
$$AO^2 = FO^2 + AF \cdot AG .$$

In a similar manner, it will be $BO^2 = FO^2 + BF \cdot BG.$ From the former it would be

[§§] The original text has "ABC" here.

$$FO^2 = AO^2 - AF \cdot AG \text{ , whence}$$

$$FO^2 - AF \cdot BG = AO^2 - AB \cdot AF \text{ .}$$

From the latter it would be

$$FO^2 = BO^2 - BF \cdot BG \text{ ,}^{***} \text{ consequently}$$

$$FO^2 - AF \cdot BG = BO^2 - AB \cdot BG \text{ ,}$$

whence it follows

$$AO^2 - AB \cdot AF = BO^2 - AB \cdot BG \text{ . Q. E. D.}$$

Corollary.

Therefore, it may be a number that

$$AO^2 - AB \cdot AF = BO^2 - AB \cdot BG = \Delta \text{ ,}$$

two lines FO and GO will be among themselves equals, and at the same time it will be $FO^2 - AF \cdot BG = \Delta$. But if therefore, it is taken that

$$AF = \frac{AO^2 - \Delta}{AB} \text{ and } BG = \frac{BO^2 - \Delta}{AB}$$

it will be $FO = GO$. But in the preceding Theorem it was

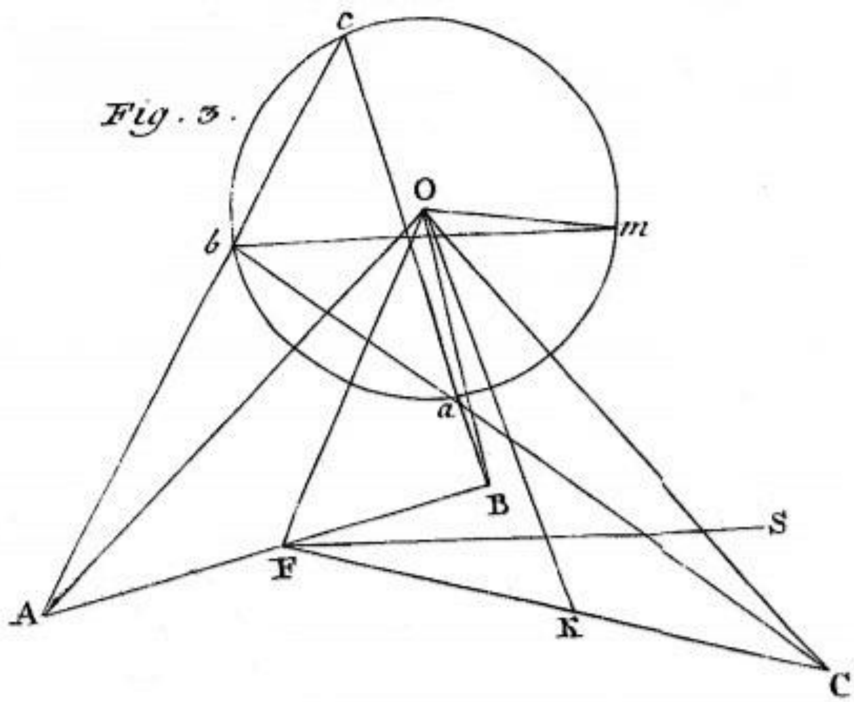
$$AF = \frac{aa - rr}{c} \text{ and } BG = \frac{bb - rr}{c} \text{ , whence}$$

$$\Delta = rr, \ AF = \frac{AO^2 - rr}{AB}, \ BG = \frac{BO^2 - rr}{AB} \text{ and } FO^2 - rr = AF \cdot BG \text{ .}$$

*** The original text has "BE" rather than "BF".

Problem.

In a given circle, with drawn center O, to inscribe triangle abc, whose three sides ab, ac, bc having been extended, cross through the given three points C, B, A.



Construction.†††

Let A, B, C be three given points, whose distances from the center O of the circle are $AO = a, BO = b, CO = c$ ‡‡‡; let the radius of the circle be $= 1$. Now from point B may the interval $BF = \frac{bb-1}{AB}$ be taken, and it will be $FO^2 - 1 = \frac{BF(aa-1)}{AB}$. Then, with line FC having been joined, on it may be captured the interval

$$FK = \frac{FO^2 - 1}{FC} = \frac{BF(aa-1)}{AB \cdot FC}, \text{ and it will be}$$

$$KO^2 - 1 = \frac{FK(cc-1)}{FC}.$$

††† In the paper immediately following this paper (E543) in *Acta Academiae Scientarum Imperialis Petropolitinae* 4, 1783, Euler's assistant Nikolai Fuss has a similar construction. Fuss uses a second concentric circle instead of the cosine function in his construction.

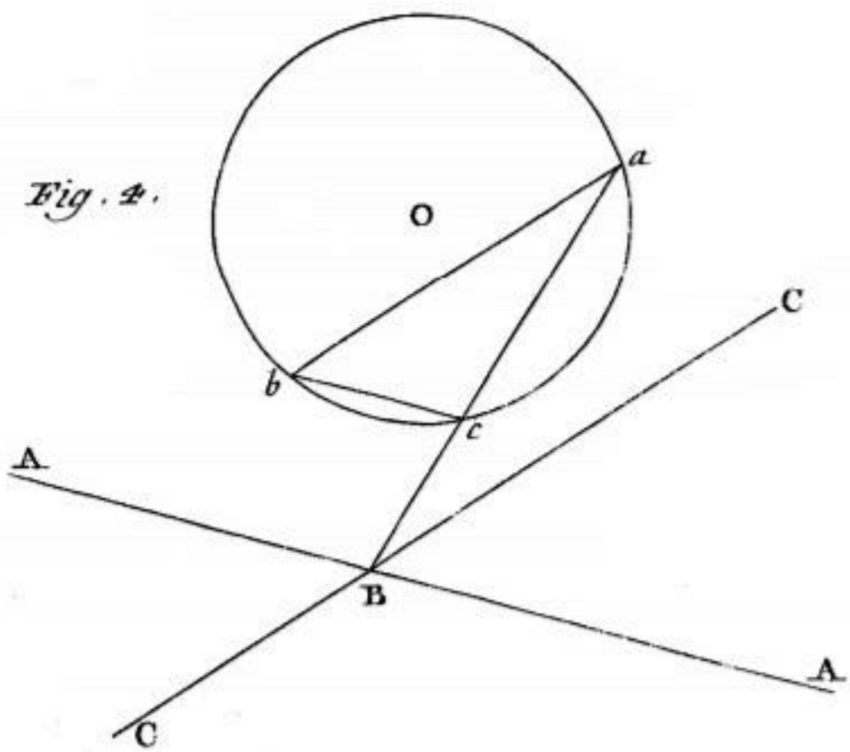
‡‡‡ At the beginning of the construction the lower case a, b, c represent lengths, but near the end they are points on the circle forming the desired triangle.

Now the radius Om is drawn from the center O, so that the cosine of angle $KOm = \frac{\cos BFC}{KO}$. Then let angle BFC be bisected by line segment FS, parallel to which line segment mb is drawn from point m, and b will be one of the angles of the desired triangle, to which if from point A line segment Ab[§§§] is drawn, which having been extended will cut the circle at c[****]. From this point c towards B the line segment cB is drawn cutting the circle at a; then certainly the side ba having been extended will cross through the third given point C, and it will result in abc the desired triangle.

Corollary 1.

Let two of the given points A and C be infinitely separated on the lines ABA, CBC crossing one another at B. From B the line Bca is drawn, cutting off an arc ca from the circle, on which the angles that may lie on the circumference of which are equal to angle CBA, then drawing from a the line ab parallel to CBC itself, and bc parallel to ABA itself, abc will be the desired triangle.

Fig. 4.



[§§§] This is uppercase in the original document, but clearly referring to the lowercase b it references.
[****] In the original text, this is an uppercase C, but we have changed it here and in the instances which follow.

Corollary 2.

Let all three given points fall at infinite distances on the lines OA, OB, OC; next the line bc may be drawn parallel to line OA at a distance from the center $OX = \cos BOC;$ then the desired triangle will contain line ba having been drawn parallel to OC itself and ca parallel to OB itself.

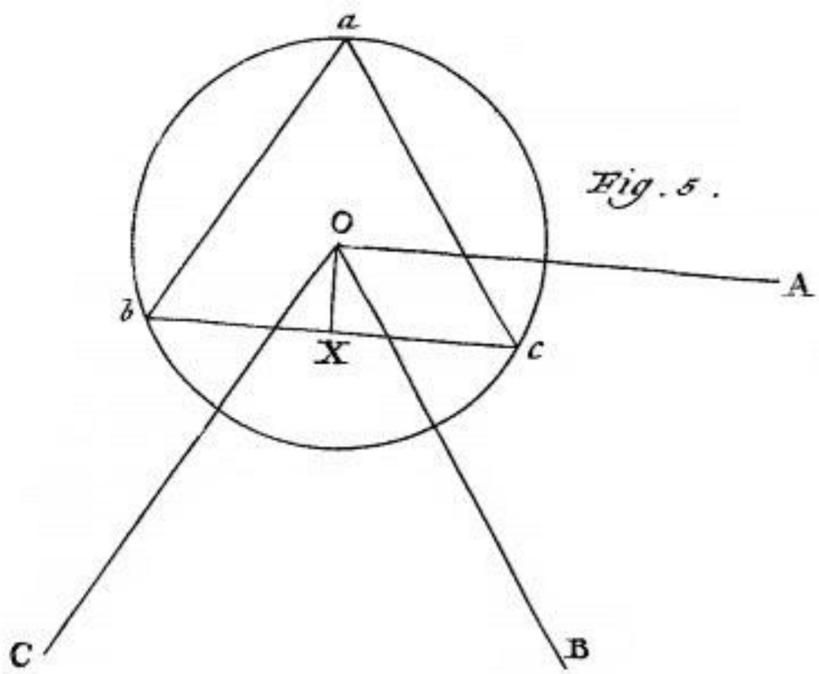


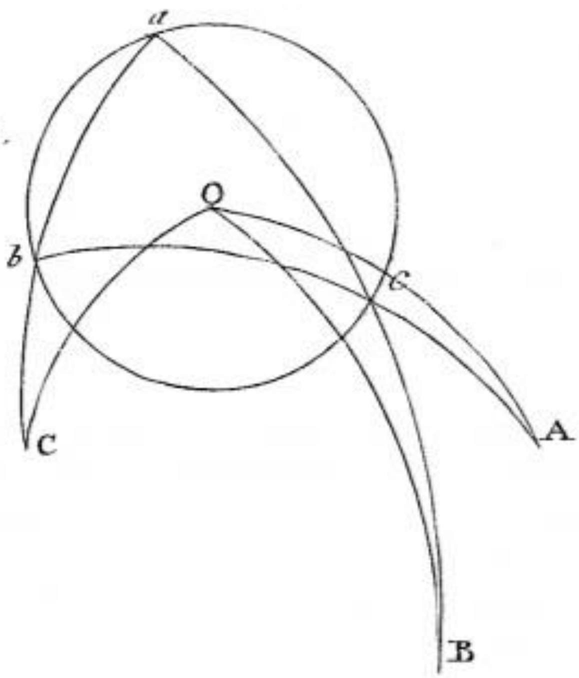
Fig. 5.

Comment 1.

Moreover, it should properly be noted here that the construction above supplies two given solutions, as the angle KOm is taken on the right side or on the left. Then since the three points A, B, C are permutable among themselves. the construction given here can be set up in six different ways, all of which ought to give a pair of solutions, on which matter no explanation is given.

Comment 2.

Fig. 6.



This Problem is also able to be solved for the sphere, so that a spherical triangle *abc* ought to be inscribed in a minor circle described on a sphere, having been set up, such that its sides *ab*, *ac*, *bc*, having been extended cross through the given three points, *C*, *B*, *A*, on the surface of the sphere. For the plane is understood to be touching the sphere in the center *O* of the circle, on which let a planar triangle be constructed in the prescribed manner; and its translation to the surface of the sphere will be very easy, although all of the angles are the same around the center on the surface of the sphere as on a plane, nevertheless the distances of the given points *A*, *B*, *C* and of the angles *a*, *b*, *c* of the triangle from the center *O* to the touchings may be different.

Problematis cuiusdam Pappi Alexandrini constructio (E543, *The Construction of a Certain Problem of Pappus of Alexandria*) was originally published in *Acta Academiae Scientarum Imperialis Petropolitinae* 4, 1783, pp. 91-96, and also appeared in *Opera Omnia*: Series 1, Volume 26, pp. 237 - 242 According to the records, it was presented to the St. Petersburg Academy on October 31, 1782.

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