On Lambert Series

And Their Significant Properties*

1773

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Translator's Preface

Originally published in 1779, Euler's "De Serie Lambertina" provides one of the early examples of the Lambert W function, a special function used in the solution to certain transcendental equations. Following the work of Johann Heinrich Lambert in 1759, who discussed a series solution to the general polynomial in series, and then particularly the solution of the general trinomial, Euler describes a symmetric form of the trinomial and its series solution. Euler investigates the series' special cases and general properties, and its use in solving certain transcendental equations. He provides several proofs of the validity of the series expansion to solve the trinomial, and in doing so he reveals several notable series expansions of functions such as the natural logarithm and the factorial.

I originally heard about the W function while doing some routine algebra, and became interested in reading Euler's original work. At the time, I knew no Latin, and put a fair amount of effort into avoiding learning, but eventually I had exhausted my available resources and decided that I would make the translation myself. This translation is the result of several months of work, first learning the language, then actually translating. I have tried to provide context for some of the more opaque elements of "De Serie Lambertina" with endnotes, containing my own understanding of the content, which I hope is enough to allow readers with a modest mathematical background to follow along.

Learning the Latin language is no simple task. If it were not for the help of my brother Charlie, starting me on the right path and giving me advice about translation, I cannot imagine I would have had as much success as I did. In this process, I also met many incredible people, who were kind enough to answer my many questions about Euler's writing. In particular, I would like to thank Sebastian Koppehel, the community on Latin StackExchange, and the Latin subreddit. As well, I would like to thank the inimitable Ian Bruce for setting an inspiring example for all those with an interest in translation and historical mathematics and physics. Finally, I would like to thank Euleriana and the Euler Archive for the opportunity to have my translation reviewed and published, so that Euler's work might reach a broader audience.

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*Euler, L. *De serie Lambertina, plurimisque eius insignibus proprietatibus.* Acta Academiae Scientarum Imperialis Petropolitinae 1779, 1783, pp. 29-51. Enestrom number E532.

§. 1.

With this name may be addressed that most notable series, by means of which the most clever and sharp Lambert first described the expression of the roots of a trinomial equation in *Acta Helvetica Volum. III.*[1] This series however, if its terms may be altered for a moment, can be expressed in the following form:

$$\begin{aligned} S = 1 + nv + \frac{1}{2}n(n+\alpha+\beta)v^2 & \\ + \frac{1}{6}n(n+\alpha+2\beta)(n+2\alpha+\beta)v^3 & \\ + \frac{1}{24}n(n+\alpha+3\beta)(n+2\alpha+2\beta)(n+3\alpha+\beta)v^4 & \\ + \text{etc.} & \end{aligned}$$

of which series the sum, S, thus depends on the resolution of the trinomial equation:

$$x^\alpha - x^\beta = (\alpha-\beta)vx^{\alpha+\beta}, \text{ in order that } S = x^n$$

where, as that equation can have several roots, x should be understood to be the maximum or minimum of the roots, as the circumstances require. Moreover, it seemed proper to provide this series with the present form, as the letters α and β may become interchangeable, in such a way that whatever may be observed from one may also be valid for the other.

§. 2. The special properties of this series therefore depend on this: in order that its sum may always be equal to the power of the exponent n, to which any determined quantity is raised.[2] From which, if for the value of n itself with whatever $n = p$ the series sum is taken to be $= P$; and further for another value whatsoever $n = q$ the series sum is taken to be $= Q$; then, because we have $P = x^p$ and $Q = x^q$, it shall be clear that $P^q = Q^p$, or $\frac{\log P}{\log Q} = \frac{p}{q}$; and in these circumstances, as long as the series sum for a unique case of the exponent n is known, then the sums for any other values whatsoever can always be assigned, supposing the remaining quantities α, β, and v keep their values. It is thus to be desired most, in order that that significant property from the innate character itself of the series may be shown.

§. 3. Here then, before all else, an important case ought to be noted, for which $n = 0$ and the sum $S = 1$. Thus when we have $S = x^n$, it is well known that in the case of $n = 0$, the formula $\frac{x^n-1}{n}$ reduces towards the hyperbolic logarithm of x, by which reason this case of ours provides at once a summation worthy of remembering:[3]

$$\begin{aligned} \log x = v + \frac{1}{2}(\alpha+\beta)v^2 + \frac{1}{6}(\alpha+2\beta)(2\alpha+\beta)v^3 & \\ + \frac{1}{24}(\alpha+3\beta)(2\alpha+2\beta)(3\alpha+\beta)v^4 & \\ + \frac{1}{120}(\alpha+4\beta)(2\alpha+3\beta)(3\alpha+2\beta)(4\alpha+\beta)v^5 & \\ + \text{etc.} & \end{aligned}$$

But if then the sum of the series may now be explored, denoting it by $= \Delta$, instead of $\log x = \Delta$ one will have $x = e^\Delta$, denoting by e the number whose hyperbolic logarithm is

$= 1$. Thus, knowing this value Δ for any number n the sum of the proposed series will be $e^{n\Delta}$; from which, therefore, any number of other series may be shown; having been proposed equal, evidently:

$$S = 1 + n\Delta + \frac{1}{2}n^2\Delta^2 + \frac{1}{6}n^3\Delta^3 + \frac{1}{24}n^4\Delta^4 + \text{etc.}$$

Then certainly, because $\Delta = \log x$, at the same time we have that equation:

$$e^{\alpha\Delta} - e^{\beta\Delta} = (\alpha - \beta)ve^{(\alpha+\beta)\Delta} \text{ , or}$$

$$e^{-\beta\Delta} - e^{-\alpha\Delta} = (\alpha - \beta)v$$

From which equation one may also discover the value of Δ.

§. 4. In addition, the general sum of the proposed series may thus also be expressed, as, if we let:

$$v = \frac{x^{-\beta} - x^{-\alpha}}{\alpha - \beta}$$

The series sum would be $S = x^n$, and in fact any values whatsoever may be assigned to the letters α and β, if only it is noted, as we shall see, that when from several values taken for x the same value for v can result, then for the sum $S = x^n$ either the maximum or the minimum ought to be taken. With these things being generally noted, we may go through some special cases, with the ratio of the letters α and β, by which the conception of our series will be made moderately clear.

Case I.

For which $\beta = 0$.

§. 5. Since the letters α and β are interchangeable, accordingly either α or β may disappear. Hence let $\beta = 0$, and our series subsequently takes the form

$$\begin{aligned} S = 1 + nv + \frac{1}{2}n(n+\alpha)v^2 \\ + \frac{1}{6}n(n+\alpha)(n+2\alpha)v^3 \\ + \frac{1}{24}n(n+\alpha)(n+2\alpha)(n+3\alpha)v^4 \\ + \frac{1}{120}n(n+\alpha)(n+2\alpha)(n+3\alpha)(n+4\alpha)v^5 \\ + \text{etc.} \end{aligned}$$

whose sum therefore will be $S = x^n$, as long as x is taken from the equation $x^\alpha - 1 = \alpha v x^\alpha$, from which comes $x^\alpha = \frac{1}{1-\alpha v}$ and identically $x = (1-\alpha v)^{-1/\alpha}$. With which case the Lambert series now takes its most noteworthy form.

§. 6. Now if we make this exponent n vanish, this type of series will reduce to that of the logarithm, such that

$$\log x = v + \frac{1}{2}\alpha v^2 + \frac{1}{3}\alpha\alpha v^3 + \frac{1}{4}\alpha^3 v^4 + \frac{1}{5}\alpha^4 v^4 + \text{etc.}$$

Thus using

$$x = (1-\alpha v)^{-\frac{1}{\alpha}}, \text{ we have } \log x = -\frac{1}{\alpha}\log(1-\alpha v).$$

Make note however,

$$\log(1-\alpha v) = -\alpha v - \frac{1}{2}\alpha^2 v^2 - \frac{1}{3}\alpha^3 v^3 - \frac{1}{4}\alpha^4 v^4 - \text{etc.}$$

A series which having been multiplied by with $-\frac{1}{\alpha}$ gives the series just discovered.[4]

Case II.

For which $\beta = \alpha$.

§. 7. This case is most notable, because the equation, from which the value of x should be derived, is inconsistent, clearly: $x^\alpha - x^\alpha = 0vx^{2\alpha}$, or $0 = 0$; to avoid this inconvenience we take $\alpha = \beta + \omega$, with ω being infinitely small, and our equation becomes:

$$x^{\beta+\omega} - x^\beta = \omega v x^{2\beta+\omega}, \text{ or}$$

$$\frac{x^\omega - 1}{\omega} = vx^{\beta+\omega}.$$

It is well known however that with ω vanishing one has $\frac{x^\omega - 1}{\omega} = \log x$, so that with this case $\log x = vx^{\beta+\omega} = vx^\alpha$, which is an equation from which the value of x can be elicited.

§. 8. Moreover, with β having been set to $\beta = \alpha$ we arrive at the following series:

$$\begin{aligned} S = 1 + nv &+ \frac{1}{2}n(n+2\alpha)v^2 + \frac{1}{6}n(n+3\alpha)^2v^3 \\ &+ \frac{1}{24}n(n+4\alpha)^3v^4 + \frac{1}{120}n(n+5\alpha)^4v^5 \\ &+ \frac{1}{720}n(n+6\alpha)^5v^6 + \text{etc.} \end{aligned}$$

a series which is equally most deserving of attention, because not only do the exponents increase continuously, but in fact the quantities being raised themselves appear in an arithmetic progression, a series of a sort scarcely considered by Geometers until now. Nevertheless we have learned here, that the sum of the series is $S = x^n$, as long as the value of x fits with this equation, namely: $\log x = vx^\alpha$, although it is only possible to obtain this value by approximation.

§. 9. Now if we further set $n = 0$, from what was shown before the following series is deduced:

$$\begin{aligned} \log x = v + \alpha v^2 &+ \frac{3^2}{6}\alpha\alpha v^3 + \frac{4^3}{24}\alpha^3 v^4 \\ &+ \frac{5^4}{120}\alpha^4 v^5 + \frac{6^5}{720}\alpha^5 v^6 + \text{etc.} \end{aligned}$$

Therefore, letting $\log x = vx^\alpha$, we have

$$\begin{aligned} x^\alpha = 1 &+ \frac{2^1}{1.2}\alpha v + \frac{3^2}{1.2.3}\alpha^2 v^2 + \frac{4^3}{1.2.3.4}\alpha^3 v^3 \\ &+ \frac{5^4}{1.2...5}\alpha^4 v^4 + \frac{6^5}{1...6}\alpha^5 v^5 + \text{etc.} \end{aligned}$$

We set $\alpha v = u$, as then $\alpha \log x = ux^\alpha$. Now let $x^\alpha = y$, so $\alpha \log x = \log y$; consequently our equation becomes $\log y = uy$, for which reason we obtain this summation:

$$\begin{aligned} y = 1 &+ \frac{2^1}{1.2}u + \frac{3^2}{1.2.3}uu + \frac{4^3}{1.2.3.4}u^3 \\ &+ \frac{5^4}{1.2...5}u^4 + \frac{6^5}{1...6}u^5 + \text{etc.} \end{aligned}$$

In the case where $u = \frac{\log y}{y}$.

§. 10. As in this series the exponents of the numbers are decreased from the numbers themselves by unity, we reduce them in the following way to equality. Multiplying on both sides by u and differentiating, it becomes

$$\frac{d\log y}{du} = \frac{dy}{ydu} = 1 + \frac{2^2}{1.2}u + \frac{3^3}{1.2.3}uu + \frac{4^4}{1.2.3.4}u^3 + \frac{5^5}{1.2...5}u^4 + \frac{6^6}{1...6}u^5 + \text{etc.}$$

Where moreover we set $\log y = uy$, then $\frac{dy}{y} = udy + ydu$, from which we have $\frac{dy}{du} = \frac{yy}{1-uy}$; with whatever that sum may be $= \frac{y}{1-uy}$. Multiplying further both sides by u, and because $uy = \log y$ we arrive at this most significant summation:

$$\frac{\log y}{1-\log y} = u + \frac{2^2}{1.2}u^2 + \frac{3^3}{1.2.3}u^3 + \frac{4^4}{1.2.3.4}u^4 + \frac{5^5}{1.2...5}u^5 + \text{etc.}$$

Under the condition that $u = \frac{\log y}{y}$.

§. 11. This last series merits attention on account of its elegance and utility, so we will examine its unique properties more carefully. So, firstly it is evident, if we may assume $u = 1$ or $u > 1$, that the produced series is divergent; as in the general form $\frac{n^n}{1.2...n}$ the numerator continually increases more than the denominator, and thus all terms in this way increase to infinity, whose sign is that of the imaginary sum; that which through the formula $u = \frac{\log y}{y}$ is clearly proven, since no number's logarithm can come out greater than the number itself. When however u is taken to be less than unity, that series sum most certainly is able to produce a finite value, whenever of course the formula $\frac{\log y}{y}$ takes a finite value, that which happens, when $\log y < 1$, or $y < e$. However if it is assumed that $y = e$, such that we have $u = \frac{1}{e}$, our series still has an infinite sum, even though our terms continuously decrease, and in this way finally vanish.

§. 12. In this however a notable series occurs at once, because, if u rises above $\frac{1}{e}$ only a little, they eventually rise infinitely above the terms, which agrees well with those things, which I formerly observed around the value of the product $1.2.3...n$ in *Calculo Differentiali* p. 466.[5] And if we also set $T = \frac{n^n}{1.2...n}$, such that

$$\log T = n\log n - \log 1 - \log 2 - \log 3. \ldots . - \log n$$

In the work just cited it is demonstrated that

$$\log 1 + \log 2 + \log 3 + \log 4. \ldots . + \log n = \frac{1}{2}\log 2\pi + \left(n + \frac{1}{2}\right)\log n - n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \text{etc.}$$

from which follows the product itself

$$1.2.3....n = \frac{\sqrt{2\pi} \times n^{n+\frac{1}{2}} \times e^{\frac{1}{12n} - \frac{1}{360n^3}\text{ etc.}}}{e^n}$$

as long as we have

$$T = \frac{1}{\sqrt{2n\pi}} \cdot e^{n - \frac{1}{12n} + \frac{1}{360n^3} - \text{etc.}}$$

When therefore n is a very large number, all terms of our series become $Tu^n = \frac{e^n u^n}{\sqrt{2n\pi}}$, out of which form it is clear, as soon as $eu > 1$, or $u > \frac{1}{e}$, then this term becomes infinite; but if however either $eu = 1$ or strictly < 1, or $u < \frac{1}{e}$, then that term is lessened into nothing.

§. 13. We may illustrate this summation by a unique example, having set $\log y = \frac{1}{2}$, as the series sum becomes $= 1$; then however we have $u = \frac{1}{2\sqrt{e}}$, by which case therefore we have

$$1 = u + \frac{2^2}{1.2}u^2 + \frac{3^3}{1.2.3}u^3 + \frac{4^4}{1...4}u^4 + \text{etc.}$$

Moreover with $e = 2.71828$, the values of this former series of terms will thus be found in fractional decimals that I have copied:

$$\begin{aligned} u &= 0.303269 \\ 2u^2 &= 0.183944 \\ \frac{9}{2}u^3 &= 0.125515 \\ \frac{32}{3}u^3 &= 0.090228 \\ \frac{5^4}{1.2.3.4}u^4 &= 0.066805 \\ \frac{3^2.6^2}{5}u^6 &= 0.050413 \end{aligned}$$

This series thus converges as slowly as possible, nevertheless to a total sum that is determined not to exceed unity.

On the Solution

of the equation $\log x = vx^{\alpha}$

§. 14. Because for the second case, where $\beta = \alpha$, the summation of our series depends on the equation $\log x = vx^{\alpha}$, then, for whatever value of v, the quantity x ought to be brought out: before all else it is appropriate to observe, that twin values of x can correspond to any one value of v. To show this we may make $x^{\alpha} = y$ and $\alpha v = u$, so that it has the equality: $\log y = uy$, or $u = \frac{\log y}{y}$; from which it is clear the number u cannot be positive, unless we grant that $y > 1$. However, then we always have $u < \frac{1}{e}$, according to which the maximum value of the formula $\frac{\log y}{y}$ arises by assuming $y = e$ in such a way that, whether y is taken to be either bigger or smaller than e, $u < \frac{1}{e}$ is always produced. Thus is it clear, that the series for the second case having been found cannot have a finite sum; as long as $u > \frac{1}{e}$, or $v > \frac{1}{\alpha e}$, accordingly v may be a positive quantity; for when it should have been negative, the sum would have always been finite on account of the alternating sign.

§. 15. Hence it further follows, as often as we have $u < \frac{1}{e}$, so often the value of y can have two values: one of course greater than e, the other being less, and the same value $u = \frac{\log y}{y}$ is produced in both cases. In just the same way, if we set either $y = 2$ or $y = 4$, both sides produce $u = \log\frac{2}{2}$. The same thing comes from the example, whether we set $y = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$, or $y = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$, because both sides produce $u = \frac{8}{9}\log\frac{3}{2}$. The same thing further happens, whether we assume $y = \left(\frac{4}{3}\right)^4$ or $y = \left(\frac{4}{3}\right)^3$, since both sides give $u = \frac{3^4}{4^3}\log\frac{4}{3}$.

§. 16. To a pair of values of y to be found, let p and q be such values, by which it comes about that $u = \frac{\log p}{p} = \frac{\log q}{q}$. We now set $q = pr$, and it is appropriate to make

$$\frac{\log p}{p} = \frac{\log pr}{pr} = \frac{\log p + \log r}{pr},$$

or $r\log p = \log p + \log r$, from which $\log p = \frac{\log r}{r-1}$, and thus $p = r^{\frac{1}{r-1}}$, and hence $q = r^{\frac{r}{r-1}}$, for which, so that more convenient formulas are returned, we make $\frac{1}{r-1}, = m$, such that $r = \frac{m+1}{m}$, from which both values of y, that we call p and q, will now be: either $y = p = \left(\frac{m+1}{m}\right)^m$, or otherwise $y = q = \left(\frac{m+1}{m}\right)^{m+1}$; as from both sides is produced $u = \frac{m^{m+1}}{(m+1)^m}\log\frac{m+1}{m}$.

§. 16.[6] With this exponent this question of the highest importance arises: which of these two values of y ought to be assigned to the sum of this expressed series:

$$y = 1 + \frac{2^1}{1.2}u + \frac{3^2}{1.2.3}uu + \frac{4^3}{1.2.3.4}u^3 + \frac{5^4}{1.2...5}u^4 + \text{etc.}$$

to the solution of which question let us assume first $u = \frac{1}{e}$, so that both values of y are equal to e; for nothing is uncertain from this case letting $y = e$. Certainly now, if we should have $u < \frac{1}{e}$, it's clear that the sum of the series comes out smaller than e. On account of which as for y we found two values, one greater, one lesser than e, it's evident that the lesser value must be taken for the sum of the expressed series. Thus if

$u = \frac{m^{m+1}}{(m+1)^m} \log \frac{m+1}{m}$, the value of y which ought to be assumed is $y = \left(\frac{m+1}{m}\right)^m$, which is naturally always less than e, for otherwise, $y = \left(\frac{m+1}{m}\right)^{m+1}$, which is greater than e.

Theorem.

§. 17. If the quantities x and v so depend upon each other, such that $\log x = vx$, and moreover the values of x correspond to twin values of v, one greater and one lesser than e, then in the following sums:

$$\text{I. } \frac{x^n - 1}{n} = v + \frac{n+2}{1.2}v^2 + \frac{(n+3)^2}{1.2.3}v^3 + \frac{(n+4)^3}{1.2.3.4}v^4 + \text{etc.}$$

$$\text{II. } \frac{x^n}{1 - \log x} = 1 + \frac{n+1}{1}v + \frac{(n+2)^2}{1.2}v^2 + \frac{(n+3)^3}{1.2.3}v^3 + \frac{(n+4)^4}{1.2.3.4}v^4 + \text{etc.}$$

in place of all x the lesser value should be taken, which is clearly less than e. The rationale for these two series is evident from the explanation of the second case: as they are found by assuming $\alpha = 1$ in [the series of] §8.

§. 18. The latter series is clearly produced by differentiation of the former; in fact, dividing by dv and differentiating we arrive at

$$\frac{x^{n-1}dx}{dv} = 1 + \frac{n+2}{1}v + \frac{(n+3)^2}{1.2}v^2 + \frac{(n+4)^3 v}{1.2.3}v^3 + \text{etc.}$$

Moreover letting $v = \frac{\log x}{x}$, we have $dv = \frac{dx}{xx}(1 - \log x)$, out of which

$$\frac{x^{n-1}dx}{dv} = \frac{x^{n+1}}{1 - \log x}.$$

Therefore if here in place of n we write $n - 1$, this summation arises:

$$\frac{x^n}{1 - \log x} = 1 + \frac{n+1}{1}v + \frac{(n+2)^2}{1.2}v^2 + \frac{(n+3)^3 v}{1.2.3}v^3 + \text{etc.}$$

which is our latter series itself.

§. 19. These two series moreover should be considered all the more deserving of attention, because they are far simpler and more elegant than the general Lambert series; and then above all, because no more distinct way can be seen to be evident to directly demonstrate the truth of them. For though the verity of the Lambert series itself has now been demonstrated: nevertheless the methods, by which the demonstration is supported, cannot accommodate the case of the present series by any means, in which a particularly significant paradox is observed, because one may fortify such a general proposition by a demonstration, but nevertheless it cannot always be applied to any special case.

§. 20. How the Lambert series can be derived from the trinomial equation

$$x^\alpha - x^\beta = (\alpha-\beta)vx^{\alpha+\beta},$$

I will show on another occasion, where a similar solution was extended to a polynomial of as high a degree as desired. Conversely, how the Lambert series is also able to be developed into the trinomial equation, seems to be a far more difficult problem; hence it would be valuable to have set forth such an analysis, and in order that the work may follow more easily, I propose in advance the following problem.

Problem.

To demonstrate the agreement of a proposed Lambert series, which is known, with this trinomial equation:

$$x^\alpha - x^\beta = (\alpha-\beta)vx^{\alpha+\beta}.$$

Solution.

§. 21. Setting the sum of this series $= S$, indeed I assume here that that sum is equal to a power of whatever kind, x^n, as it only presses upon us to find the relationship between the quantity x and the quantities which determine the series itself, which are α, β, and v. As is easy to see, the fact that the sum S can be produced with such a form x^n, can by no means be had by such reasoning before demonstration, (because before all else this ought to have been demonstrated). With this granted we may establish the reasoning in the following way.

§. 22. In fact, having set $S = x^n$, in the first place of the undetermined exponent n I put the fixed value $n = -\alpha$, so we obtain the following series:

$$\begin{aligned} x^{-\alpha} = 1 - \alpha v - \frac{1}{2}\alpha\beta v^2 - \frac{1}{6}\alpha.2\beta(\alpha+\beta)v^3 \\ - \frac{1}{24}\alpha.3\beta(\alpha+2\beta)(2\alpha+\beta)v^4 \\ - \frac{1}{120}\alpha.4\beta(\alpha+3\beta)(2\alpha+2\beta)(3\alpha+\beta)v^5 - \text{etc.} \end{aligned}$$

By similar reasoning, if we set $n = -\beta$, we arrive at the following series:

$$\begin{aligned} x^{-\beta} = 1 - \beta v - \frac{1}{2}\beta\alpha v^2 - \frac{1}{6}\beta.2\alpha(\beta+\alpha)v^3 \\ - \frac{1}{24}\beta.3\alpha(\beta+2\alpha)(2\beta+\alpha)v^4 \\ - \frac{1}{120}\beta.4\alpha(\beta+3\alpha)(2\beta)2\alpha)(3\beta+\alpha)v^5 - \text{etc.} \end{aligned}$$

§. 23. Now we subtract the former of these two series from the latter, and thus we obtain that equation: $x^{-\beta} - x^{-\alpha} = (\alpha - \beta)v$, because apart from the second terms all following clearly cancel each other. But if now we multiply that resulting equation by $x^{\alpha+\beta}$, the assumed trinomial equation is produced

$$x^{\alpha} - x^{\beta} = (\alpha - \beta)vx^{\alpha+\beta}.$$

§. 24. Therefore, provided that it is possible to demonstrate, that the sum of the Lambert series is equal to the power n of any quantity x, which is independent of n, the preceding Analysis would absolutely provide a sufficient proof. I will attempt to amend this defect in the following problem.

Principal Problem.

To set forth the analytical operations, which may lead one directly towards understanding of the true sum of the Lambert series.

Solution.

§. 25. As the proposed Lambert series involves four quantities α, β, v and n, we imagine the first three α, β and v as if they were constant and given, while the fourth n we consider as if it were variable; and in this way one may suppose the desired sum S as if it were some function of the quantity n, as keeping with tradition we represent it with this notation: $S = \Phi : n$, so we let

$$\begin{aligned}\Phi : n = 1 + nv &+ \frac{1}{2}n(n+\alpha+\beta)v^2 \\ &+ \frac{1}{6}n(n+\alpha+2\beta)(n+2\alpha+\beta)v^3 \\ &+ \frac{1}{24}n(n+\alpha+3\beta)(n+2\alpha+2\beta)(n+3\alpha+\beta)v^4 \\ &+ \frac{1}{120}n(n+\alpha+4\beta)(n+2\alpha+3\beta)(n+3\alpha+2\beta)(n+4\alpha+\beta)v^5 + \text{etc.}\end{aligned}$$

§. 26. Therefore as this equation must be true, whatever number we may write in the place of n, we first set $n - \alpha$ in place of n and we obtain

$$\begin{aligned}\Phi : (n-\alpha) = 1 + (n-\alpha)v &+ \frac{1}{2}(n-\alpha)(n+\beta)v^2 \\ &+ \frac{1}{6}(n-\alpha)(n+2\beta)(n+\alpha+\beta)v^3 \\ &+ \frac{1}{24}(n-\alpha)(n+3\beta)(n+\alpha+2\beta)(n+2\alpha+\beta)v^4 \\ &+ \frac{1}{120}(n-\alpha)(n+4\beta)(n+\alpha+3\beta)(n+2\alpha+2\beta)(n+3\alpha+\beta)v^5 + \text{etc.}\end{aligned}$$

And of course we may similarly obtain

$$\begin{aligned}\Phi:(n-\beta) = 1 &+ (n-\beta)v + \frac{1}{2}(n-\beta)(n+\alpha)v^2 \\ &+ \frac{1}{6}(n-\beta)(n+2\alpha)(n+\alpha+\beta)v^3 \\ &+ \frac{1}{24}(n-\beta)(n+3\alpha)(n+\alpha+2\beta)(n+2\alpha+\beta)v^4 \\ &+ \frac{1}{120}(n-\beta)(n+4\alpha)(n+\alpha+3\beta)(n+2\alpha+2\beta)(n+3\alpha+\beta)v^5 + \text{etc.}\end{aligned}$$

§. 27. We now subtract the former of these two series from the latter, and since for terms of any order, by having pulled out a common factor, we would have

$$(n-\beta)(n+\lambda\alpha) - (n-\alpha)(n+\lambda\beta) = (\lambda+1)n(\alpha-\beta),$$

with this observation, having subtracted we will find

$$\begin{aligned}&\Phi:(n-\beta) - \Phi:(n-\alpha) \\ &\quad = (\alpha-\beta)v + \frac{2}{2}(\alpha-\beta)nv^2 + \frac{3}{6}(\alpha-\beta)n(n+\alpha+\beta)v^3 \\ &\quad + \frac{4}{24}(\alpha-\beta)n(n+\alpha+2\beta)(n+2\alpha+\beta)v^4 \\ &\quad + \frac{5}{120}(\alpha-\beta)n(n+\alpha+3\beta)(n+2\alpha+2\beta)(n+3\alpha+\beta)v^5 + \text{etc.}\end{aligned}$$

§. 28. Because in this series all terms contain the factor $(\alpha-\beta)v$, by dividing by this we reach this equation:

$$\begin{aligned}\frac{\Phi:(n-\beta) - \Phi:(n-\alpha)}{(\alpha-\beta)v} = 1 &+ nv + \frac{1}{2}n(n+\alpha+\beta)v^2 \\ &+ \frac{1}{6}n(n+\alpha+2\beta)(n+2\alpha+\beta)v^3 \\ &+ \frac{1}{24}n(n+\alpha+3\beta)(n+2\alpha+2\beta)(n+3\alpha+\beta)v^4 + \text{etc.}\end{aligned}$$

a series which is the thing itself, as we noted with the nature of $\Phi:n$, for the sum which is to be substituted we arrive at this equation:

$$\Phi:(n-\beta) - \Phi:(n-\alpha) = (\alpha-\beta)v\Phi:n.$$

§. 29. All this work therefore leads to this question: what sort of value of n should be taken for $\Phi:n$, in order that this equation is satisfied? It will soon be evident however, even by brief examination, that it can be satisfied by such a substitution: $\Phi:n = Ak^n$, where indeed neither A nor k depend on n; then in fact we have

$$\Phi:(n-\alpha) = Ak^{n-\alpha} \text{ and } \Phi:(n-\beta) = Ak^{n-\beta}.$$

Whence, substituting these values, the equation found will assume that form:

$$A(k^{n-\beta} - k^{n-\alpha}) = (\alpha-\beta)vAk^n$$

Which dividing by Ak^n transforms into this: $k^{-\beta} - k^{-\alpha} = (\alpha - \beta)v$; indeed if we were to multiply throughout by $k^{\alpha+\beta}$ and in place of k we would write x, we will have deduced that trinomial equation first noted: $x^\alpha - x^\beta = (\alpha - \beta)v^{\alpha+\beta}$.

§. 30. Therefore in this way it is most rigorously shown, that the sum of a Lambert series must be expressed by such a formula, as undoubtedly $S = Ak^n$, or $S = Ax^n$, where in the case of $n = 0$ the series sum must be made $= 1$, so clearly the letter A must be made $= 1$, and the sum of the series itself is directly as is assigned: that is $S = x^n$, as long as the quantity x is drawn from that equation: $x^\alpha - x^\beta = (\alpha - \beta)vx^{\alpha+\beta}$.

§. 31. One might object to this solution, perhaps contriving to satisfy the equation

$$\Phi : (n - \beta) - \Phi : (n - \alpha) = (\alpha - \beta)v\Phi : n$$

by other means, aside from the value $\Phi : n = k^n$, which indeed cannot be denied, for these types of equations generally admit several solutions. Certainly, while that value may be sufficient, it absolutely satisfies the requirement, and precisely so, that neither A nor k are dependent on n: nonetheless the same may also be confirmed from the principles of the Analysis of the infinite in the following way.

§. 32. Since $S = \Phi : n$ is a function of n, with this variable quantity having been assumed, from well-known principles it will be[7]

$$\Phi(n - \alpha) = S - \frac{\alpha dS}{dn} + \frac{\alpha^2 ddS}{1.2.dn^2} - \frac{\alpha^3 d^3S}{1.2.3.dn^3} + \frac{\alpha^4 d^4S}{1.2.3.4.dn^4} - \text{etc.}$$

and in a similar way

$$\Phi(n - \beta) = S - \frac{\beta dS}{dn} + \frac{\beta^2 ddS}{1.2.dn^2} - \frac{\beta^3 d^3S}{1.2.3.dn^3} + \frac{\beta^4 d^4S}{1.2.3.4.dn^4} - \text{etc.}$$

these having been substituted we are led to that infinite differential equation:

$$\begin{aligned}(\alpha - \beta)vS = (\alpha - \beta)\frac{dS}{dn} - (\alpha\alpha - \beta\beta)\frac{ddS}{1.2.dn^2} \\ + (\alpha^3 - \beta^3)\frac{d^3S}{1.2.3.dn^3} - (\alpha^4 - \beta^4)\frac{d^4S}{1.2.3.4.dn^4} + \text{etc.}\end{aligned}$$

from which the quantity S may be derived.

§. 33. However, since in each of the terms of this equation the variable S takes up a single dimension everywhere, it has been shown in the integral calculus, such an equation cannot otherwise be satisfied, except for by a value of this sort: $S = Ce^{\lambda n}$: this having been settled, if we set $e^\lambda = k$, it becomes $S = Ck^n$, precisely as we previously had assumed, for now, we consider it appropriate to say, nothing further can be desired concerning the Lambert series.[8]

More Useful confirmation of the given Solution.

§. 34. If we only consider the condition just found, that we must have

$$\Phi : (n - \beta) - \Phi : (n - \alpha) = (\alpha - \beta)v\Phi : n,$$

certainly it can be satisfied in a much more general way. In particular if each of the letters p, q, r, s etc. are roots of this equation: $x^{-\beta} - x^{-\alpha} = (\alpha - \beta)v$, it is clear that the value:

$$\Phi : n = Ap^n + Bq^n + Cr^n + Ds^n + \text{etc.}$$

satisfies the same condition. Thus the value of the sum S of the Lambert series will obviously be contained in this formula, of which since it was defined and it only depends upon the quantities α, β, v, and n, it will be sought for, the way those indeterminate letters A, B, C, D etc. may be defined, in order that they make $S = \Phi : n$.

§. 35. Here, however, immediately two cases present themselves, wherein either a single root determines the sum S, or in fact all the roots coincide, which two cases therefore are worth examining carefully. Wherein first I observe, if all the roots p, q, r, s, etc. are to be used at the same time, that they ought to be applied without doubt with equal regard, since there is no reason, why preference may be granted to any of them: For this reason let the coefficients A, B, C, D, etc. be equal between themselves, and so

$$S = A(p^n + q^n + r^n + s^n + \text{etc.});$$

therefore since with the case $n = 0$ it must make $S = 1$, if the number of roots is set $= i$, with this case it is $S = Ai$, thus $A = \frac{1}{i}$.

§. 36. Additionally, however, our series is so composed, that taking $v = 0$ also produces in our sum $S = 1$. Now in fact with the case $v = 0$ our equation becomes $x^{-\beta} - x^{-\alpha} = 0$, or $x^{\alpha-\beta} - 1 = 0$, whose only root is $= 1$, and the sum of all the roots is always $= 0$, the only exceptional case being where $\alpha - \beta = 1$. Thus as long as it is assumed $n = 1$, we have $S = \frac{1}{i}(p + q + r + s + \text{etc.}) = 0$, since the sum is still $= 1$, so that hypothesis opposes the truth.

§. 37. The same issue also shines through most clearly, if we set $n = 1$ in general, as you see with which case we let $S = \frac{1}{i}(p + q + r + s + \text{etc.})$, where $p + q + r + s + \text{etc.}$ is the sum of the roots of the trinomial equation

$$x^{-\beta} - x^{-\alpha} = (\alpha - \beta)v$$

and so it will be equal to the coefficient of the second term, after the equation has been reduced in order, which frequently fails it, the series sum also being equal to nothing; which since it contradicts the truth, it has been demonstrated sufficiently, that not all roots of the trinomial equation can agree with the sum S to be established.

§. 38. Therefore with this case remote, in which all the roots may have come out to be equal, the first case remains, wherein the sum S is determined by only one of those roots, as in the solution we had assumed. It is clear moreover that the root shall be either the maximum or the minimum: here in fact the same distinction comes about, in the solution of all equations from recurrent series[9], as likewise from that method only one root of the equation, either the maximum or minimum, is typically found, as long as the solution we have given already has been put together beyond any doubt. However, on the occasion of the method which we have used, it would do no harm to add the following problem.

Problem.

To find all functions of the variable quantity n, with which the general condition may be satisfied:

$$\Phi : n = a\Phi : (n+\alpha) + b\Phi : (n+\beta) + c\Phi : (n+\gamma) + \text{etc.}$$

Solution.

§. 39. But by the reasoning we found in the preceding problem, it will be easily seen, in order to satisfy that condition, we ought to set $\Phi : n = Ak^n$, where A and k are constant quantities. Moreover, making this substitution produces the following equation:

$$Ak^n = Aak^{k+\alpha} + Abk^{n+\beta} + Ack^{n+\gamma} + Adk^{n+\delta} + \text{etc.}$$

which dividing by Ak^n gives

$$1 = ak^\alpha + bk^\beta + ck^\gamma + dk^\delta + \text{etc.}$$

where k denotes some root of this equation, of which precisely each one of the roots together are sufficient for the previously written condition. One may even combine these different solutions with each other in whatever way. So if p, q, r, s etc. should be roots of this equation, it is sufficient for our problem in general, to have set

$$\Phi : n = Ap^n + Bq^n + Cr^n + \text{etc.}$$

where the letters A, B, C, D etc. inside remain our choice; and this is the general solution to the problem of this analysis, something which will often be able to bring forth noteworthy use.

§. 40. But let us return to the Lambert series, and also let us show, in what way these innumerable other series may be shown to be connected.

Problem.

Let a Lambert series be proposed, which for brevity we give in the form below:

$$S = 1 + Av + Bv^2 + Cv^3 + Dv^4 + \text{etc.}$$

of which it has been noted the sum $= x^n$, where it should be assumed for x either the maximum or the minimum root of this trinomial equation: $x^\alpha - x^\beta = (\alpha-\beta)vx^{\alpha+\beta}$, from there that the innumerable other series are made connected, whose sum may be equally so assigned.

Solution.

§. 41.[10] Therefore these letters A, B, C, D etc. for the sake of brevity have been set to the following forms:

$$\begin{aligned}
&A = n; B = \frac{1}{2}n(n+\alpha+\beta); C = \frac{1}{6}n(n+\alpha+2\beta)(n+2\alpha+\beta);\\
&D = \frac{1}{24}n(n+\alpha+3\beta)(n+2\alpha+2\beta)(n+3\alpha+\beta);\\
&E = \frac{1}{120}n(n+\alpha+4\beta)(n+2\alpha+3\beta)(n+3\alpha+2\beta)(n+4\alpha+\beta)\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

With these values having thus been noted one may establish two well known principal ways from which place the other series ought to be established: that is, one by differentiation, the other by integration.

§. 42. Since if we let $(\alpha-\beta)v = x^{-\beta} - x^{-\alpha}$ we will have

$$(\alpha-\beta)dv = -\beta x^{-\beta-1}dx + \alpha x^{-\alpha-1}dx$$

or

$$(\alpha-\beta)dv = \frac{dx(\alpha x^\beta - \beta x^\alpha)}{x^{\alpha+\beta+1}};$$

hence therefore if our series is differentiated, and we divide by dv, we are led to the following summation:

$$\begin{aligned}
\frac{(\alpha-\beta)nx^{n+\alpha+\beta}}{\alpha x^\beta - \beta x^\alpha} &= A + 2Bv + 3Cvv\\
&\quad + 4Dv^3 + 5Ev^4 + 6Fv^5 + \text{etc.}
\end{aligned}$$

§. 43. We also could have multiplied the proposed series by any power of v before, as is found by differentiation; just as that to be multiplied by v^λ, so we have

$$v^\lambda x^n = v^\lambda + Av^{\lambda+1} + Bv^{\lambda+2} + Cv^{\lambda+3} + Dv^{\lambda+4} + \text{etc.}$$

this differentiated series also divided by dv gives

$$\lambda v^{\lambda-1}x^n + nv^\lambda x^{n-1}\frac{dx}{dv} = \lambda v^{\lambda-1} + (\lambda+1)Av^\lambda$$
$$+ (\lambda+2)Bv^{\lambda+1} + (\lambda+3)Cv^{\lambda+2}$$
$$+ (\lambda+4)Dv^{\lambda+3} + (\lambda+5)Ev^{\lambda+4} + \text{etc.}$$

dividing which expression by $v^{\lambda-1}$ gives the summation:

$$\lambda x^n + nvx^{n-1}\frac{dx}{dv} = \lambda + (\lambda+1)Av$$
$$+ (\lambda+2)Bv^2 + (\lambda+3)Cv^3 + (\lambda+4)Dv^4 + \text{etc.}$$

by which way it is clear

$$\frac{nx^{n-1}dx}{dv} = \frac{(\alpha-\beta)nx^{\lambda+\alpha+\beta}}{\alpha x^\beta - \beta x^\alpha}$$

with which substituted value that sum becomes

$$= \lambda x^n + \frac{nx^{n+\alpha} - nx^{n+\beta}}{\alpha x^\beta - \beta x^\alpha}$$
$$= \frac{x^n}{\alpha x^\beta - \beta x^\alpha}\left((\lambda\alpha - n)x^\beta - (\lambda\beta - n)x^\alpha\right).$$

§. 44. But if now we multiply this series again by v^λ, and we differentiate again, innumerable new series are discovered, whose summations again follow the rule. And repeating in this way, one may continue further; however to follow this labor onwards is unnecessary.

§. 45. In a similar way by integration, we may draw out new series. If we multiply the proposed by

$$dv = \frac{\alpha x^{-\alpha-1}dx}{\alpha-\beta} - \frac{\beta x^{-\beta-1}dx}{\alpha-\beta},$$

and we integrate both sides, we come to the following summation:

$$\frac{\alpha x^{n-\alpha}}{(\alpha-\beta)(n-\alpha)} - \frac{\beta x^{n-\beta}}{(\alpha-\beta)(n-\beta)} = v + \frac{1}{2}Avv$$
$$+ \frac{1}{3}Bv^3 + \frac{1}{4}Cv^4 + \frac{1}{5}Dv^5 + \text{etc.} + \text{const.}$$

where to determine such a constant, the case $v = 0$ is considered, such that $x = 1$, and hence the constant $= \frac{n}{(n-\alpha)(n-\beta)}$.

§. 46.[11] We could have also multiplied by v^λ before integration; in truth with this way we have fallen into excessively laborious calculations, from which it is enough for us, to have uncovered a source, out of which innumerable new series of whatever kind may be drawn.[12]

NOTES

1. Johann Heinrich Lambert (1728-1777) published "Observationes Variae in Mathesin Puram" (Various observations on pure mathematics) in *Acta Helvetica Physico-Mathematico-Botanico-Medica* Vol. III, 1758 (pp. 128-168). Within the paper he considers a general series solution for a real root of a polynomial of any degree. Here I will try to reproduce his reasoning.

   In §29 he considers the general polynomial,

$$0 = x^m - Ax^{m-1} + Bx^{m-2} - ... + Hx^2 - Ix + K,$$

where $x = \alpha, \beta, \gamma, \delta, etc$ are the m roots of the equation.

   Take the sums of the roots, their squares, their cubes, and so on, and denote these sums by Σr for the sum of roots, Σr^2 for the sum of the squares of the roots, etc. So,

$$\begin{gathered}\alpha + \beta + \gamma + \delta + ... = \Sigma r\\ \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + ... = \Sigma r^2\\ \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + ... = \Sigma r^3\\ ...\end{gathered}$$

Next, substitute the roots into the original expression as:

$$\begin{aligned}0 &= \alpha^m - A\alpha^{m-1} + B\alpha^{m-2} - ... + H\alpha^2 - I\alpha + K.\\ 0 &= \beta^m - A\beta^{m-1} + B\beta^{m-2} - ... + H\beta^2 - I\beta + K.\\ 0 &= \gamma^m - A\gamma^{m-1} + B\gamma^{m-2} - ... + H\gamma^2 - I\gamma + K.\\ 0 &= \delta^m - A\delta^{m-1} + B\delta^{m-2} - ... + H\delta^2 - I\delta + K.\\ &...\end{aligned}$$

Add all of these equations together, to obtain

$$0 = \Sigma r^m - A\Sigma r^{m-1} + B\Sigma r^{m-2} - ... + H\Sigma r^2 - I\Sigma r + mK.$$

And thus, we can solve for Σr^m,

$$\Sigma r^m = A\Sigma r^{m-1} - B\Sigma r^{m-2} + ... - H\Sigma r^2 + I\Sigma r - mK.$$

Take m to be $1, 2, 3, ...$, and we have expressions for the different sums,

$$\begin{aligned}\Sigma r &= A\\ \Sigma r^2 &= A\Sigma r - 2B\\ \Sigma r^3 &= A\Sigma r^2 - B\Sigma r + 3C\\ \Sigma r^4 &= A\Sigma r^3 - B\Sigma r^2 + C\Sigma r - 4D\\ \Sigma r^5 &= A\Sigma r^4 - B\Sigma r^3 + C\Sigma r^2 - D\Sigma r + 5E\\ &...\end{aligned}$$

   We can substitute recursively, as

$$\begin{aligned}\Sigma r &= A\\ \Sigma r^2 &= A^2 - 2B\\ \Sigma r^3 &= A^3 - 3AB + 3C\\ \Sigma r^4 &= A^4 - 4A^2B + 2B^2 + 4AC - 4D\\ &...,\end{aligned}$$

but these expressions still do not give us the roots directly.

   Instead, Lambert makes a critical assumption: assume there exists a root which is purely real, and which has a larger real part than all other roots' real parts. (Lambert's assumption becomes the condition for convergence of his series solution.) Then, taking the ratio of two sums, in the limit as $m \to \infty$, all terms go to zero but the leading powers.

$$\lim_{m\to\infty} \frac{\alpha^m + \beta^m + \gamma^m + \delta^m + ...}{\alpha^{m-1} + \beta^{m-1} + \gamma^{m-1} + \delta^{m-1} + ...} = \frac{\alpha^m}{\alpha^{m-1}} = \alpha.$$

The series representation of this root can be found as:

$$S = \frac{A\Sigma r^{m-1} - B\Sigma r^{m-2} + ... - H\Sigma r^2 + I\Sigma r - mK}{A\Sigma r^{m-2} - B\Sigma r^{m-3} + ... + H\Sigma r^2 - I\Sigma r + (m-1)K}.$$

In using this to solve for the trinomial,

$$x^m + px = q$$

Lambert obtains the series solution,

$$x = \frac{q}{p} - \frac{q^m}{p^{m+1}} + m\frac{q^{2m-1}}{p^{2m+1}} - m\frac{3m-1}{2!}\frac{q^{3m-2}}{p^{3m+1}} + m\frac{(4m-1)(4m-2)}{3!}\frac{q^{4m-3}}{p^{4m+1}} - ...$$

which I've put into summation-notation form:

$$S = \sum_{k=0}^{\infty}(-1)^k \frac{q^{k(m-1)+1}}{p^{km+1}} \frac{1}{k!} \prod_{j=0}^{k-2}(km-j).$$

Details of the actual derivation can be found in "Observationes" §38.

This method of solving for the roots of an equation is due to Daniel Bernoulli ("Observationes de seriebus recurrentibus", Commentarii Academiae Scientiarum Imperialis Petropolitanae, Tome III pp. 85-100 (1728)). It is referenced for example by Euler in Chapter 17 of *Introductio in analysin infinitorum* vol. I. See 9.

Lambert's solution was used by Euler here to obtain his series solution for a symmetric trinomial, that is, a trinomial symmetric in parameters α and β of the form:

$$x^\alpha - x^\beta = (\alpha - \beta)vx^{\alpha+\beta}.$$

2. This is to say, because the function is symmetric in the values α and β, Euler's sum above can be used to determine the value not only of x, but x raised to any power n.

3. To understand Euler here, note that in the original series, we may rearrange to obtain $\frac{S-1}{n}$ on the left hand side, and $v + \frac{1}{2}(n + \alpha + \beta)v^2 + ...$ on the right; then we may freely set $n = 0$ and obtain a non-zero result on the right hand side, and an indeterminate on the left. Then, in the limit, the expression $\lim_{n\to 0} \frac{x^n - 1}{n} = \log x$.

4. The reasoning here may be slightly confusing. Euler simply means that, plugging in $\beta = 0$ to the original trinomial, we get $x^\alpha - 1 = \alpha v x^\alpha$, which rearranged gives us the value of x in terms of α and v, $x = (1 - \alpha v)^{-1/\alpha}$. Taking the log of both sides, we get $\log x = \log[(1 - \alpha v)^{-1/\alpha}]$. Expanding the right side as a series (as shown), then plugging that series back into the equation, we find $\log x = -\frac{1}{\alpha}(-\alpha v - -1/2\alpha^2 v^2 - ...)$, and distributing the $-1/\alpha$ gives the original expression for $\log x$.

5. The cited article is §159 of Volume 2 of Euler's *Institutiones calculi differentialis*. It is found in chapter 6, the entirety of which has been translated by Ian Bruce, and is available for free from 17centurymaths.com. The article expresses a sum

$$s = \log 1 + \log 2 + \log 3 + \log 4 + \ .\ .\ .\ + \log x$$

Which is equivalent to

$$s = \frac{1}{2}\ln(2\pi) + (x + \frac{1}{2})x \ln x - x + \frac{\mathfrak{A}}{1 \cdot 2x} - \frac{\mathfrak{B}}{3 \cdot 4x^3} + \frac{\mathfrak{C}}{5 \cdot 6x^5} - \frac{\mathfrak{D}}{7 \cdot 8x^7} + \text{etc.}$$

Where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, ...$ are the *Bernoulli numbers* for $n = 2, 4, 6, 8, ...$ respectively (with their respective signs being included in the sum, not in the numbers). That is,

$$\mathfrak{A} = \frac{1}{6}$$
$$\mathfrak{B} = \frac{1}{30}$$
$$\mathfrak{C} = \frac{1}{42}$$
$$\mathfrak{D} = \frac{1}{30}$$
$$\mathfrak{E} = \frac{5}{66}$$

etc.

From which we obtain the given series for the factorial,

$$n! = \frac{\sqrt{2\pi}}{e^n} n^{n+\frac{1}{2}} \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \text{etc.}\right).$$

This series is a form of *Stirling's series*, related to Stirling's approximation of the factorial, originally stated by Abraham de Moivre (1667-1754). It's interesting to note here, that Euler is well-known for his development of the Gamma function $\Gamma(x)$, which also interpolates the factorial. He first put the function in the infinite product form,

$$n! = \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^n}{1+\frac{n}{k}}$$

Which he discovered in 1729, though by January of 1720 he had developed the integral form,

$$n! = \int_0^1 (-\ln s)^n ds.$$

6. This is the first of several sections which are misnumbered in this paper. This is the only misprint which I keep as-is, in order to maintain order for cross-references with the original.

7. Euler here uses a Taylor expansion of the function $\Phi(n-\alpha)$,

$$\Phi(n-\alpha) = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} \frac{d^k \Phi(n)}{dn^k}$$

Then substituting $\Phi(n) = S$,

$$\Phi(n-\alpha) = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} \frac{d^k S}{dn^k}$$

And similarly for $\Phi(n-\beta)$.

8. This might warrant some further explanation. We take the equation

$$\Phi(n-\beta) - \Phi(n-\alpha) = (\alpha-\beta)v\Phi(n), \tag{1}$$

and we have $S = \Phi(n)$. Using the series expansions of $\Phi(n-\alpha)$ and $\Phi(n-\beta)$ this equation becomes

$$\sum_{k=0}^{\infty} \frac{(-\beta)^k}{k!} \frac{d^k S}{dn^k} - \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k!} \frac{d^k S}{dn^k} = (\alpha-\beta)vS. \tag{2}$$

This is an infinite-degree constant-coefficient linear ordinary differential equation.

We assume a solution of the form $S = Ce^{\lambda n}$. Then $\frac{d^k S}{dn^k} = \lambda^k S$, so (2) becomes

$$\sum_{k=0}^{\infty} \frac{(-\lambda\beta)^k}{k!} S - \sum_{k=0}^{\infty} \frac{(-\lambda\alpha)^k}{k!} S = (\alpha-\beta)vS,$$

or, dividing out the term S,

$$\sum_{k=0}^{\infty} \frac{(-\lambda\beta)^k}{k!} - \sum_{k=0}^{\infty} \frac{(-\lambda\alpha)^k}{k!} = (\alpha-\beta)v.$$

From here, recall the Taylor series expansion of the exponential is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

We can apply this to the two terms on the left-hand side and obtain

$$(\alpha-\beta)v = e^{-\lambda\beta} - e^{-\lambda\alpha}, \tag{3}$$

which is an identical expression to the one Euler obtains in §29, with k in place of e^{λ}, and the remainder of the derivation is the same as his.

However, the requirement is not simply to show that this is a solution, but that it is the *only* solution to the above ODE. Euler defers to the results of integral calculus, and says nothing more on the matter. I have done some research on infinite-degree linear constant-coefficient equations, the principle reference being R. D. Carmichael's *Linear Differential Equations of Infinite Order* (1935). I haven't found a reference from e.g.

Institutionum Calculi Integralis relating to this topic yet. Hopefully this has provided some context for Euler's results in this section.

9. See Chapter 17 of Euler's *Introductio in analysin infinitorum*, E101, Lausanne: Marcum-Michaelem Bousquet, Volume 1. Opera Omnia Series 1, Volume 8, pp.1-392.

10. Originally misprinted §31, but because it does not alter the later references, I have fixed the number.

11. Originally misprinted as §56.

12. Euler ends with a more poetic passage; it might be translated as *... from which it is enough for us, to have revealed a fountain, out of which innumerable new series of any kind may be drunk.* The dual connotation lends well to a more flourished sentence, uncharacteristic of Euler's writing.