INVESTIGATIONS ON A NEW TYPE OF MAGIC SQUARE

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Translation by Andie Ho and Dominic Klyve[1]

1. A very curious question that has taxed the brains of many inspired me to undertake the following research that has seemed to open a new path in Analysis and in particular in the area of combinatorics. This question concerns a group of thirty-six officers of six different ranks, taken from six different regiments, and arranged in a square in a way such that in each row and column there are six officers, each of a different rank and regiment. But after spending much effort to resolve this problem, we must acknowledge that such an arrangement is absolutely impossible, though we cannot give a rigorous proof.

2. To better explain the state of the aforementioned question, I will denote the six different regiments by the Latin letters

$$a, \quad b, \quad c, \quad d, \quad e, \quad f,$$

and the six different ranks by the Greek letters

$$\alpha, \quad \beta, \quad \gamma, \quad \delta, \quad \varepsilon, \quad \zeta.$$

It is clear that the characteristics of each officer is determined by two letters, one Latin and the other Greek. The former denotes his regiment and the other his rank, and there are, effectively, thirty-six combinations of two of these letters as follows:

$$\begin{array}{cccccc} a\alpha & a\beta & a\gamma & a\delta & a\varepsilon & a\zeta \\ b\alpha & b\beta & b\gamma & b\delta & b\varepsilon & b\zeta \\ c\alpha & c\beta & c\gamma & c\delta & c\varepsilon & c\zeta \\ d\alpha & d\beta & d\gamma & d\delta & d\varepsilon & d\zeta \\ e\alpha & e\beta & e\gamma & e\delta & e\varepsilon & e\zeta \\ f\alpha & f\beta & f\gamma & f\delta & f\varepsilon & f\zeta \end{array}$$

where each expresses the characteristics of an officer. Thus, it is a matter of writing these thirty-six terms in the thirty-six entries of a square in such a way that in each row and column we find six Latin and six Greek letters.

3. Thus we will have three conditions to meet, the first of which demands that in each row we find six Greek and six Latin letters; second, that the same twelve characters are found in all the columns, and finally, that

[1]This is a working draft of a translation by Andie Ho and Dominic Klyve.

all thirty-six paired terms are found in the square, in other words, that no term appears twice. For if we only have to satisfy the first two conditions, it would be easy to find several solutions. Here is one:

$$\begin{array}{cccccc} a\alpha & b\zeta & c\delta & d\varepsilon & e\gamma & f\beta \\ b\beta & c\alpha & f\varepsilon & e\delta & a\zeta & d\gamma \\ c\gamma & d\varepsilon & a\beta & b\zeta & f\delta & e\alpha \\ d\delta & f\gamma & e\zeta & c\beta & b\alpha & a\varepsilon \\ e\varepsilon & a\delta & b\gamma & f\alpha & d\beta & c\zeta \\ f\zeta & e\beta & d\alpha & a\gamma & c\varepsilon & b\delta \end{array}$$

But this arrangement contains the flaw that the terms $b\zeta$ and $d\varepsilon$ appear twice, and that the terms $b\varepsilon$ and $d\zeta$ are missing entirely.

4. Since all our efforts to construct such a square with thirty-six entries have been futile, in order to extend my research, instead of six different regiments and ranks, I will use some number n so that there are n Latin letters

$$a, \quad b, \quad c, \quad d, \quad \text{etc.}$$

and as many Greek letters

$$\alpha, \quad \beta, \quad \gamma, \quad \delta \quad \text{etc.}$$

that can be combined in nn different ways and arranged in a square with nn entries so that each row and column contains all the Latin and Greek letters and no term is encountered twice in the square.

5. Because each line of the square contains all these different letters and since the letters in each line are always the same, it is clear that such an arrangement will satisfy the condition of ordinary magic squares. [2] For to produce all the numbers in the natural order we only have to give the Latin letters a, b, c, d, e, etc. the values $0, n, 2n, 3n, 4n, \ldots, (n-1)n$, and the Greek letters $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. the values $1, 2, 3, 4, 5, ..., n$. But since in these squares we only take into account the sum of all the numbers in each row and column, it is not necessary that all the numbers appear in each line, provided that the sum is the same throughout. This is also the reason why we can construct ordinary magic squares with 36 entries.

6. In order to make the operations that I am going to do easier, I will put in place of the Latin and Greek letters the natural numbers $1, 2, 3, 4, 5$, etc., which I will name, in order to distinguish between them, one the *Latin numbers* and the other the *Greek numbers*; so as not to confuse the two, I will attach the Greek numbers to the Latin numbers in the form of exponents, as will be seen in the following forty-nine-entry square

[2] see E748, "De Quadratis Magicis", elsewhere in this volume

1^1	2^6	3^4	4^3	5^7	6^5	7^3
2^2	3^7	1^5	5^4	4^1	7^6	6^3
3^3	6^1	5^6	7^5	1^2	4^7	2^4
4^4	5^2	6^7	1^6	7^3	2^1	3^5
5^5	1^3	7^1	2^7	6^4	3^2	4^6
6^6	7^4	4^2	3^1	2^5	5^3	1^7
7^7	4^5	2^3	6^2	3^6	1^4	5^1

in which I have arranged the Latin numbers in their natural order in the first row and column so that these numbers represent both the indices of these two lines and those of their exponents. I have also set the Greek numbers, or exponents, equal to the Latin numbers in the first column, as I will continue to do throughout, since the interpretation of these numbers is completely arbitrary.

7. It is easy to be convinced that all the terms written in the preceding square perfectly satisfy the three requisite conditions and relationships above. To have the reader become familiar with the point of view from which we must envision the majority of the methods that have led us to the following research, we are going to begin with the analysis of the construction of the given square. For this, we will again take the fundamental Latin square which, in omitting the exponents, will have the following form:

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	6	5	7	1	4	2
4	5	6	1	7	2	3
5	1	7	2	6	3	4
6	7	4	3	2	5	1
7	4	2	6	3	1	5

where each of the seven rows and columns contain all of the seven numbers $1, 2, 3, 4, 5, 6, 7$.

8. Having thus established this *Latin square*, we still must find a method for joining the Greek numbers, or exponents, to each Latin number in the square. First, starting with the exponent 1, since it must be found in each row and column, we take from each column seven different numbers, all from different rows; in other words, the numbers that we take from each column must all be taken from different heights, something that must also be done for the other exponents $2, 3, 4, 5$, etc. Here we must note that since we are assuming that the exponents from the first column are known and that we always set them equal to the Latin numbers from this column, the first terms of the formulas we just described will always follow the order of the natural numbers $1, 2, 3, 4, 5, 6, 7$.[3]

9. Since in the following research everything that we use to determine the entry of the exponents, that is, the ranks of the arranged officers depends on these formulas, I will name them *guiding formulas*. We must have one for each exponent. Thus, in the square with 49 entries shown above in the 6th paragraph, the

[3]An example here will serve far better than this explanation. To find the guiding formula for 1, start with the first column, and find the number of which 1 is an exponent, and find the corresponding number in each successive column. E.g., in the second column, 1 appears as an exponent of 6, and in the third column, 1 appears as an exponent of 7. 1's guiding formula is, therefore, 1,6,7 etc.

guiding formulas are

for the exponent	1	it is	1	6	7	3	4	2	5
" "	2	"	2	5	4	6	1	3	7
" "	3	"	3	1	2	4	7	5	6
" "	2	"	2	5	4	6	1	3	7
" "	2	"	2	5	4	6	1	3	7
" "	2	"	2	5	4	6	1	3	7
" "	2	"	2	5	4	6	1	3	7

This is what we must understand from the term *guiding formulas*, which we will continue to use throughout. First of all it is obvious that in order to construct a complete square, we must have such a guiding formula for each Greek number, or exponent. Next, all these guiding formulas must necessarily agree in such a way that in writing one under another, we find all the different numbers in each column, since otherwise a given Latin or principal number would receive two different exponents.

10. Having established a square of Latin numbers for any case, the first operation consists of looking for the guiding formulas for each exponent. If it happens that for one of these numbers we cannot find such a formula, then we can state with certainty the Latin square is incapable of producing a complete square. Furthermore, if all the guiding formulas for all the exponents are found, but it is impossible to choose them so that they agree with each other in the manner in which I have just described and shown in the example above, it is a reliable indicator that the Latin square will not produce a solution to the problem. However, we must take care to only come to this conclusion after having been thoroughly convinced that we have found and examined all the possible guiding formulas for the proposed square.

11. Thus, the formation of the guiding formulas is the first and principal object in this research; but I must confess that up until now I had no sure method for conducting this investigation. It even seemed that we would have to be content with a type of trial and error, which I am going to explain for the Latin square with 49 entries shown above.

In order to find, for example, the guiding formula for the exponent 4 of this square, we randomly choose the first four terms which I will take as they have been given

$$4 \quad 7 \quad 3 \quad 5$$

and which are taken from the first four columns and from the four rows that correspond to the indices $4, 6, 1, 2$. It is clear that the three last terms of our formula,

$$1 \quad 2 \quad 6$$

must be taken from the last three columns and from the three rows that correspond to the indices $3, 5, 7$. Thus, the pieces of the 3rd, 5th, and 7th rows give us the following terms:

$$\begin{matrix} 1 & 4 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{matrix}$$

which clearly results in the last three terms of our guide in the order 6, 1, 2, as we have assigned to them above. If we had not known the first four terms, we see by what we have just said that it would be necessary to examine all the possible combinations in the same manner.

12. After having explained in general the operations that we must undertake in order to construct such complete squares, I move on to some more particular research which naturally will change according to the nature of the Latin squares, which can be formed in as many different ways as there are entries in the square. We can easily see how quickly the number of possible ways to construct it becomes so large that we can no longer count them. That is why I will be content here to cover briefly several simple, regular types of Latin squares that will lead to much more complicated types.

13. First of all, the most simple Latin square is doubtless the one where all the numbers

$1, 2, 3, 4, \ldots, n$ are in their natural order, observing that having reached the last one, we start again with 1. The squares of this first type, of a classification that arose, will have in general the following form for any number of entries nn:

$$\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \ldots & n \\
2 & 3 & 4 & 5 & 6 & \ldots & n & 1 \\
3 & 4 & 5 & 6 & \ldots & n & 1 & 2 \\
4 & 5 & 6 & \ldots & n & 1 & 2 & 3 \\
5 & 6 & \ldots & n & 1 & 2 & 3 & 4 \\
6 & \ldots & n & 1 & 2 & 3 & 4 & 5
\end{array}$$

The squares of this first type, which occur for all the numbers in the rows and columns n, will be known from this point forward as *single step Latin squares*.

14. Following this classification, the second type will be *double step Latin squares*, which come from taking the numbers from the first row, arranging them in their natural order two at a time and transposing them in the second row, which becomes

$$2 \quad 1 \quad 4 \quad 3 \quad 6 \quad 5 \quad 8 \quad 7 \quad \text{etc.}$$

From this and from the first row, we construct the third and fourth rows by adding 2 to each of their terms, the fifth and sixth rows by adding 2 to the terms in the third and fourth, and so on. The *double step Latin squares* formed in this way will have in general the following form:

$$\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{etc.} \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & \text{etc.} \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \text{etc.} \\
4 & 3 & 6 & 5 & 8 & 7 & 10 & 9 & \text{etc.} \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \text{etc.} \\
6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 & \text{etc.}
\end{array}$$

by which we can easily see that this second type can only occur for squares in which the number of entries in each line is even.

15. For the third class, I will show the *triple step Latin squares*, where we consider three numbers together in the first row and alter them in three different ways, before forming the following lines that we obtain three at a time by adding 3 to the three preceding terms, as we can see in the following general form:

1	2	3	4	5	6	7	8	9	etc.
2	3	1	5	6	4	8	9	7	etc.
3	1	2	6	4	5	9	7	8	etc.
4	5	6	7	8	9	10	11	12	etc.
5	6	4	8	9	7	11	12	10	etc.
6	4	5	9	7	8	12	10	11	etc.
7	8	9	10	11	12	13	14	15	etc.

which shows us that this construction only applies when the number of entries in a line is divisible by 3.

16. In the same way, we can form squares of the fourth class, proceeding to the *quadruple step* by taking the terms from the first row separately and four at a time and applying all the necessary transpositions to form the first four rows. From these, we form the following four by adding 4 to each term, and so on for the other rows. But since the first four terms

1 2 3 4

give several different transpositions, we will have several general forms of squares of this type, from which it will suffice to take the first member (I call any square sections within a square a "member of a square"), assuming that it is easy to derive the general form from it, the transpositions being the same in all the other members, or single squares, of which the large Latin square is composed. The Latin square in this class, must always have a number of entries divisible by $4^2 = 16$. Here are four similar transpositions:

I				II				III				IV			
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	1	4	3	2	1	4	3	2	3	4	1	2	4	1	3
3	4	1	2	3	4	2	1	3	4	1	2	3	1	4	2
4	3	2	1	4	3	1	2	4	1	2	3	4	3	2	1

where it would be superfluous to form or to assemble the general forms of the squares composed of the same members. One can easily see that we only have to alter the blocks of four following the same laws as the first row.

We also see that this classification could very well lead us to other regular squares; but we will stop here to carefully develop in the following sections the four types that we have just established and to derive complete squares from them.

FIRST SECTION

THE GENERAL FORM OF SINGLE STEP LATIN SQUARES

1	2	3	4	5	6	...	n
2	3	4	5	6	...	n	1
3	4	5	6	...	n	1	2
4	5	6	...	n	1	2	3
5	6	...	n	1	2	3	4
6	...	n	1	2	3	4	5

CASE OF $n = 2$

17. We start with the simplest case, where $n = 2$ and the Latin square is

1	2
2	1

from which we cannot extract any guiding formula. Consequently this case is impossible, since we cannot derive any other square. And indeed, if we satisfy the first two conditions of the question from §3, we arrive at the square

1^1	2^2
2^2	1^1

where the two terms 1^1 and 2^2 are found twice, while the two others, 2^1 and 1^2, are missing entirely. Thus, if the question concerns an assembly of four officers of two different ranks and regiments, we immediately see that it would be impossible to arrange them in a square in the prescribed manner.

CASE OF $n = 3$

18. We move on to the case of $n = 3$, and our Latin square will be

1	2	3
2	3	1
3	1	2

whose diagonal with different terms, 1 3 2, immediately gives a guiding formula for the exponent 1. Since all the numbers decrease from 1, it is clear that the guiding formulas will follow the same order and consequently will be

for the	exponent	1	1	3	2
“	“	2	2	1	3
“	“	3	3	2	1

In inserting the exponents following the system of guides, we will obtain the following complete square:

$$\begin{array}{ccc} 1^1 & 2^3 & 3^2 \\ 2^2 & 3^1 & 1^3 \\ 3^3 & 1^2 & 2^1 \end{array}$$

which is the only solution that can take place for single step squares with nine entries, since the formula 1, 3, 2 is the only guide for the exponent 1 and since the fundamental square, or proposed Latin square, is the only one for the given case.

CASE OF $n = 4$

19. Let us consider the case of $n = 4$, which leads us to the following Latin square:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{array}$$

but here we quickly see that it is impossible to find any guide for the exponent 1. In examining the square according to the prescribed rules, we will see that it is the same for all the other exponents. Thus, we must conclude that this Latin square will not give us any complete squares for the case $n = 4$. But we must note that this Latin square is not the only possible one in this case, since we can form three others, amongst which is found one that leads us to some very beautiful solutions. Thus, this is the only single step square with 16 entries that does not meet the required conditions.

20. The same problem is encountered in all cases where the number n is even, and this observation leads us to the following theorem:

For all cases where the number n is even, the single step Latin square will never give a solution to the proposed question.

To demonstrate this, we only have to show that it is impossible to find a guide for the exponent 1 for all single step squares in which the number of horizontal or vertical entries is even. Let us suppose that such a guiding formula is

$$1 \quad a \quad b \quad c \quad d \quad e \quad \text{etc.},$$

where the letters a, b, c, d, etc., of which we have $n-1$, denote the numbers $2, 3, 4, \ldots, n$, in any order, which is determined by the rows corresponding to the indices $\alpha, \beta, \gamma, \delta, \varepsilon$ etc., which also denote the numbers 2, 3, 4, 5, etc., so that the sum of all the numbers $\alpha, \beta, \gamma, \delta$ etc. must be equal to the sum of the letters a, b, c, d, etc.

Thus, in our Latin square, all the numbers of the rows increase arithmetically by 1. Noting that in continuing on to numbers beyond n we must begin again with 1, it follows that, because the second number a of the supposed guide is taken from the second column and from the row that corresponds to the index α, we will have

$$a = \alpha + 1.$$

Likewise, since the third term, b, of this guide is taken from the third column and from the row corresponding to the index β, we will have

$$b = \beta + 2$$

Following this line of reasoning, we find that for the other terms we will have

$$c = \gamma + 3 \quad d = \delta + 4 \quad e = \epsilon + 5 \quad f = \zeta + 6 \quad \text{etc.},$$

always observing that having reached a number greater than n, we will put the excess above n in its place. Now let the sum of all the letters

$$\alpha + \beta + \gamma + \delta + \text{etc.} = S,$$

The sum of the letters $a + b + c + d +$ etc. will be equal to $S + 1 + 2 + 3 + \cdots + (n-1)$.. In other words, we will have

$$a + b + c + d + \text{etc.} = S + \frac{1}{2}n(n-1)$$

Thus, the sum of the Latin letters, $a+b+c+d+$ etc., and of the Greek letters, $\alpha\beta+\gamma+\delta+$ etc., as we have observed above, must be equal. In other words, the difference must be a multiple of n. If we set this equal to λn, it leads us to the equation

$$\frac{1}{2}n(n-1) = \lambda n,$$

which gives

$$\lambda = \frac{1}{2}(n-1).$$

Consequently, since λ is a whole number, this equality does not hold unless $n-1$ is an even number or n is an odd number. In this way, the truth of our theorem is rigorously demonstrated, and it would be useless to try to consider Latin squares in any case where n is an even number.

CASE OF $n = 5$

21. We return to our squares. The case of $n = 5$ leads us to the following single step Latin square:

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

from which we can easily extract the three following guides for the exponent 1:

1	3	5	2	4
1	4	2	5	3
1	5	4	3	2

By adding 1 to each of the terms of these guides, we will obtain those for the exponent 2. Adding 1 to this once more gives those for the exponent 3, and so on. In this fashion, we will be able to construct the following three squares suitable for directing the inscription of the exponents:

1	3	5	2	4	1	4	2	5	3	1	5	4	3	2
2	4	1	3	5	2	5	3	1	4	2	1	5	4	3
3	5	2	4	1	3	1	4	2	5	3	2	1	5	4
4	1	3	5	2	4	2	5	3	1	4	3	2	1	5
5	2	4	1	3	5	3	1	4	2	5	4	3	2	1

22. With these three complete systems of guides, we can form three complete squares of twenty-five entries each and get just as many solutions for the problem concerning an assembly of twenty-five officers of five different ranks and regiments. Here are the three complete squares:

		I					II					III		
1^1	2^5	3^4	4^3	5^2	1^1	2^3	3^5	4^2	5^4	1^1	2^4	3^2	4^5	5^3
2^2	3^1	4^5	5^4	1^3	2^2	3^4	4^1	5^3	1^5	2^2	3^5	4^3	5^1	1^4
3^3	4^2	5^1	1^5	2^4	3^3	4^5	5^2	1^4	2^1	3^3	4^1	5^4	1^2	2^5
4^4	5^3	1^2	2^1	3^5	4^4	5^1	1^3	2^5	3^2	4^4	5^2	1^5	2^3	3^1
5^5	1^4	2^3	3^2	4^1	5^5	1^2	2^4	3^1	4^3	5^5	1^3	2^1	3^4	4^2

The construction of these three squares is all the easier after having written in the exponent 1, the others proceeding in their natural order down the columns.

23. We must still note in regard to the guiding formulas that their terms progress arithmetically, the first increasing by 2, the second by 3, the third by 4, the fourth by 5, and so on. Next, the exponents of the first rows of the three complete squares are

in the first	1	5	4	3	2,
in the second	1	3	5	2	2,
in the third	1	4	2	5	3,

which match the three guiding formulas. Finally, the first of these three types of squares, changing the order of the rows, gives the very remarkable following square

1^1	2^5	3^4	4^3	5^2
3^3	4^2	5^1	1^5	2^4
5^5	1^4	2^3	3^2	4^1
2^2	3^1	4^3	5^4	1^3
4^4	5^3	1^2	2^1	3^5

in which not only the rows and columns contain the different Greek and Latin letters, but where even the diagonals (and their completed parallels), such as

$$3^1 \quad 1^4 \quad 4^5 \quad 2^1 \quad 5^2$$

satisfy the prescribed conditions.

CASE OF n = 7

24. The case of n = 7 gives us the following single step Latin square with forty-nine entries

1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6

where considering the ascending, arithmetically increasing guides (§20) gives us the following guiding formulas for the exponent 1,

1	3	5	7	2	4	6
1	4	7	3	6	2	5
1	5	2	6	3	7	4
1	6	4	2	7	5	3
1	7	6	5	4	3	2

where the first increases by 2, the second by 3, the third by 4, the fourth by 5, and the fifth by 6. But we must not believe that these are all the guides for the exponent 1, since in examining the square more carefully, we find the following additional fourteen:

1	3	6	2	7	5	4
1	3	7	6	4	2	5
1	4	6	3	2	7	5
1	4	7	5	3	2	6
1	4	7	2	6	5	3
1	4	2	7	6	3	5
1	5	4	2	7	3	6
1	5	7	3	6	4	2
1	6	4	7	3	5	2
1	6	4	3	7	2	5
1	6	5	2	4	7	3
1	6	2	5	7	4	3
1	7	4	6	2	5	3
1	7	5	3	6	2	4

25. All these guiding formulas have been found in a very cumbersome manner explained above (§8); but the reigning order in the single step squares gives us very simple methods for finding several such formulas as long as we have found one. This will be the subject of the following problem:

Having found a guiding formula for any single step square in which the number n is odd, find a reliable method with which we can find several other guiding formulas.

26. Let

$$1 \quad a \quad b \quad c \quad d \quad e \quad \text{etc.}$$

be the guiding formula that we have found, which corresponds to the exponent 1 and whose term that corresponds to some index t be equal to x, so that in taking $t = 1$, we get also $x = 1$. Thus, we must remark 1) that in giving t all the possible values from 1 to n, the term x must also receive all these different values; and 2) that since t is the index of the column from which the number x is taken, the index from the row will be, as we see by the construction of the square, equal to $x - t + 1$, which also corresponds to the term x. Since the numbers a, b, c, d, etc. must be taken from different rows, it follows that the formula $xt + 1$, and therefore xt must include all the different values, just like the numbers t and x.

27. This noted, let

$$1 \quad A \quad B \quad C \quad D \quad E \quad \text{etc.}$$

be a new guiding formula that we want to derive from the given one and whose index of any term X is equal to T. We understand by what we have said above that by giving T all the possible values, not only the term X, but also the difference XT must likewise receive all the different values. These conditions will obviously be filled by taking

$$T = x \quad \text{and} \quad X = t.$$

Thus we always get a new guide by exchanging the two numbers t and x, which results in the rule for the formation of a new guide: Take x for the index and t for the term that corresponds to it.

This new guide will thus be formed *per inversionem*, by inversion.

28. We can also form another new guide by taking

$$T = t \quad \text{and} \quad X = \alpha + t - x$$

If we let t take all the values from 1 to n, it is clear that for all α the term $\alpha + tx$ will also receive all the different values. Thus, since $XT = \alpha x$, this formula will also receive all the possible values. But for this newly found guide to correspond to the exponent 1, it is necessary that when we have $t = 1$ and $x = 1$, we also have $X = 1$, which gives us $\alpha = 1$.

Thus, we will always obtain another new guide by taking

$$T = t \quad \text{and} \quad X = 1 + t - x,$$

This is the second rule that I propose.

29. By combining the two rules that I have just explained, it will be easy to extract from one given guide several new formulas that we can represent in the following manner:

	I	II	III	IV	V	VI
$T = t$	t	x	$1+t-x$	x	$1+t-x$	$1+x-t$
$X = x$	$1+t-x$	t	t	$1+x-t$	$2-x$	x
	VII	VIII	IX	X	XI	XII
	$2-x$	$1+x-t$	$2-x$	$2-t$	$2-t$	$2-t$
	$1+t-x$	$2-t$	$2-t$	$1+x-t$	$2-x$	$2-x$

Here are eleven different rules with which we can derive eleven new unique guides from one single proposed guiding formula.

30. We will clarify the two principal rules, and those in the preceding §derived from them, with an example. Let us take at random one of the guiding formulas from above *, for example

$$1 \quad 4 \quad 2 \quad 7 \quad 6 \quad 3 \quad 5$$

to which we can alternately apply the first and the second rule, or equally the second and the first. The two series of guides which come from it are the following:

given guide	1	4	2	7	6	3	5	given guide	1	4	2	7	6	3	5
1^{st} rule	1	3	6	2	7	5	4	2^{nd} "	1	6	2	5	7	4	3
2^{nd} "	1	7	5	3	6	2	4	1^{st} "	1	3	7	6	4	2	5
1^{st} "	1	6	4	7	3.	5	2	2^{nd} "	1	7	4	6	2	5	3
2^{nd} "	1	4	7	5	3	2	6	1^{st} "	1	5	7	3	6	4	2
1^{st} "	1	6	5	2	4	7	3	2^{nd} "	1	5	4	2	7	3	6
2^{nd} "	1	4	6	3	2	7	5	1^{st} "	1	4	6	3	2	7	5
1^{st} "	1	5	4	2	7	3	6	2^{nd} "	1	6	5	2	4	7	3
2^{nd} "	1	5	7	3	6	4	2	1^{st} "	1	4	7	5	3	2	6
1^{st} "	1	7	4	6	2	5	3	2^{nd} "	1	6	4	7	3	5	2
2^{nd} "	1	3	7	6	4	2	6	1^{st} "	1	7	5	3	6	2	4
1^{st} "	1	6	2	5	7	4	3	2^{nd} "	1	3	6	2	7	5	4
2^{nd} "	1	4	2	74	6	3	5	1^{st} "	1	4	2	7	6	3	5

They are perfectly equal with the sole difference that by starting with the second rule, the order of guides is reversed.

31. Here are eleven new guides, all originating from a single one, and even from any one amongst them. Thus, all the new guides are found amongst the fourteen ones above (§24). There are only two that have escaped this operation, namely

$$1 \quad 4 \quad 7 \quad 2 \quad 6 \quad 5 \quad 3 \quad \text{and} \quad 1 \quad 6 \quad 4 \quad 3 \quad 7 \quad 2 \quad 5$$

Both are of a special nature, since each produces itself by the first rule, while by the second rule, each produces the other.

32. After having found nineteen different guiding formulas for the case of $n = 7$, we can derive a complete square from each one, making nineteen different squares. By taking any one of them, letting it be $1, a, b, c, d, e, f$, and continuing these numbers according to their natural order, we will have the guides for the following exponents, 2, 3, 4, 5, 6, and 7, and in this way we will obtain the following square of guides

$$\begin{array}{ccccccc}
1 & a & b & c & d & e & f \\
2 & a+1 & b+1 & c+1 & d+1 & e+1 & f+1 \\
3 & a+2 & b+2 & c+2 & d+2 & e+2 & f+2 \\
4 & a+3 & b+3 & c+3 & d+3 & e+3 & f+3 \\
5 & a+4 & b+4 & c+4 & d+4 & e+4 & f+4 \\
6 & a+5 & b+5 & c+5 & d+5 & e+5 & f+5 \\
7 & a+6 & b+6 & c+6 & d+6 & e+6 & f+6
\end{array}$$

It is clear that each row and column contains all the numbers from 1 to 7, whatever the order of the numbers a, b, c, d, e, and f.

33. In order to facilitate the construction of the sought after complete square, we assign the proper exponents to the first row, which is always the series of natural numbers 1, 2, 3, 4, 5, 6, 7. Then in the proposed guiding formula let

$$1 \quad a \quad b \quad c \quad d \quad e \quad f$$

be the term that corresponds to the index $t = x$. We must join the exponent 1 to this term x of the square. Then since the exponents ascend, going down each column following the natural order, the following term, $x + 1$, will have the exponent 2 and in general, the term $x + \lambda$ will have the exponent $\lambda + 1$. Let us take λ so that $x + \lambda = t$, which gives us $\lambda = tx$. Thus, the number t in the first row will have the exponent

$\lambda + 1 = t + 1 - x.$

Thus, we give t the values 1, 2, 3, 4, etc., and the exponents of the first row will be the following:

$1, 3 - a, 4 - b, 5 - c, 6 - d, 7 - e, 8 - f.$

34. We have seen above that this formula is also a guide which results from the first one in accordance with the second rule. This is because in order to construct the complete square, we can first take each guiding formula to represent the exponents that it is necessary to give to the numbers in the first row. Next, going down the columns, all we have to do is increase the upper exponents following their natural order. In this way, if the proposed guiding formula $1, a, b, c, d, e, f$ is also the series of exponents in the first row, the complete square that comes from it will have the following form:

1^1	2^a	3^b	4^c	5^d	6^e	7^f
2^2	3^{a+1}	4^{b+1}	5^{c+1}	6^{d+1}	7^{e+1}	1^{f+1}
3^3	4^{a+2}	5^{b+2}	6^{c+2}	7^{d+2}	1^{e+2}	2^{f+2}
4^4	5^{a+3}	6^{b+3}	7^{c+3}	1^{d+3}	2^{e+3}	3^{f+3}
5^5	6^{a+4}	7^{b+4}	1^{c+4}	2^{d+4}	3^{e+4}	4^{f+4}
6^6	7^{a+5}	1^{b+5}	2^{c+5}	3^{d+5}	4^{e+5}	5^{f+5}
7^7	1^{a+6}	2^{b+6}	3^{c+6}	4^{d+6}	5^{e+6}	6^{f+6}

35. Having found nineteen guiding formulas in all for the case $n = 7$, we can form the same number of complete squares from them. So if a question concerns forty-nine officers of seven different ranks and regiments, we can derive a large number of different solutions from this, all taken from one single step Latin square. We can even derive several other solutions from the same source. Since the number of guiding formulas is so considerable, having used whichever one we wish for the exponent 1, we can derive the formulas for the other exponents for the other squares, so that however these different formulas are arranged in a square, the numbers in the columns are all different. We can easily see that in this way, we will obtain many more new complete squares created from several guiding formulas together. It will suffice to clarify this mix of guides with a single example.

Forumula of the exponent							Type of the guide						
1	4	7	2	6	5	3	1	4	7	2	6	5	3
2	7	5	4	1	3	6	1	6	4	3	7	2	5
3	6	1	5	4	2	7	1	4	6	3	2	7	5
4	1	3	6	2	7	5	1	5	7	3	6	4	2
5	2	6	3	7	4	1	1	5	2	6	3	7	4
6	3	2	7	5	1	4	1	5	4	2	7	3	6
7	5	4	1	3	6	2	1	6	5	4	2	7	3

We easily understand from this single example that we can find many other equally suitable combinations, the number of which would be very difficult to determine.

36. By inserting the exponents according to these guides, the complete square that results will take this form:

1^1	2^5	3^4	4^2	5^6	6^7	7^3
2^2	3^6	4^7	5^3	6^1	7^4	1^5
3^3	4^1	5^2	6^4	7^5	1^6	2^7
4^4	5^7	6^5	7^6	1^2	2^3	3^1
5^5	6^3	7^1	1^7	2^4	3^2	4^6
6^6	7^2	1^3	2^1	3^7	4^5	5^4
7^7	1^4	2^6	3^5	4^3	5^1	6^2

Here, we see that the exponents in the rows are no longer guiding formulas as in the preceding nineteen types, and that we cannot find any order, since we encounter a mixture of seven different squares. This observation is of little importance because the consideration of the regular squares leads us to believe that the exponents of the first rows must have in general the property of the guides.

Moreover, it is undoubtedly very surprising that while the case of $n = 7$ gives us a prodigious number of solutions, which will be added to below, the case of $n = 6$ will not provide us with any, although even the preceding case, $n = 5$, leads us to three different solutions.

CASE OF $n = 9$

37. Now let $n = 9$. The single step Latin square to which the following research will correspond, will take on this form

1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	1
3	4	5	6	7	8	9	1	2
4	5	6	7	8	9	1	2	3
5	6	7	8	9	1	2	3	4
6	7	8	9	1	2	3	4	5
7	8	9	1	2	3	4	5	6
8	9	1	2	3	4	5	6	7
9	1	2	3	4	5	6	7	8

38. Since it would be too laborious to search for all the guiding formulas that could take place in the Latin square, the number of which will undoubtedly be enormous, I will be content to consider only those that progress arithmetically, excluding those whose difference is 3 or 6, since they are not relatively prime to the number $n = 9$. In general, the difference between these progressions, like the difference between the numbers x and t, or xt, must not have any divisor in common with the number n, because a formula chosen without heeding this rule will not contain all the values from 1 to n, or will not be able to be arranged in the class of guides. By excluding these two cases, the formulas that progress arithmetically will be

1	3	5	7	9	2	4	6	8
1	6	2	7	3	8	4	9	5
1	9	8	7	6	5	4	3	2

From these we can form three complete squares of eighty-one entries by taking the guiding formulas for the following exponents of the same type, since we avoided counting the others.

39. By taking any one of the three guides, which we will let be

$$1 \quad a \quad b \quad c \quad d \quad \text{etc.},$$

for the exponent 1, we will see by what we have said above (§23 and §33) that the exponents in the first row, which are $1, 3-a, 4-b, 5-c, 6-c$, etc., also constitute a guide. Consequently, we can take each of these three guides that we have found for the exponents in the first row, which gives us the following three complete squares:

I

1^1	2^3	3^5	4^7	5^2	6^2	7^4	8^6	9^8
2^2	3^4	4^6	5^8	6^1	7^3	8^5	9^7	1^9
3^3	4^5	5^7	6^9	7^2	8^4	9^6	1^8	2^1
4^4	5^6	6^8	7^1	8^3	9^5	1^7	2^9	3^2
5^5	6^7	7^9	8^2	9^4	1^6	2^4	3^1	2^4
6^4	7^3	8^1	9^3	1^5	2^7	3^9	4^2	5^4
7^7	8^9	9^2	1^4	2^6	3^8	4^1	5^3	6^5
8^8	9^1	1^3	2^5	3^7	4^9	5^2	6^4	7^6
9^9	1^2	2^4	3^6	4^8	5^1	6^3	7^5	8^7

II

1^1	2^6	3^2	4^7	5^3	6^8	7^4	8^9	9^5
2^2	3^7	4^3	5^8	6^4	7^9	8^5	9^1	1^6
3^3	4^8	5^4	6^9	7^5	8^1	9^6	1^3	2^7
4^4	5^9	6^5	7^1	8^6	9^2	1^7	2^3	3^8
5^5	6^1	7^6	8^2	9^7	1^3	2^8	3^4	4^9
6^6	7^2	8^7	9^3	1^8	2^4	3^9	4^5	5^1
7^7	8^3	9^8	1^4	2^9	3^5	4^1	5^6	6^2
8^8	9^4	1^9	2^5	3^1	4^6	5^2	6^7	7^3
9^9	1^5	2^1	3^6	4^2	5^7	6^3	7^8	8^4

III

1^1	2^9	3^8	4^7	5^6	6^5	7^4	8^3	9^2
2^2	3^1	4^9	5^3	6^7	7^6	8^5	9^4	1^3
3^3	4^2	5^1	6^9	7^8	8^7	9^6	1^5	2^4
4^4	5^3	6^2	7^1	8^9	9^8	1^7	2^6	3^5
5^5	6^4	7^3	8^2	9^1	1^9	3^8	3^7	4^6
6^6	7^5	8^4	9^3	1^2	2^1	3^9	4^8	5^7
7^7	8^6	9^5	1^4	2^3	3^2	4^1	5^9	6^8
8^8	9^7	1^6	2^5	3^4	4^3	5^2	6^1	7^9
9^9	1^8	2^7	3^6	4^5	5^4	6^3	7^2	8^1

40. These are three complete squares derived from the three regular guides that we proposed to examine. To better clarify the usage of the rules discussed above (§26, 27 and 28) for the formation of the guides and to be able to more easily judge their number, we are going to choose one of the guides at random. By applying the two rules successively we will obtain the twelve following guides

	adopted guide	1	6	5	9	2	4	8	7	3
	Where the inverse is	1	5	9	6	3	2	8	7	4
from which we obtain	by the 2nd rule	1	6	8	5	4	3	9	2	7
		1	7	4	8	3	5	9	2	6
	by the 1st rule	1	8	6	5	4	2	9	3	7
		1	8	5	3	6	9	2	4	7
	by the 2nd rule	1	4	7	9	2	5	8	6	3
		1	4	8	2	9	7	6	5	3
	by the 1st rule	1	5	9	2	6	8	3	7	4
		1	4	9	2	8	7	6	3	5
	by the 2nd rule	1	7	4	3	9	8	5	2	6
		1	8	4	3	7	9	2	6	5

We must stop at the sixth pair, since if we applied the first rule to it again, we would obtain the same formulas, one of which reproduces the other by inversion. Thus, our two rules have given us all eleven new guides.

41. One very important remark still to be made is that in making use of the third rule, which we have been able to forego in the preceding section of $n = 7$, since it would not have been of any help, we can still find another dozen new guides. This rule can be stated in the following manner:

Suppose that for the proposed guide the index is t and the term that corresponds to it is x; then for the new formula we can take the index $T = 2t - 1$ and the same term $X = 2x - 1$.

The reason for this is obvious, 1) since by taking $t = 1$ and $x = 1$, it becomes

$$T = 1 \quad \text{and} \quad X = 1$$

2), since if x takes on every value, $2x$ and thus $2x-1$ will take on every values; and 3), since if $x-t$ contains all the values from 1 to 9, $X-T=2(x-t)$ will also under go all the corresponding changes.

42. It will be good to clarify this new rule that is so full of guides when considered alongside the two preceding ones, by using an example. To do this, we will take the guide chosen above, which will give us the following dozen.

adopted guide		1	6	5	9	2	4	8	7	3
provided by the 3^{rd} rule		1	7	2	6	9	4	8	5	3
from which we obtain the 1^{st} rule		1	3	9	6	8	4	2	7	5
and following	by the 2^{nd} rule	1	5	2	8	6	3	9	4	7
		1	9	4	8	7	3	6	2	5
	by the 1^{st} rule	1	3	6	8	2	5	9	4	7
		1	8	6	3	9	7	5	4	2
	by the 2^{nd} rule	1	9	7	6	4	2	8	5	3
		1	4	7	2	6	9	3	5	8
	by the 1^{st} rule	1	6	9	5	8	4	3	7	2
		1	4	7	2	8	5	3	9	6
	by the 2^{nd} rule	1	6	4	9	7	3	5	2	8
		1	8	6	3	7	2	5	9	4

43. Once again we apply this third rule to the first of the new dozen guides that we have just found. We obtain, with the help of the two preceding rules, the new dozen that follow.

From the adopted guide		1	7	2	6	9	4	8	5	3
we obtain by the 3^{rd} rule		1	7	4	6	3	9	2	5	8
and from that, by the 1^{st} rule		1	7	5	3	8	4	2	9	6
and following	by the 2^{nd} rule	1	5	9	8	3	7	6	4	2
		1	5	8	2	7	3	6	9	4
	by the 1^{st} rule	1	9	5	8	2	7	6	4	3
		1	4	6	9	2	7	5	3	8
	by the 2^{nd} rule	1	3	8	6	4	9	2	5	7
		1	8	7	5	4	9	3	6	2
	by the 1^{st} rule	1	7	2	5	8	4	9	3	6
		1	9	7	5	4	8	3	2	6
	by the 2^{nd} rule	1	5	2	9	7	3	8	6	4
		1	3	6	9	2	8	5	7	4

Here, as everywhere else, we continue until the guides are duplicated, which occur here at the sixth pair.

44. If we apply the third rule to the first of these guides, that is

$$1 \quad 7 \quad 4 \quad 6 \quad 3 \quad 9 \quad 2 \quad 5 \quad 8,$$

we get

1 8 4 3 7 9 2 6 5

which is already encountered in the first dozen. Thus our three rules have only given us three dozen guides, although for this case there are certainly a good many more. In view of the fact that among all those that have just been found, none of the inverses appear. However, we must find several of them for this case, since in the preceding case, where $n = 7$, there are at least two similar guides.

45. In order to completely convince us, we search for a guide that has the property of reproducing itself by inversion:

	Such as this	1	8	5	9	3	7	6	2	4
which produces	by the 1st rule	1	8	5	9	3	7	6	2	4
we will therefore have	by the 2nd rule	1	4	8	5	3	9	2	7	5
	by the 1st rule	1	7	5	2	4	9	8	3	6
	by the 2nd rule	1	5	8	3	2	7	9	6	4
	by the 1st rule	1	5	4	9	2	8	6	3	7
	by the 2nd rule	1	7	9	5	4	8	2	6	3

The first guide adopted as invertible has thus led us to give five other new guides where we see there are also non-invertible formulas that are found in the close relationship with those that are invertible and that are not encountered at all in the preceding dozens of formulas.

46. By examining the first of the related guides we will see that it can give us another half dozen new formulas. This guide of the preceding order

1 8 5 9 3 7 6 2 4

gives us, by the 3rd rule, this

1 4 6 2 9 3 8 7 5

which, being invertible, will give us the following guides.

The invertible guide	1	4	6	2	9	3	8	7	5
produces by the 2nd rule	1	8	7	3	6	4	9	2	5
by the 1st rule	1	8	4	6	9	5	3	2	7
by the 2nd rule	1	4	9	8	62		5	7	3
by the 1st rule	1	6	9	2	7	5	8	4	3
by the 2nd rule	1	6	4	3	8	2	9	5	7

where we have continued the operations as before up until the reproduction of an invertible guide.

47. In the same manner, by applying the third rule to the first guide of the preceding order, we obtain the following invertible formula

1 5 7 6 2 4 3 9 8,

from which is derived, by the alternating application of the second and first rules, the following guides.

Invertible guide	1	5	7	6	2	4	3	9	8
2^{nd} rule	1	7		8	4	3	5	9	2
by the 1^{st} rule	1	9	6	5	7	3	2	4	8
by the 2^{nd} rule	1	3	7	9	8	4	6	5	2
by the 1^{st} rule	1	9	2	6	8	7	3	5	4
by the 2^{nd} rule	1	3	2	8	7	9	5	4	6

48. If we wanted to iterate these operations by reapplying the third rule to the first guide of the new order, we will return to the first half dozen and then to the previous guides, and thus this source of guides seems to have been exhausted by the three rules employed. Having found thus far three classes of 12 and three others of 6 guides, we have 54 of them in all, and with the first three going in arithmetic progression, 57 different guiding formulas, each of which can give us a complete square. By combining them, as we can do in the manner given above *, we easily see that the total number of possible solutions must become incomparably greater.

49. Even with the 57 formulas that we have just found, we still lack most of the possible guides, since in employing the first direct method that I have just given above *, we can easily find the following 8 that are not in any particular order:

1	3	5	8	2	9	6	4	7
1	3	5	9	8	4	2	7	6
1	3	6	8	2	4	9	7	5
1	3	6	8	4	2	9	5	7
1	3	6	9	4	8	2	5	7
1	3	6	2	9	8	4	7	5
1	3	6	9	7	4	2	5	8
1	3	7	6	2	9	5	4	8

thus we can conclude that the number of all the guides will be at least four times as big.

ODD MAGIC SQUARES WHOSE DIAGONALS AND PARALLELS ALSO FILL THE PRESCRIBED CONDITIONS

50. Let n be any odd number and d be the difference of the terms of a guiding formula that proceeds in arithmetic progression

$1, 1+d, 1+2d, 1+3d$, etc.

and whose terms, subtracting the number n from all those that are greater than this number must produce all the different values from 1 to n, after having continued it up to the term $1 + (n1)d$. This said, it is clear that the difference d must be a prime number and consequently, when n is a prime number, we can give all the values less than n to d; if n has a factor p, it is necessary to exclude all the progressions whose difference d is $p, 2p, 3p, 4p$, etc. Even this essential condition is not sufficient to give this progression the property of guides. Since the index $t = 1 + \lambda$ corresponds to the term $x = 1 + \lambda d$, it is necessary, as we were shown elsewhere (§26), that the formula $xt = \lambda(d1)$ also produces all the different numbers. From this, it is obvious that $d1$ must be relatively prime to n. Thus, it must always exclude the value $d = 1$, and the values $d = p + 1, d = 2p + 1, d = 3p + 1$, etc., every time that n has a factor p.

51. Now it is not difficult to determine in general, for each number n, the number of values that the difference d can receive. If n is a prime number, the number of values of d, which we always take to be smaller than n, will be $n2$, and the number of formulas in progression which will occur will also be $n2$. If n is a product of two different factors, like $n = pq$, the number of all the values of d will be

$(p2)(q2)$.

And in general if n is a product of several different factors p, q, r, s, etc., the number of values of d will be

$(p2)(q2)(r2)(s2)$ etc.

But when n has two or more factors that are the same, the form of the expression for the number of values of d will be a bit different. If $n = p^\alpha q^\beta r^\gamma s^\delta$, the number of values that we can give to d will be

$$p^{\alpha-1} q^{\beta-1} r^{\gamma-1} s^{\delta-1} \text{etc.} (p-2)(q-2)(r-2)(s-2) \text{etc.}$$

52. After these reflections, in general it will be easy to construct a magic square such that not only the rows and columns, but even the two diagonals and all their parallels (each completed by its corresponding parallel from the other side

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) includes the different terms. To this effect, I must remark that whatever the form of such a square represented by the Latin and Greek letters, like we had at the beginning, we can always reduce it to numbers, so that the first column contains all the terms in their natural order, as we have supposed so far. Everything comes back to seeing what manner we should transpose the other columns of the complete squares, so that the required property extends to the diagonals and to all their parallels.

53. Since we are only considering here the guiding formulas that go in arithmetic progression, we see that in the rows there will be as many Latin numbers as Greek (or exponents), also go in arithmetic progression, and that in letting d stand for the difference between that of the Latin numbers and δ for that of the progression of the Greek numbers, the first row will be

$$1^1 \quad (1+d)^{1+\delta} \quad (1+2d)^{1+2\delta} \quad \text{etc.}$$

Therefore, since for the following lines we only have to add unity to the Latin and Greek numbers, the complete square will have the following form:

$$\begin{array}{lllll} 1^1 & (1+d)^{1+\delta} & (1+2d)^{1+2\delta} & (1+3d)^{1+3\delta} & \text{etc.} \\ 2^2 & (2+d)^{2+\delta} & (2+2d)^{2+2\delta} & (2+3d)^{2+3\delta} & \text{etc.} \\ 3^3 & (3+d)^{3+\delta} & (3+2d)^{3+2\delta} & (3+3d)^{3+3\delta} & \text{etc.} \\ 4^4 & (4+d)^{4+\delta} & (4+2d)^{4+2\delta} & (4+3d)^{4+3\delta} & \text{etc.} \\ & & \text{etc.} & & \end{array}$$

54. Now, since the Latin numbers of each row must include all the possible numbers, it follows that the difference d must be taken as we have shown above, that is so that neither d nor $d1$ has any divisors in common with the number n. This same condition also extends to the difference of the progression of the exponents δ and requires that δ and $\delta 1$ both be prime to the number n. Next, it is obvious that the two differences d and δ must not be equal, for if they were equal, all the terms would already be found in the first column. This second condition suffices when the number n is prime; but if it is not prime, it is necessary that the number $d\delta$ be relatively prime to n.

55. These three conditions filled, we will have satisfied the first of the principal prescribed conditions for the construction of squares with different diagonals and parallels. That is, we will obtain a square whose rows and columns contain all the different numbers as we have constructed in the 1st part of this section. This leaves only to see in what manner we can meet the other condition, the one concerning the diagonals and their parallels.

56. To this effect, let us consider the first diagonal, which descends from the left to the right. Since the Latin numbers of which it is composed form this progression

$1, 2+d, 3+2d, 4+3d, 5+4d, 6+5d$, etc.,

whose difference is $d+1$, we see that this diagonal includes all the different numbers every time that $d+1$ is a number prime to n. Since all the parallels of this diagonal have the same difference $d+1$, the required property also extends to the parallels. It is the same for the Greek number, or exponents, which will also receive all the possible values, provided that the difference of their progressions, $\delta+1$, is prime to the number n.

57. Let us also consider the second diagonal, which rises from left to right. We will see that both the Latin and Greek numbers of this diagonal and of its parallels form arithmetic progressions whose difference for the former is $d1$ and for the latter is $\delta 1$. Provided that both $d1$ and $\delta 1$ are numbers prime to n, all the terms that are found in this diagonal and in all these parallels will also be different. Moreover, this last condition is already included in the nature of the guides.

58. Thus, we have laid out all the conditions that are demanded by the construction of the squares that are the object of this 2nd part. They reduce to the three following: 1st) that the numbers $d, d+1, and d1$ are prime to the number n; 2nd) that the numbers $\delta, \delta+1,$ and $\delta 1$ are also prime to the number n; and 3rd) that the number $d\delta$ does not have any divisors in common with n.

59. Let us suppose that p is any divisor or factor of the number n. It will be necessary to exclude these values of d:

$d = \lambda p, d = \lambda p + 1, d = \lambda p1,$

and the values of the letter δ which follow:

$$\delta = \lambda p \quad \delta = \lambda p1 \quad \delta = \lambda p + 1.$$

Let $p = 3$. It is necessary to exclude the values of d and δ from all the possible numbers. From this we see that, in all the cases where the number n is divisible by 3, it will be impossible to construct a square whose diagonals and the parallels satisfy the required conditions.

60. When the number n is prime, the number of all the different values that we can give to the differences d and δ will be equal to $n3$. Next, if n is a product of two different prime numbers, like $n = pq$, the number of values of d and δ will be

(p - 3)(q - 3).

And in general, if $n = p^\alpha q^\beta r^\gamma$ etc., the same number will be expressed by this formula

$$p^{\alpha-1}q^{\beta-1}r^{\gamma-1} \text{ etc. } (p-3)(q-3)(r-3) \text{ etc.}$$

61. After these general remarks, let us develop several particular cases. Since we have just excluded the values of n that are multiples of 3, we take $n = 5$, for which the possible values for d and δ will be 2 and 3, from which one can be taken for d and the other for δ. Thus, let $d = 2$ and $\delta = 3$. The resulting square will have the form

1^1	3^4	5^2	2^5	4^3
2^2	4^5	1^3	3^1	5^4
3^3	5^1	2^4	4^2	1^5
4^4	1^2	3^5	5^3	2^1
5^5	2^3	4^1	1^4	3^2

It is obvious that in changing the values of d and of δ, that is, in letting $d = 3$ and $\delta = 2$, we could form another square; but it is not worth the effort to distinguish it from this one.

Let $n = 7$. The possible values of d and δ will be 2, 3, 4, 5. Choosing two that are different gives six different combinations:

$$\begin{array}{ll} d = 2 \text{ and } \delta = 3, & d = 3 \text{ and } \delta = 4, \\ d = 2 \text{ and } \delta = 4, & d = 3 \text{ and } \delta = 5, \\ d = 2 \text{ and } \delta = 5, & d = 4 \text{ and } \delta = 5, \end{array}$$

and the squares that result are the following:

I. If $d = 2$ and $\delta = 3$

1^1	3^4	5^7	7^3	2^6	4^2	6^5
2^2	4^5	6^1	1^4	3^7	5^3	7^6
3^3	5^6	7^2	2^5	4^1	6^4	1^7
4^4	6^7	1^3	3^6	5^2	7^5	2^1
5^5	7^1	2^4	4^7	6^3	1^6	3^2
6^6	1^2	3^5	5^1	7^4	2^7	4^3
7^7	2^3	4^6	6^2	1^5	3^1	5^4

II. If $d = 2$ and $\delta = 4$

1^1	3^5	5^2	7^6	2^3	4^7	6^4
2^2	4^6	6^3	1^7	3^4	5^1	7^5
3^3	5^7	7^4	2^1	4^5	6^2	1^6
4^4	6^1	1^5	3^2	5^6	7^3	2^7
5^5	7^2	2^6	4^3	6^7	1^4	3^1
6^6	1^3	3^7	5^4	7^1	2^5	4^2
7^7	2^4	4^1	6^5	1^2	3^6	5^3

III. If $d = 2$ and $\delta = 3$

1^1	3^6	5^4	7^2	2^7	4^5	6^3
2^2	4^7	6^5	1^3	3^1	5^6	7^4
3^3	5^1	7^6	2^4	4^2	6^7	1^5
4^4	6^2	1^7	3^5	5^3	7^1	2^6
5^5	7^3	2^1	4^6	6^4	1^2	3^7
6^6	1^4	3^2	5^7	7^5	2^3	4^1
7^7	2^5	4^3	6^1	1^6	3^4	5^2

IV. If $d = 3$ and $\delta = 4$

1^1	4^5	7^2	3^6	6^5	2^7	5^4
2^2	5^6	1^3	4^7	7^4	3^1	6^5
3^3	6^7	2^4	5^1	1^5	4^2	7^6
4^4	7^1	3^5	6^2	2^6	5^3	1^7
5^5	1^2	4^6	7^3	3^7	6^4	2^1
6^6	2^3	5^7	1^4	4^1	7^5	3^2
7^7	3^4	6^1	2^5	5^2	1^6	4^3

V. If $d = 3$ and $\delta = 5$

1^1	4^6	7^4	3^2	6^7	2^5	5^3
2^2	5^7	1^5	4^3	7^1	3^6	6^4
3^3	6^1	2^6	5^4	1^2	4^7	7^5
4^4	7^2	3^7	6^5	2^3	5^1	1^6
5^5	1^3	4^1	7^6	3^4	6^2	2^7
6^6	2^4	5^2	1^7	4^5	7^3	3^1
7^7	3^5	6^3	2^1	5^6	1^4	4^2

VI. If $d = 4$ and $\delta = 5$

1^1	5^6	2^4	6^2	3^7	7^5	4^3
2^2	6^7	3^5	7^3	4^1	1^6	5^4
3^3	7^1	4^6	1^4	5^2	2^7	6^5
4^4	1^2	5^7	2^5	6^3	3^1	7^6
5^5	2^3	6^1	3^6	7^4	4^2	1^7
6^6	3^4	7^2	4^7	1^5	5^3	2^1
7^7	4^5	1^3	5^1	2^6	6^4	3^2

63. The nature of these squares gives us this advantage: we can begin the inscription of its terms in any entry of the square that we want. To demonstrate the multiplicity of the forms that come from it, we take the first of the six squares that we have just constructed and we fill the entries in the following manner:

4^7	6^3	1^6	3^2	5^5	7^1	2^4
5^1	7^4	2^7	4^3	6^6	1^2	3^5
6^2	1^5	3^1	5^4	7^7	2^3	4^6
7^3	2^6	4^2	6^5	1^1	3^4	5^7
1^1	3^7	5^3	7^6	2^3	4^5	6^1
2^5	4^1	6^4	1^7	3^3	5^6	7^2
3^6	5^2	7^5	2^1	4^4	6^7	1^3

64. If we want to apply all of this to ordinary magic squares, we only have to replace the Latin numbers with the values

$$0, \quad 7, \quad 14, \quad 21, \quad 28, \quad 35, \quad 42$$

and replace the Greek numbers with the following

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7,$$

in any order, and then to replace each term of the preceding square by the sum of the two Latin and Greek numbers transformed in this manner. Thus, in the complete square that we have just found, let us set

in place of the Latin numbers	1	2	3	4	5	6	7
the following values	14	42	0	35	21	7	28
and in place of the Greek numbers	1	2	3	4	5	6	7
we substitute these	5	4	1	7	2	3	6

and we will obtain the following ordinary magic square:

41	8	17	4	23	33	49
26	35	48	36	10	18	2
11	16	5	28	34	43	38
29	45	39	9	19	7	27
21	6	22	31	46	37	12
44	40	14	20	1	24	32
3	25	30	47	42	13	15

In this square, not only all the rows and columns, but also all the diagonals and their completed parallels, like for example

$$8 \quad 26 \quad 38 \quad 7 \quad 46 \quad 20 \quad 30,$$

will produce the same sum, that is, 175.

65. To give another idea of the case where the number n is not prime, but not divisible by 3, let us consider the one of

$$n = 35 = 5 \cdot 7,$$

in which the number of all the values that we can give to the letters d and δ will be 8. Here, by letting $n = pq$, with $p = 5$ and $q = 7$, the formula that expresses the number of values is

$$(p - 3)(q - 3) = 2 \cdot 4 = 8,$$

which agrees very well. The values that the letters d and δ can receive are the following 8

$$2, \quad 3, \quad 12, \quad 17, \quad 18, \quad 23, \quad 32, \quad 33.$$

Next, by excluding the numbers d and δ, whose difference $d - \delta$ is divisible by 5 or by 7, the admissible combinations will be

$d = 2$ and $\delta = 3$,	$d = 12$ and $\delta = 23$,
$d = 2$ and $\delta = 18$,	$d = 17$ and $\delta = 18$,
$d = 2$ and $\delta = 33$,	$d = 17$ and $\delta = 23$,
$d = 3$ and $\delta = 12$,	$d = 17$ and $\delta = 33$,
$d = 3$ and $\delta = 32$,	$d = 23$ and $\delta = 32$,
$d = 12$ and $\delta = 18$,	$d = 33$ and $\delta = 33$,

from which we can form twelve different squares of 1225 entries in which all the prescribed conditions will be met; but we will not voluntarily give the construction of even one of them.

End of the First Section

SECOND SECTION

THE GENERAL FORM OF DOUBLE STEP LATIN SQUARES

1	2	3	4	5	6	...	...	$n-1$	n
2	1	4	3	6	5	...	...	n	$n-1$
3	4	5	6	...	...	$n-1$	n	1	2
4	3	6	5	...	...	n	$n-1$	2	1
5	6	...	...	$n-1$	n	1	2	3	4

etc.

66. In establishing the classes of the regular squares, we have already remarked above, in the preceding section, that this type entirely excludes the odd numbers n. We will see in the following that the values of n must not only be even, but also evenly even. That is, the number of entries in the double step square must be divisible into squares of four entries. But before we come to the demonstration of this truth, we must determine the general relation found between the different numbers of the square and their position. To this effect, I observe first of all that because the terms of the first row are also the indices of the columns that correspond to them, and also because the terms in the first column are the indices of the rows, each entry of the square will be determined by two indices, one vertical and the other horizontal. Thus, let t be the vertical index of any term x, and let u be the horizontal index. We will be concerned with finding the relation between the three letters t, u, and x. To this effect, it is necessary to carefully distinguish between the cases where one of the two numbers t and u is odd and where both are even. We will see that the first case gives

$$x = t + u - 1$$

,

and the other

$$x = t + u - 3$$

,

which allows us to see that the two indices t and u can be switched without changing the value of x, since it depends only on the sum of the two letters. After this observation, we can propose our aforementioned theorem, stated in the following manner: None of the double step squares will give any guiding formulas unless the number of the horizontal or vertical terms is divisible by 4.

67. To demonstrate this theorem, let the series

$$a \quad b \quad c \quad d \quad e \quad \text{etc.}$$

be any guiding formula for the indeterminate exponent a; let

$$\alpha, \quad \beta, \quad \gamma, \quad \delta, \quad \epsilon \quad \text{etc.}$$

be the series of horizontal indices indicated by the letter u, and the series of vertical indices marked by t, which always progresses in the order of the natural numbers,

$$1, \quad 2, \quad 3, \quad 4, \quad \text{etc.},$$

and it is necessary, in accordance with the nature of the guides shown in the preceding section, that both of the series include all the different numbers from 1 to n. Having the two indices, the vertical one t and the horizontal one u, we will easily be able to derive, by the preceding rules, the value of each term of our guide.

68. It is clear that for the first term we will always have $a = \alpha$. For the second term, b, we have $t = 2$ and $u = \beta$. Thus, by distinguishing the two cases of the value of β which can be odd or even, we will have $b = \beta + 1$ for the former and $b = \beta 1$ for the other. For the third term, c, because $t = 3$ is odd and $u = \gamma$, we will always have $c = \gamma + 2$. For the fourth term, d, where $t = 4$ is even and $u = \delta$, it is again necessary to distinguish the two cases of the value of δ. If it is odd, we will have $d = \delta + 3$, and if it is even, $d = \delta + 1$, and so on for the others. Thus, for the guiding formula

$$a \quad b \quad c \quad d \quad e \quad f \quad g \quad \text{etc.}$$

for a square whose horizontal indices are

$$\alpha, \quad \beta, \quad \gamma, \quad \delta, \quad \epsilon, \quad \zeta, \quad \eta \quad \text{etc.}$$

and whose vertical indices are

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7 \quad \text{etc.}$$

we will have the following terms:

$$\begin{array}{rcl} a & = & \alpha \\ b & = & \begin{cases} \beta+1 & \beta \text{ odd} \\ \beta-1 & \beta \text{ even} \end{cases} \\ c & = & \gamma+2 \\ d & = & \begin{cases} \delta+3 & \delta \text{ odd} \\ \delta+1 & \delta \text{ even} \end{cases} \\ e & = & \epsilon+4 \\ f & = & \begin{cases} \zeta+5 & \zeta \text{ odd} \\ \zeta+3 & \zeta \text{ even} \end{cases} \\ g & = & \eta+6 \\ h & = & \begin{cases} \theta+1 & \theta \text{ odd} \\ \theta-1 & \theta \text{ even} \end{cases} \\ i & = & \iota+8 \end{array}$$

69. Thus we see that the determination of the letters a, b, c, d, etc., by the indices $\alpha, \beta, \gamma, \delta$, etc., will be completely regular, if of the alternating letters β, δ, ζ, etc., none are even. We will thus have

$$a = \alpha, \quad b = \beta + 1, \quad c = \gamma + 2, \quad d = \delta + 3, \quad e = \varepsilon + 4, \quad f = \zeta + 5, \quad \text{etc.},$$

the number of terms always being equal to n. Let us suppose for an instant that all these alternating letters are odd. Let the sum of the series of horizontal indices be

$$\alpha + \beta + \gamma + \delta + \varepsilon + \text{etc.} = \Sigma$$

and the sum of the terms of the guide be

$a + b + c + d + e$+etc.$= S$.

By adding all the terms, we will have this equation

$S = E + 1 + 2 + 3 + 4 + 5 + \ + (n-1) = \Sigma + \frac{1}{2}n(n-1)$.

Since both of our series must include all the different numbers from 1 to n, it follows that the two sums S and Σ must be equal, or their difference must be a multiple of the number n, which we will call λn, from which we get

$$S = \Sigma + \lambda n$$

Thus, it must be the case that

$$\frac{1}{2}n(n-1) = \lambda n.$$

But we have already said above that double step squares entirely exclude the odd values of n. Thus, by supposing that n is even and equals $2k$, k being any whole number, we will have

$$k(2k-1) = 2\lambda k$$

in other words $\lambda = k - \frac{1}{2}$ or $k = \lambda + \frac{1}{2}$, which is impossible.

70. But this conclusion originates from the supposition that all the alternating letters $\beta, \delta, \zeta, \theta$, etc. are odd, and it is only for this case that finding the guides becomes entirely impossible, whatever value is given to n. Thus for the double step square to have guides, it is absolutely necessary that among the letters $\beta, \delta, \zeta, \theta$, etc., we can find some that are even ones. To see the results of this, we will suppose that there is only one of them, which diminishes the sum of the series of the horizontal indices by 2, and we will have

$\frac{1}{2}n(n-1) - 2 = \lambda n;$

or, by setting $n = 2k$ it must be

$k(2k-1) - 2 = k\lambda k,$

so thus it is obvious that k must be an even number. Let $k = 2m$ and thus $n = 4m$. Our equation will become

$m(4m-1) - 1 = 2\lambda m$

or

$1 = m(4m-1) - 2\lambda m = m(4m - 2\lambda - 1).$

Since this equation only holds when $m = 1$ and $\lambda = 1$, it is clear that the case can only exist when $n = 4$.

71. Let us suppose in general that among the alternating numbers $\beta, \delta, \zeta, \theta$, etc., there are π even ones. Since the number of all these letters is $\frac{1}{2}n$, it is obvious that π will not be greater than $\frac{1}{2}n$. Next, since each even value of the letters reduces the sum by two, our equation will be

$\frac{1}{2}n(n-1) - 2\pi = \lambda n,$

or, by letting $n = 2k$, we will have

$k(2k-1) - 2\pi = 2\lambda k,$

which will only occur when k is an even number, equals $2m$, and thus $n = 4m$. Thus, our equation will be

$m(4-1) - \pi = 2\lambda m,$

from which we extract the even numbers from the alternating letters, giving

$\pi = m(4m - 2\lambda - 1)$

in other words, equal to a product of two factors, m and $4m - 2\lambda - 1$. Since π cannot be greater than $\frac{1}{2}n = 2m$ and since the coefficient of m, $4m - 2\lambda - 1$, is an odd number, it is necessary that $4m - 2\lambda - 1 = 1$

from which we get

$\lambda = 2m - 1$ and $\pi = m$.

Thus, it is necessary that half of the letters $\beta, \delta, \zeta, \theta$, etc., are even and that the number n be divisible by 4. Consequently, the oddly even numbers, 2, 6, 10, 14, etc., will be entirely excluded from this section, since they will never permit guides, which it is necessary to prove.

72. For this reason, we will establish for this entire section that the number n is divisible by 4, by setting $n = 4m$. In these cases, the preceding demonstration allows us to see the possibility of guides. We will principally consider the guides that correspond to the first exponent, 1, and which, because $a = 1$, will have in general the form

$$1 \quad b \quad c \quad d \quad e \quad f \quad g \quad \text{etc.},$$

which corresponds to the series of the horizontal indices

$$1, \quad \beta, \quad \gamma, \quad \delta, \quad \varepsilon, \quad \zeta, \quad \text{etc.},$$

the series of vertical indices being the natural numbers

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad \text{etc.}$$

This said, we have seen that if t indicates the vertical index and u the horizontal index, we will have the term of the guide

$x = t + u - 1,$

except in the sole case where the numbers t and u are even, for which we will have

$x = t + u - 3;$

so that in both cases, x is an odd number.

73. We have shown that for the number $n = 4m$, the case where t and u are even must always be encountered m times. From this it follows that there are also m cases where odd values of u correspond to even values of t. By the same reasoning, for the case of t is odd we will have for u m even numbers and as many odd. We clarify this with the following example where $m = 2$ and $n = 8$:

Vertical Indices	$t = 1$,	2,	3,	4,	5,	6,	7,	8.
Horizontal Indices	$u = 1$,	6,	2,	5,	7,	4,	8,	3.

Here, the even indices $u = 6$ and 4 correspond to the even indices $t = 2$ and 6. The odd indices $t = 3$ and 7 correspond to the even indices $u = 2$ and 8. Next, the odd indices $t = 1$ and 5 correspond to the odd indices $u = 1$ and 7, and the even $t = 8$ and 4 to the odd ones $u = 3$ and 5. From these two series we can form, by the formulas $x = t + u - 1$ and $x = t + u - 3$, the following guide

$$1 \quad 5 \quad 4 \quad 8 \quad 3 \quad 7 \quad 6 \quad 2,$$

where all the terms are different.

74. It is also easy to examine each proposed formula to find out whether it is a guide. Having these numbers x and the horizontal indices t, we only have to find the indices u for either of the formulas given for x. The last one, $x = t + u - 3$ or $u = x - t + 3$, only holds when t is even and x odd. When all the numbers found in this manner for u are different, the proposed formula will always be a real guide. Here are a few examples:

I.	Indices	t	=	1	2	3	4								
	Formula	x	=	1	3	4	2								
	Indices	u	=	1	4	3	2								
II.	Indices	t	=	1	2	3	4	5	6	7	8				
	Formula	x	=	1	3	5	7	4	2	8	6				
	Indices	u	=	1	4	3	6	8	5	2	7				
II.	Indices	t	=	1	2	3	4	5	6	7	8	9	10	11	12
	Formula	x	=	1	4	6	8	10	12	2	3	5	7	9	11
	Indices	u	=	1	3	4	5	6	7	8	10	9	12	11	2

where we see that the series u includes all the different values so that the proposed formulas for x are truly guides.

75. By carefully considering the last two examples, we will see that it is easy to examine the guides for all the numbers divisible by 4 We only have to split them into two equal groups, each containing $2m$ terms. We will see, by the prescribed rules, if the series that we find for u includes all the different values. Here is the example of two general guides for all the numbers $n = 4m$:

First guide.

$$\text{First half}\ \left\{\begin{array}{ccccccccc} t & = & 1 & 2 & 3 & 4 & 5 & 6 & \dots & 2m \\ x & = & 1 & 4 & 6 & 8 & 10 & 12 & \dots & 4m \\ \hline u & = & 1 & 3 & 4 & 5 & 6 & 7 & \dots & 2m+1 \end{array}\right.$$

$$\text{Second half}\ \left\{\begin{array}{ccccccccc} t & = & 2m+1 & 2m+2 & 2m+3 & 2m+4 & \dots & 4m \\ x & = & 2 & 3 & 5 & 7 & \dots & 4m-1 \\ \hline u & = & 2m+2 & 2m+4 & 2m+3 & 2m+6 & \dots & 2 \end{array}\right.$$

where we will easily be assured that in the two halves found for u, all the different numbers really appear.

Second guide.

$$\text{First half}\ \left\{\begin{array}{ccccccccc} t & = & 1 & 2 & 3 & 4 & 5 & 6 & \dots & 2m \\ x & = & 1 & 3 & 5 & 7 & 9 & 11 & \dots & 4m-1 \\ \hline u & = & 1 & 4 & 3 & 6 & 5 & 8 & & 2m+1 \end{array}\right.$$

$$\text{Second half}\ \left\{\begin{array}{ccccccccc} t & = & 2m+1 & 2m+2 & 2m+3 & 2m+4 & \dots & 4m \\ x & = & 4 & 2 & 8 & 6 & \dots & 4m-2 \\ \hline u & = & 2m+4 & 2m+1 & 2m+6 & 2m+3 & \dots & 4m-1 \end{array}\right.$$

In the last half, the penultimate term of u is $4m+2$, or 2, from which we see that among the values of u we find all the numbers from 1 to $4m$.

76. Having found a single guiding formula, we can extract several others from it using the similar rules that we employed in the preceding section. To make this more clear, we are going to consider any guide

$$1 \quad a \quad b \quad c \quad d \quad e \quad f \quad \text{etc.},$$

where x is the term that corresponds to the index t. We have seen that in taking u for the horizontal index, we must distinguish two cases: one where t is even and x is odd, which gives $u = 3 + x - t$; and the other $u = 1 + x - t$, which includes all the other values. To make things easier, we can represent both formulas with the ambiguous formula

$$u = x - t + 2 \pm 1,$$

where the sign on top holds when x is odd and t even; in all other cases it is necessary to use the bottom sign.

77. It is the nature of all guiding formulas to have the following two properties:

1st) that while the letter t varies according to all the values from 1 to $n = 4m$, the letters x must also receive all the values; 2nd) that while the two letters t and x vary according to all their values, the formula

$u = x - t + 2 \pm 1$ also receives all the possibly values. From this follows this third property, that while the letters x and t vary from 1 to n, the formula $t - x + 2 \pm 1$ will also give all the different values, provided that care is taken regarding the ambiguity of the signs, the upper of which only holds when the number t is odd and x even; especially since the two letters are so closely related to each other, while one varies according to all the values, the other also takes the same variations and they can be exchanged, at least in this respect.

78. We see now in what manner we can derive the new guide from the one we had assumed known. Then let

$$1 \quad A \quad B \quad C \quad D \quad \text{etc.}$$

be such a guide, where the term that corresponds to the index T is X. It is necessary that, while T varies through all the values, X also undergoes the same variations, just as $X - T + 2 \pm 1$ and $T - X + 2 \pm 1$, provided that we observe the prescribed rules with respect to the ambiguity of the signs. Having already remarked that the two letters t and u can be switched, it follows that we will find a new guide in letting the same term x correspond to the index $u = x - t + 2 \pm 1$, that is to say by taking

$$T = x - t + 2 \pm 1 \quad \text{and} \quad X = x.$$

Thus, having for the case $n = 8$ this guiding formula

$$1 \quad 3 \quad 5 \quad 8 \quad 2 \quad 4 \quad 6 \quad 7,$$

which corresponds, for u, to

$$1 \quad 4 \quad 3 \quad 5 \quad 6 \quad 7 \quad 8 \quad 2,$$

we will derive a new guide by placing the second term, 3, to the fourth place, as assigned by the number u which appears below it; the third term 5, to the third place, and so on for the others, which gives the new guide

$$1 \quad 7 \quad 5 \quad 3 \quad 8 \quad 2 \quad 4 \quad 6.$$

79. Next, we will find yet another new guide by switching the letters t and x and taking

$T = x$ and $X = t$.

In this way, the first property has already been satisfied. The other, which concerns the formula $X - T + 2 \pm 1$, will also be perfectly satisfied. For this formula, being at present $t - x + 2 \pm 1$, will go through all the values, provided that we observe that the upper sign only holds when t is even and x is odd. We see that this rule agrees with the first of those that we gave in §27 of the preceding section and that we had characterized by the term inversion in such a way that we can still employ the same rule without any changes in this section. The guiding formula of the preceding example,

$$1 \quad 3 \quad 5 \quad 8 \quad 2 \quad 4 \quad 6 \quad 7,$$

by the inversion will give

$$1 \quad 5 \quad 2 \quad 6 \quad 3 \quad 7 \quad 8 \quad 4.$$

80. Another rule is derived from the first case by taking

$$T = t \quad \text{and} \quad X = t - x + 2 \pm 1,$$

which results in $U = X - T + 2 \pm 1$, where the ambiguity of the signs is the opposite of before, so that in place of T and X we substitute their values, obtaining

$$U = -x + 4,$$

a formula that, without ambiguity, will go through all the possible variations, while t and thus also x go through all the values. This rule is analogous to the second from the preceding section, where we also had

$$T = t \quad \text{and} \quad X = t - x + 1,$$

which takes place in all the cases, except in those where t is odd and x is even, which requires us to use the value

$$X = t - x + 3.$$

For example, let

	the	t =1	2	3	4	5	6	7	8
	the	x =1	3	5	8	2	4	6	7
we will have for		X=1	8	7	5	6	3	4	2

81. By the average of these two rules, we can derive several other guides from each known one and almost always twelve new ones as well, as was concluded in the previous section (see the example in §42 and following), provided that we always employ the second rule with the indicated correction. To clarify all of this using an example, let us take the guide that we have made use of thus far. By writing its inverse under it and then applying alternately the second and the first rule, we will obtain the following twelve new guides, including the proposed guide.

The given guide		1	3	5	8	2	4	6	7	I
generates, by inversion		1	5	2	6	3	7	8	4	
and following[1])	by the 2nd rule	1	8	7	5	6	3	4	2	II
	[applied to I]	1	6	4	7	3	8	2	5	
	by the 1st rule	1	7	5	3	8	2	4	6	III
	[applied to I]	1	8	6	7	4	5	3	2	
	by the 2nd rule	1	4	7	2	8	5	6	3	IV
	[applied to III]	1	3	8	6	4	2	5	7	
	be the 1st rule	1	6	2	5	7	4	8	3	V
	[applied to II]	1	4	8	2	6	7	3	5	
	by the 2nd rule	1	5	4	8	7	3	2	6	VI
	[applied to V]	1	7	6	3	2	8	5	4	

where we have continued the operations up to the reproduction of the last guides, which occurs at the sixth pair.

82. Let us apply the same operations to a guide with twelve terms, by adopting a form of those that go in arithmetic progression. The dozen that we obtain by the two rules will be:

Proposée[1])	1	3	5	7	9	11	4	2	8	6	12	10	I
renversée	1	8	2	7	3	10	4	9	5	12	6	11	
par la 2de regle	1	12	11	10	9	8	6	7	4	5	2	3	II
[appliquée à I]	1	7	4	10	3	9	6	12	5	11	8	2	
par la 1re regle	1	12	5	3	9	7	2	11	6	4	10	8	III
[appliquée à I]	1	11	12	9	10	7	8	6	5	4	3	2	
par la 2de regle	1	3	11	2	9	12	8	10	6	7	4	5	IV
[appliquée à III]	1	4	6	8	10	12	2	3	5	7	9	11	
par la 1re regle	1	7	8	2	9	3	10	4	11	5	12	6	V
[appliquée à II]	1	4	2	11	12	9	10	7	5	8	3	6	
par la 2de regle	1	8	10	3	9	4	12	5	11	6	2	7	VI
[appliquée à V]	1	11	4	6	8	10	12	2	5	3	9	7	

of which the last pair is reproduced by the first rule.

83. Each of the guides found for the exponent 1 gives, as we have seen above, guides suitable for all the other exponents and present different terms for all the columns. It is clear, by the construction of the Latin square, by increasing the terms of the guide for the exponent 1 by 2, we will obtain another guide for the exponent 3, and another for the exponent 5 by increasing the terms of that one by 2. In general, from a guide for the exponent a, which we let be

$$a\ b\ c\ d\ e \text{ etc.},$$

we will derive a guide for the exponents $a+2$ by adding 2 to each term of the previous one. Thus, for the case of $n = 8$, each guide for the exponent 1, to which we have assigned a dozen, will give suitable guides for the odd exponents; for example

for the exponent	1	1	3	5	7	4	2	8	6
,, ,,	3	3	5	7	1	6	4	2	8
,, ,,	5	5	7	1	3	8	6	4	2
,, ,,	7	7	1	3	5	2	8	6	4

where each column contains different numbers, either all odd or all even.

84. The formation of the guides for the exponent 2 and the other evens is not as obvious; however, since in the Latin square the second row is derived from the first by increasing all the odd terms and decreasing the even ones by unity, we can conjecture that by doing the same thing with respect to the proposed guide, we will get the guide for the exponent 2, because all the odd terms produce all the even ones in this way. Conversely, when all the even terms are decreased by unity, they produce the odds. But it is still necessary to show that the resulting formula is indeed a guide.

85. In this guide for the exponent 1, let the term that corresponds to the index t equal x, and x' be that which corresponds to the same index in the guide for the exponent 2. In the same way, let u be the horizontal index of the same term x in the first guide, and u' that of the term x' in the other. We will have, observing the prescribed rules concerning the ambiguity of the signs,

$$u = x - t + 2 \pm 1 \quad \text{and} \quad u' = x' - t + 2 \pm 1.$$

Thus, there will be four cases to consider, depending on whether the two numbers t and x are even or odd. The values of u and of u' will be expressed for each case in the following manner:

I.	II.	III.	IV.
$t = 2i$	$t = 2i$	$t = 2i + 1$	$t = 2i + 1$
$x = 2k$	$x = 2k + 1$	$x = 2k$	$x = 2k + 1$
$u = 2k - 2i + 1$	$u = 2k - 2i + 4$	$u = 2k - 2i$	$u = 2k - 2i + 1$
$x' = 2k - 1$	$x' = 2k + 2$	$x' = 2k - 1$	$x' = 2k + 2$
$u' = 2k - 2i + 2$	$u' = 2k - 2i + 3$	$u' = 2k - 2i + 1$	$u' = 2k - 2i + 2$

86. From this we see that the second and third cases give even values for u and that the values of u' are less by one. Thus, it is obvious that all the even values of u produce all the odd values for u'. Next, the first and fourth cases, where the values of u are odd, give values greater than one for u'. Thus, all the odd values of u produce all the even values for u'. In this manner, all the values of u, each different, also produce all the possible values for u', and the resulting formula is indubitably a guide, since it bears the properties.

87. Having found the guide for the exponent 2 in the manner that we have just shown, we will form the guides for all the other even exponents by the first rule. By this method we will easily construct, from each proposed guide for the exponent 1, a complete system of guides similar to the one we show here for the guide

$$1 \quad 3 \quad 5 \quad 7 \quad 4 \quad 2 \quad 8 \quad 6.$$

For the exponent		1	1	3	5	7	4	2	8	6
,,	,,	2	2	4	6	8	3	1	7	5
,,	,,	3	3	5	7	1	6	4	2	8
,,	,,	4	4	6	8	2	5	3	1	7
,,	,,	5	5	7	1	3	8	6	4	2
,,	,,	6	6	8	2	4	7	5	3	1
,,	,,	7	7	1	3	5	2	8	6	4
,,	,,	8	8	2	4	6	1	7	5	3

where we see that in each column the terms are all different. Consequently, by joining the exponents in the proposed Latin square to all the numbers in the explained manner, no term will be encountered twice, and the square will be complete.

88. By more carefully considering the complete system of guides that we have formed, we will see that all the columns correspond perfectly to those in the double step Latin square. The only other difference is in their order, which is changed, that is to say that the horizontal indices that follow the natural order 1 2 3 4 5 6 7 8 in the square are 1 3 5 7 4 2 8 6 here. Then by considering any column whose index is t and the

largest term x marked by the exponent 1, if we denote the terms that follow x in descending order by

$$x', \quad x'', \quad x''' \quad \text{etc.}$$

and give them the exponents

$$2, \quad 3, \quad 4 \quad \text{etc.},$$

the term $x^{(\varphi)}$ will have the exponent $\varphi + 1$. By taking φ so that it becomes

$$x^{(\varphi)} = t,$$

which is the term corresponding to the same index in the first row of the Latin square, it will be necessary to give this term the exponent $\varphi + 1$. Since the values x, x', x'', x''', $\ldots$ $x^{(\varphi)}$ have the same order as in the Latin square, we will always have $t = x + \varphi - \left\{\begin{smallmatrix}0\\2\end{smallmatrix}\right.$ or $\varphi = t - x + \left\{\begin{smallmatrix}0\\2\end{smallmatrix}\right.$ and thus

$$\varphi + 1 = t - x + \left\{\begin{smallmatrix}1\\3\end{smallmatrix}\right. = t - x + 2 \pm 1,$$

where the ambiguity of the signs follows the same laws that we assigned above.

89. From this, it is clear that the exponents of the first row of our Latin square also constitute a guiding formula taken from the proposed guide by the second rule. To construct a complete square we can start with the first row by placing the exponents according to any guide and by continuing the inscription of the others in descending order according to the column of the same square that starts with the same number. Thus, since we take from the proposed guide

$$1 \quad 3 \quad 5 \quad 7 \quad 4 \quad 2 \quad 8 \quad 6$$

by the second rule

$$1 \quad 8 \quad 7 \quad 6 \quad 4 \quad 5 \quad 2 \quad 3\ ,$$

we can start with this guide, by combining it with the first row of the single step square to the terms to which it will serve as exponents. The others will be inserted in the manner that we have just explained and that we will clarify with the example of the following square:

1^1	2^8	3^7	4^6	5^4	6^5	7^2	8^3
2^2	1^7	4^8	3^5	6^3	5^6	8^1	7^4
3^3	4^2	5^1	6^8	7^6	8^7	1^4	2^5
4^4	3^1	6^2	5^7	8^5	7^8	2^3	1^6
5^5	6^4	7^3	8^2	1^8	2^1	3^6	4^7
6^6	5^3	8^4	7^1	2^7	1^2	4^5	3^8
7^7	8^6	1^5	2^4	3^2	4^3	5^8	6^1
8^8	7^5	2^6	1^3	4^1	3^4	6^7	5^2

But we must note that this beautiful property of exponents of the first row can only exist when the system of guides is formed from a single proposed guide.

90. However, it is easy to combine several guides together to form such a complete system, as we have shown in §35 of the previous section. I add once more, regarding this section, that after having taken the guides for the odd exponents of any guiding formula for the exponent 1, we can derive those for the even exponents from another formula, provided that the terms follow the same order with regards to being odd or even. Thus, for the previous example, having derived the odd guides of the formula 1 3 5 7 4 2 8 6, we will be able to take those that direct the inscription of the even exponents from this one: 1 5 7 3 8 4 6 2, which is also a guide and whose terms, with regards to being odd or even, follow the same order. Here is the complete

system.

For the exponent	1	1	3	5	7	4	2	8	6
,, ,,	2	2	6	8	4	7	3	5	1
,, ,,	3	3	5	7	1	6	4	2	8
,, ,,	4	4	8	2	6	1	5	7	3
,, ,,	5	5	7	1	3	8	6	4	2
,, ,,	6	6	2	4	8	3	7	1	5
,, ,,	7	7	1	3	5	2	8	6	4
,, ,,	8	8	4	6	2	5	1	3	7

which, used in the manner shown, will give the following complete square

1^1	2^6	3^7	4^2	5^8	6^5	7^4	8^3
2^2	1^7	4^6	3^5	6^3	5^4	8^1	7^8
3^3	4^8	5^1	6^4	7^2	8^7	1^6	2^5
4^4	3^1	6^8	5^7	8^5	7^6	2^3	1^2
5^5	6^2	7^3	8^6	1^4	2^1	3^8	4^7
6^6	5^3	8^2	7^1	2^7	1^8	4^5	3^4
7^7	8^4	1^5	2^8	3^6	4^3	5^2	6^1
8^8	7^5	2^4	1^3	4^1	3^2	6^7	5^6

where the exponents of the first row takes this order

$$1 \quad 6 \quad 7 \quad 2 \quad 8 \quad 5 \quad 4 \quad 3\ ,$$

which is clearly not a guide, since the values of u will be

$$1 \quad 5 \quad 5 \quad 7 \quad 4 \quad 2 \quad 6 \quad 6\ ,$$

and they are not all different.

91. After these general reflections, which can be applied to all double step Latin squares, no matter how large the number n is, provided that it is divisible by 4, we are going to develop some particular cases where $n = 4$ and $n = 8$, omitting the rest, which would take too long. Since, for the case $n = 8$ we have already shown several examples, we will be content with finding all the guides. We have seen that each of them can furnish a complete system and that two different guides can also lead to a complete system, provided that the terms keep the same order with respect to being odd or even. With these systems formed, whose number obviously surpasses by far that of the first guides, the construction of the squares no longer holds the least difficulty.

CASE OF $n = 4$

92. The double step Latin square in this case is

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

which only produces the following two guides for the exponent 1

$$1 \quad 4 \quad 2 \quad 3 \quad \text{and} \quad 1 \quad 3 \quad 4 \quad 2$$

of which one, by applying the two prescribed rules to it, produces the other. From these two guides, we can form the two following complete systems.

			I				II			
For	the exponent	1	1	4	2	3	1	3	4	2
"	"	2	2	3	1	4	2	4	3	1
"	"	3	3	2	4	1	3	1	2	4
"	"	4	4	1	3	2	4	2	1	3

and writing the exponents according to the guides, we get the two complete squares here:

I				II			
1^1	2^3	3^4	4^2	1^1	2^4	3^2	4^3
2^2	1^4	4^3	3^1	2^2	1^3	4^1	3^4
3^3	4^1	1^2	2^4	3^3	4^2	1^4	2^1
4^4	3^2	2^1	1^3	4^4	3^1	2^3	1^2

It is easy to be convinced that whatever other Latin square that we would want to find, we can never get any other complete squares that satisfy the prescribed conditions. However, both of the squares that we have just formed can have their columns transposed so that the prescribed properties are also met by the diagonals. Here are two examples:

I				II			
1^1	3^4	4^2	2^3	1^1	4^3	2^4	3^2
2^2	4^3	3^1	1^4	2^2	3^4	1^3	4^1
3^3	1^2	2^4	4^1	3^3	2^1	4^2	1^4
4^4	2^1	1^3	3^2	4^4	1^2	3^1	2^3

CASE OF $n = 8$

93. The fundamental Latin square is

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	5	6	7	8	1	2
4	3	6	5	8	7	2	1
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	1	2	3	4	5	6
8	7	2	1	4	3	6	5

for which the two general formulas give the two following guides

1	3	5	7	4	2	8	6
1	4	8	6	2	3	5	7

the first of which begins with the four odd numbers. It is not difficult to find all the guides whose odd and even terms keep the same order, which are the following four:

1	3	5	7	4	2	8	6
1	5	7	3	4	8	2	6
1	5	7	3	8	4	6	2
1	7	5	3	8	2	4	6

94. Let us apply the two rules given and demonstrated above (§78 and following) to these four guides, which will give us the four dozen following:

First dozen.

Original	1	3	5	7	4	2	8	6	I
inverted	1	6	2	5	3	8	4	7	
2^{nd} rule	1	8	7	6	4	5	2	3	II
[applied to I]	1	5	4	8	3	7	6	2	
1^{st} rule	1	8	5	3	2	7	6	4	III
[applied to I]	1	7	8	5	6	4	3	2	
2^{nd} rule	1	3	7	2	6	8	4	5	IV
[applied to III]	1	4	6	8	2	3	5	7	
1^{st} rule	1	5	6	2	7	3	8	4	V
[applied to II]	1	4	2	7	8	5	3	6	
2^{nd} rule	1	6	8	3	7	4	2	5	VI
[applied to V]	1	7	4	6	8	2	5	3	

Second dozen

Second dozen.

Original	1	5	7	3	4	8	2	6	I
inverted	1	7	4	5	2	8	3	6	
2^{nd} rule	1	6	5	2	4	7	8	3	II
[applied to I]	1	4	2	8	6	7	5	3	
1^{st} rule	1	3	8	2	7	5	6	4	III
[applied to I]	1	4	8	5	3	2	6	7	
2^{nd} rule	1	8	6	3	7	2	4	5	IV
[applied to III]	1	7	6	8	3	5	4	2	
1^{st} rule	1	8	5	7	6	3	2	4	V
[applied to II]	1	6	4	7	8	3	5	2	
2^{nd} rule	1	3	7	6	2	4	8	5	VI
[applied to V]	1	5	2	6	8	4	3	7	

Third dozen

Third dozen.

Original	1	5	7	3	8	4	6	2	I
inverted	1	8	4	6	2	7	3	5	
[2nd rule	1	6	5	2	8	3	4	7	II
applied to I	1	3	2	7	6	8	5	4	
1st rule	1	3	2	8	7	5	4	6	III
applied to I	1	4	6	7	3	2	8	5	
2nd rule	1	8	4	5	7	2	6	3	IV
applied to III	1	7	8	6	3	5	2	4	
1st rule	1	7	5	8	6	4	2	3	V
applied to II	1	6	8	3	4	7	5	2	
2nd rule	1	4	7	5	2	3	8	6	VI
applied to V]	1	5	6	2	4	8	3	7	

Fourth dozen

Fourth dozen.

Original	1	7	5	3	8	2	4	6	I
inverted	1	6	4	7	3	8	2	5	
[2nd rule	1	4	7	2	8	5	6	3	II
applied to I	1	5	2	6	3	7	8	4	
1st rule	1	3	5	8	2	4	6	7	III
applied to I	1	4	8	2	6	7	3	5	
2nd rule	1	8	7	5	6	3	4	2	IV
applied to III	1	7	6	3	2	8	5	4	
1st rule	1	5	4	8	7	3	2	6	V
applied to II	1	8	6	7	4	5	3	2	
2nd rule	1	6	2	5	7	4	8	3	VI
applied to V]	1	3	8	6	4	2	5	7	

95. Here are 48 guiding formulas, which exhaust our Latin square. All the formulas that we can extract by the ordinary method are found in the previous four dozen. By employing only one of these formulas, we can construct a complete square and thus 48 different solutions, without counting those that are given by combining several of the guides whose odd and even terms keep the same order and whose number is without a doubt very considerable. To facilitate the same combinations and at the same time to estimate the number of all the different solutions, we are going to distribute these 48 guides into different categories, according to the order observed with respect to the odd and even terms, the latter of which we designate by the letter p and the others by the letter i. We will obtain the following types:

I.	i	i	i	i	p	p	p	p
	1	3	5	7	4	2	8	6
	1	5	7	3	4	8	2	6
	1	5	7	3	8	4	6	2
	1	7	5	3	8	2	4	6

V.	i	i	p	i	p	p	i	p
	1	3	2	7	6	8	5	4
	1	7	4	5	2	8	3	6
	1	7	6	3	2	8	5	4
	1	7	8	5	6	4	3	2

II.	i	i	i	p	p	p	p	i
	1	3	5	8	2	4	6	7
	1	3	7	2	6	8	4	5
	1	3	7	6	2	4	8	5
	1	7	5	8	6	4	2	3

VI.	i	p	p	i	p	i	i	p
	1	4	2	7	8	5	3	6
	1	6	4	7	8	3	5	2
	1	6	8	3	4	7	5	2
	1	8	6	7	4	5	3	2

III.	i	i	p	p	p	p	i	i
	1	3	8	6	4	2	5	7
	1	5	2	6	8	4	3	7
	1	5	6	2	4	8	3	7
	1	7	4	6	8	2	5	3

VII.	i	p	i	i	p	i	p	p
	1	4	7	5	2	3	8	6
	1	8	5	3	2	7	6	4
	1	8	5	7	6	3	2	4
	1	8	7	5	6	3	4	2

IV.	i	p	p	p	p	i	i	i
	1	4	2	8	6	7	5	3
	1	4	6	8	2	3	5	7
	1	4	8	2	6	7	3	5
	1	8	4	6	2	7	3	5

VIII.	i	p	i	p	p	i	p	i
	1	4	7	2	8	5	6	3
	1	6	5	2	4	7	8	3
	1	6	5	2	8	3	4	7
	1	8	7	6	4	5	2	3

IX.	i	i	p	p	i	i	p	p
	1	3	2	8	7	5	4	6
	1	3	8	2	7	5	6	4
	1	5	2	6	3	7	8	4
	1	5	4	8	3	7	6	2
	1	5	4	8	7	3	2	6
	1	5	6	2	7	3	8	4
	1	7	6	8	3	5	4	2
	1	7	8	6	3	5	2	4

X.	i	p	p	i	i	p	p	i
	1	4	6	7	3	2	8	5
	1	4	8	5	3	2	6	7
	1	6	2	5	3	8	4	7
	1	6	2	5	7	4	8	3
	1	6	4	7	3	8	2	5
	1	6	8	3	7	4	2	5
	1	8	4	5	7	2	6	3
	1	8	6	3	7	2	4	5

96. In considering any class of guides with λ formulas, it is clear that, since we can combine each of them with each of the guides of another order of λ guides, we will extract $\lambda\lambda$ different solutions. Since we have in all eight categories of four guides each, each of which can be combined with something of the same category, we can derive sixteen solutions from each category, thus giving 128 solutions from eight categories. By adding the two categories of eight guides, each of which gives 64 solutions, the number of all the possible solutions will be 256, each of which will also satisfy the problem. But we must remark that the quadruple step Latin squares will give an even greater number of them, without counting those that we can extract from several transformations explained above and which will be explained more clearly in the following. Combined with the different solutions for the cases of $n = 3$, $n = 4$, $n = 5$, and $n = 7$, this must increase our surprise with regards to the case of $n = 6$, the impossibility of which seems to be increasingly confirmed.

End of the Second Section

THIRD SECTION

1	2	3	4	5	6	7	8	9	etc.
2	3	1	5	6	4	8	9	7	etc.
3	1	2	6	4	5	9	7	8	etc.
4	5	6	7	8	9	10	11	12	etc.
					etc.				

96a. Here it is obvious that the number n must be divisible by 3. Thus, we will write $n = 3m$ throughout, where m is the number of members of which each row and column is composed. Thus, the simplest case will be the one where $m = 1$, in other words, $n = 3$, and where the Latin square contains a single member of the general triple step square

1	2	3
2	3	1
3	1	2

whose construction has been sufficiently explained in §18 of the first section.

97. The first question to present itself here is whether all the cases of this triple step square always give guiding formulas. I must remark that when the square is composed of two members, it will never permit guides, so that the case of $n = 6$ must still be excluded in this type of single step square. We can convince ourselves of this truth using the ordinary method for finding guides; it will acquire greater certainty, since we can give a rigorous proof. This demonstration is taken from principles very different from those whose impossibility we have proven in the preceding cases, in which the number n is oddly even, but which cannot be applied in this section due to the multiplicity of the different cases that we would be obliged to consider.

98. To make this proof clearer and easier, I will denote the first member of the proposed triple step square, that is

1	2	3
2	3	1
3	1	2

by the letter A, which will be comprised of three rows and columns. The letter a will denote each number contained in the small square, that is 1, 2, or 3. Likewise, I will express the second member of the square, that is

4	5	6
5	6	4
6	4	5

by the letter B and each of the numbers that it contains by b. Assuming this, the two member Latin square, that is to say the case of $n = 6$, can be represented by

$$\begin{matrix} A & B \\ B & A \end{matrix}$$

where each of the rows and columns has six terms.

99. I now observe that, if this square gives a guide, it will contain three as and three bs, some being taken from the first column, AB, and the others from the second column, BA. Since all the terms of a guide must be taken from different rows and from different columns, each term that we put in the guide excludes a row and another column. Thus, when we want to take all three as from the first column, since they would be taken from the letter A, the first row would be excluded, as would the first column, and thus, the three bs must be taken from the second part of the second column, that is from the member A, the remaining one and the one that does not contain any bs. Now suppose that we take two as and one b from the first column, that is, three terms. It must be that the other column gives just as many, that is, one a and two bs. Since the two as are taken from the member A of the first row and the b is taken from the member B of the second row, it is clear that the remaining term of the first row can only be b and those of the second row can only be aa, the first column being excluded. In place of the missing terms a b b, we will obtain a a b. Thus, we already see rather clearly that in taking one a and two bs from the first column, it would be equally impossible to derive the remaining terms a a b from the second. Consequently, it is proven that the case $n = 6$ does not permit any guides.

100. If for the case of $n = 9$ or $m = 3$ we denote the third member of the general square, that is

$$\begin{matrix} 7 & 8 & 9 \\ 8 & 9 & 7 \\ 9 & 7 & 8 \end{matrix}$$

by the letter C and the three numbers 7, 8, 9 that it contains by the letter c, we will have the square

$$\begin{matrix} A & B & C \\ B & C & A \\ C & A & B \end{matrix}$$

to examine. The guiding formula, if there is one, will be comprised of three as, three bs, and three cs. Then by taking the three as from the first column, the first row will be excluded. Consequently, we can only take the three cs from the second column, which excludes the second row. Because there are still three bs in the third column, we easily see that this case permits guides; we can even derive them using other methods.

101. By examining the case of $n = 12$ or $m = 4$ in the same manner and by designating the fourth member of the general square

$$\begin{matrix} 10 & 11 & 12 \\ 11 & 12 & 10 \\ 12 & 10 & 11 \end{matrix}$$

by the letter D and the terms that it contains, 10, 11, 12 by d, so that the square to examine is

$$\begin{array}{cccc} A & B & C & D \\ B & C & D & A \\ C & D & A & B \\ D & A & B & C \end{array}$$

we will see that no matter how we choose the lower case letters from the lines of this square, it will never permit any guides. It seems that we can rigorously draw the same conclusion for all the cases where n is an even number, so that this section only applies to odd multiples of 3, like 3, 9, 15, 21, etc.

102. The elegant proof for the case of $n = 6$, shown in §98 and §99, undertakes a digression to quintuple step Latin squares, or to septuple or to any other odd numbered step, for which we will be able to show with the same ease that those comprised of two members will never permit guides. For the case of *, denoting the two members of which it is composed by A and B, and the five terms that they contain by a and b, we will be able to derive from the square

$$\begin{array}{cc} A & B \\ B & A \end{array}$$

or

$$\begin{array}{cccccccccc} a & a & a & a & a & b & b & b & b & b \\ . & . & . & . & . & . & . & . & . & . \\ a & a & a & a & a & b & b & b & b & b \\ b & b & b & b & b & a & a & a & a & a \\ . & . & . & . & . & . & . & . & . & . \\ b & b & b & b & b & a & a & a & a & a \end{array}$$

a guiding formula that contains, in any order, five as and five bs.

103. Thus, if we wanted to take all five as from the first column, the first row would be excluded from it and only the term A, which does not include b, would remain in the second column. By taking four as and one b from the first column, the second column will only give one b and four as, though we must have one a and four bs in order to complete the guide. The same drawback is encountered by taking three as and two bs from the first column, for instead of the two as and three bs that we still need, the second column will only give two bs and three as. Thus, we see that there are no guides to be expected from this. The reason depends on whether the number of lower case letters is odd, and it seems that we can state that the same impossibility exists in all the cases where the number of the members A, B, etc. is even.

104. But in all the cases where the number of lower case letters is even, this impossibility ceases entirely. Let us suppose that there is a quadruple step square that consists of two members, A and B, each of which has its lower case letter, a and b, four times, which would be the case of $n = 8$. It is necessary to take from the square * a guide that comprises, in any order, four as and four bs, which is not the least bit difficult. We only have to take two as and two bs from the first column. Since the first member B in the second column still gives two bs and the other, A, two as, the guide will be complete. At the same time we see that in all these cases we must always take two as and two bs from each column. This reasoning holds true for all even numbers.

105. Let us return to our triple step squares. To search for guides from them, let us consider any term x, which corresponds to the vertical index t and to the horizontal index u. By comparing this term to the sum of its indices, $t+u$, we will soon see that there are two possible relationships between them; one

$$x = t + u - 1$$

and the other

$$x = t + u - 4,$$

where the difference depends on the divisibility of the numbers t and u by 3. These numbers can be reduced to three categories that we will be able to represent by $3\lambda+1, 3\lambda+2, 3\lambda+3$, or simply by 1, 2, 3, which can also denote the three categories. Next, because of the ambiguity of the numbers 1 and 4 in the two expressions of x, we will set

$$x = t + u - w.$$

That said, the following table will serve to determine the relationship between x and its indices and the values of w for all the possible values of t and u.

If	t	=	1	1	1	2	2	2	3	3	3
	u	=	1	2	3	1	2	3	1	2	3
we will get	w	=	1	1	1	1	1	4	1	4	4
	x	=	1	2	3	2	3	1	3	1	2

where we see that we have $w = 4$ when one of the indices t or u is equal to 3 and when neither of them is equal to 1.

106. Having thus found $x = t + u - w$, inversely we take

$$u = x - t + w,$$

from which we will be able to find the horizontal index u of each term x and the vertical index that corresponds to it. From there, we will be able to assign the real value of w for all the values of t and x, as we can see in this table:

If	x	=	1	1	1	2	2	2	3	3	3
	t	=	1	2	3	1	2	3	1	2	3
we will get	w	=	1	4	4	1	1	4	1	1	1

Consequently, there are three cases where $w = 4$, which we will write separately like this:

$$w = 4,\ \text{si} \left\{ \begin{array}{cc|c|c|c} x & = & 1 & 1 & 2 \\ t & = & 2 & 3 & 3 \end{array} \right.$$

107. This last table will be of great help in determining whether a proposed formula is a guide. We only have to write the formula or the series for x and t, one under the other, and to derive the values of u according to the table. If we find them all different, it is a sure sign that the proposed formula is a guide. To clarify this with an example, let us take for the case of $n = 9$

This progression for x	1	3	5	7	9	2	4	6	8
and in its subscipt the series of t	1	2	3	4	5	6	7	8	9
on the condition of the propsed formula, we will have $u =$	1	2	6	4	5	9	7	8	3

which, since it includes all the different values, shows that the arithmetic progression

$$1 \quad 3 \quad 5 \quad 7 \quad 9 \quad 2 \quad 4 \quad 6 \quad 8$$

is in fact a guide.

108. Having found a single guide, we will be able, by the methods similar to those that we used in the preceding sections, to derive a great number of other formulas that are also guides. Assuming that for a new guide the term X corresponds to the vertical index T and to the horizontal index U, since we now have $x = t + u - w$, we see that the two indices t and u are permutable, so that taking

$$T = u \qquad \text{and} \qquad U = t,$$

we will have

$$X = x.$$

Thus, in the preceding example, having the values of u in front of us, we only have to arrange them in their natural order and to write its number x under each one, in this manner

$$\begin{array}{ccccccccccc} T & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ X & = & 1 & 3 & 8 & 7 & 9 & 5 & 4 & 6 & 2 \end{array}$$

and this formula will surely be a new guide, since all the Us, being the same as the ts, have different values.

109. We will also be able, as in the preceding sections, to switch the two letters t and x, by taking

$$T = x \qquad \text{and} \qquad X = t,$$

from which we will take a new guide, the inverse, as above. Thus, the formula proposed above,

$$1 \quad 3 \quad 5 \quad 7 \quad 9 \quad 2 \quad 4 \quad 6 \quad 8$$

will give this new guide by inversion

$$1 \quad 6 \quad 2 \quad 7 \quad 3 \quad 8 \quad 4 \quad 9 \quad 5,$$

and that which we had taken from the proposed formula by the other rule,

$$1 \quad 3 \quad 8 \quad 7 \quad 9 \quad 5 \quad 4 \quad 6 \quad 2,$$

leads to the following inverse

$$1 \quad 9 \quad 2 \quad 7 \quad 6 \quad 8 \quad 4 \quad 3 \quad 5.$$

110. Having the formula

$U = X - T + w = t - x + w$

for U in accordance with the rule where $T = X$ and $X = t$, since these expressions go through all the values while t and x undergo the necessary variations, it follows that, taking

$T = t$,

we will be able to set

$X = t - x + w$.

This is what the second rule consists of, which differs from those of the preceding sections only in the value of w, which will always be equal to 1, except in the three cases shown in §106, namely

$$\begin{array}{r|c|c|c} t = & 1 & 1 & 2 \\ x = & 2 & 3 & 3 \end{array}$$

when $w = 4$. With these two rules, if we have found several guides by the ordinary methods, we will be able to derive several others.

111. But we will soon discover a large variety among the formulas that we want to transform by these rules. There are some that remain the same by both. One example is

1 3 2 7 9 8 4 6 5.

which is the diagonal of the proposed square. It is produced by the first as well as the second of our rules. There are also formulas that, by either of the rules produce a single new guide. For example, the arithmetic progression, decreasing by unity

1 9 8 7 6 5 4 3 2.

which by the first rule reproduces itself, while the second gives

1 6 5 7 3 2 4 9 8.

which is reproduced by the inverse.

112. Let us develop the proposed arithmetic progression

1 3 5 7 9 2 4 6 8.

which, with the help of our two rules [§109 and §110] gives, as we will see, four new guides

Proposed	1	3	5	7	9	2	4	6	8
inverted	1	6	2	7	3	8	4	9	5
2nd rule	1	3	8	7	9	5	4	6	2
	1	9	2	7	6	8	4	3	5

Thus, here are, with the previous ones, seven guides for the case of $n = 9$, which have the wonderful property of having terms follow the same order with respect to their divisibility by 3. Now it is easy to find still more of them that follow the same law in this respect, which we are going to look at side by side.

1	3	2	7	9	8	4	6	5
1	3	5	7	9	2	4	6	8
1	3	8	7	9	5	4	6	2
1	6	2	7	3	8	4	9	5
1	6	5	7	3	2	4	9	8
1	6	8	7	3	5	4	9	2
1	9	2	7	6	8	4	3	5
1	9	5	7	6	2	4	3	8[1])
1	9	8	7	6	5	4	3	2

all of which we had found, except the two following ones

1	6	8	7	3	5	4	9	2
1	9	5	7	6	2	4	3	8[1])

which produce each other by the first rule as well as by the second. It is important to have reviewed these nine formulas that have the same order with respect to the terms divisible by 3. For we will see in the following section that to form a complete magic square, we can use 2 and even 3 similar guides for the different exponents with regards to our three types of numbers. Thus, we see that these nine formulas are capable of producing a prodigious number of different squares.

113. But there are also a number of guides that in this manner give up to a dozen new ones, as we can see by the following one chosen at random.

Proposed	1	3	8	6	7	9	2	5	4
inverted	1	7	2	9	8	4	5	3	6

from which we find by the	2^{nd} rule	1	3	5	2	8	7	9	4	6
		1	5	2	8	7	3	6	9	4
	1^{st} rule	1	4	2	8	3	9	6	5	7
		1	3	6	9	2	7	5	4	8
	2^{nd} rule	1	8	2	9	6	7	5	4	3
		1	3	7	8	4	9	6	5	2
	1^{st} rule	1	3	9	8	7	5	6	2	4
		1	9	2	5	8	7	3	4	6
	2^{nd} rule	1	3	4	9	8	2	5	7	6
		1	6	2	3	7	9	8	5	4

giving twelve, none of which had been known to us previously.

114. After these rules for the construction of the guides for the exponent 1, there is still the matter of deriving guides for the other exponents, in other words, in what manner we must construct the complete systems. For this, I observe that in general having found for any exponent a the guide

$$a \quad b \quad c \quad d \quad e \quad \text{etc.},$$

we will extract from it, in accordance with the form of the Latin square, the guide for the exponent $a+3$ by increasing each term from the first by 3. By carefully considering the form of the proposed square, we can even suspect that if any term of the proposed guide is of the form $3\alpha+1$, then the one for the exponent 2 will be of the form $3\alpha+2$ and for the exponent 3 the form $3\alpha+3$. Then if a term of the proposed guide for the exponent 1 has the form $3\alpha+2$, the corresponding term of the guide for the exponent 2 will have the form $3\alpha+3$ and that of the exponent 3 the form $3\alpha+1$. Finally, if the term of the proposed guide is of the form $3\alpha+3$, that of the exponent 2 will be of the form $3\alpha+1$ and that for the exponent 3 of the form $3\alpha+2$. This conjecture, which will be represented in the following table for clarification

form of the term	for the exponents 2	3
$3a+1$	$3a+2$	$3a+3$
$3a+2$	$3a+3$	$3a+1$
$3a+3$	$3a+1$	$3a+2$

can even be rigorously proven in the following manner.

115. In the proposed guide for the exponent 1, let x be any term that corresponds to the vertical index t and to the horizontal index u, so that

$$u = x - t + w.$$

Next, in the guide for the exponent 2, let x be a term that corresponds to the same vertical index t, but to the horizontal index u, so that

$$u = x - t + w.$$

Finally, in the guide for the exponent 3, let x be a term that corresponds to the same vertical index t, but to the horizontal index

$$u = x - t + w.$$

Here we must note that the horizontal indices u, u, u must be taken from the same formula explained above. That said, it will be necessary to prove that while the index u goes through all its values (as is the nature of guides), the two other indices u and u also go through all their values. This will be clarified by the following table which represents all the possible cases with respect to the two given values, t and x, or which we have abbreviated $\alpha - \beta = \gamma$.

t	$=$	$3\beta +$	1	2	3	1	2	3	1	2	3
x	$=$	$3\alpha +$	1	2	3	2	3	1	3	1	2
u	$=$	$3\gamma +$	1	1	1	2	2	2	3	3	3
x'	$=$	$3\alpha +$	2	3	1	3	1	2	1	2	3
u'	$=$	$3\gamma +$	2	2	2	3	3	3	1	1	1
x''	$=$	$3\alpha +$	3	1	2	1	2	3	2	3	1
u''	$=$	$3\gamma +$	3	3	3	1	1	1	2	2	2

From this table, it is obvious that each time that $u = 3\gamma + 1$, we will have

$$u = 3\gamma + 2 \quad \text{and} \quad u = 3\gamma + 3.$$

In the same manner, when $u = 3\gamma + 2$, we will have

$$u = 3\gamma + 3 \quad \text{and} \quad u = 3\gamma + 1.$$

Finally, when $u = 3\gamma + 3$, we will have

$$u = 3\gamma + 1 \quad \text{and} \quad u = 3\gamma + 2.$$

Thus, we see that since u goes through all the values, u as well as u must also go through all the values. Thus, the rule given above gives us two other guides for the following exponents 2 and 3 for each guide for the exponent 1. From this we can form the guides for the exponents 4, 5, 6 by adding 3 to each term of the first three, and those for the exponents 7, 8, 9 by doing the same thing vis-à-vis the three previous ones.

116. In this manner, the formation of a complete system of guides from a single proposed one for the exponent 1 of the fundamental Latin square will not be the least bit difficult. Let us take, in order to give an example for the case of $n = 9$, the guide that goes in arithmetic progression

$$1 \quad 3 \quad 5 \quad 7 \quad 9 \quad 2 \quad 4 \quad 6 \quad 8\ ;$$

The complete system will be

1	3	5	7	9	2	4	6	8
2	1	6	8	7	3	5	4	9
3	2	4	9	8	1	6	5	7
4	6	8	1	3	5	7	9	2
5	4	9	2	1	6	8	7	3
6	5	7	3	2	4	9	8	1
7	9	2	4	6	8	1	3	5
8	7	3	5	4	9	2	1	6
9	8	1	6	5	7	3	2	4

and the complete square that results from the system will have the following form

1^1	2^3	3^8	4^7	5^9	6^5	7^4	8^6	9^2
2^2	3^1	1^9	5^8	6^7	4^6	8^5	9^4	7^3
3^3	1^2	2^7	6^9	4^8	5^4	9^6	7^5	8^1
4^4	5^6	6^2	7^1	8^3	9^8	1^7	2^9	3^5
5^5	6^4	4^3	8^2	9^1	7^9	2^8	3^7	1^6
6^6	4^5	5^1	9^3	7^2	8^7	3^9	1^8	2^4
7^7	8^9	9^5	1^4	2^6	3^2	4^1	5^3	6^8
8^8	9^7	7^6	2^5	3^4	1^3	5^2	6^1	4^9
9^9	7^8	8^4	3^6	1^5	2^1	6^3	4^2	5^7

117. In this square we have taken the first three guides for the exponents 1, 2, 3 by the same formula. But we would have been able to employ different guides, provided that their terms followed the same order with regards to the divisibility by 3. Having thus shown nine different guiding formulas above that all follow the same law, we could form 729 different complete squares. To clarify this with an example, let us take

*

The complete system of guides will be

1	3	5	7	9	2	4	6	8
2	1	9	8	7	6	5	4	3
3	5	7	9	2	4	6	8	1
4	6	8	1	3	5	7	9	2
5	4	3	2	1	9	8	7	6
6	8	1	3	5	7	9	2	4
7	9	2	4	6	8	1	3	5
8	7	6	5	4	3	2	1	9
9	2	4	6	8	4	3	5	7

from which we construct the following complete square:

1^1	2^9	3^5	4^7	5^6	6^2	7^4	8^3	9^8
2^2	3^1	1^6	5^8	6^7	4^3	8^5	9^4	7^9
3^3	1^2	2^7	6^9	4^8	5^4	9^6	7^5	8^1
4^4	5^3	6^8	7^1	8^9	9^5	1^7	2^6	3^2
5^5	6^4	4^9	8^2	9^1	7^6	2^8	3^7	1^3
6^6	4^5	5^1	9^3	7^2	8^7	3^9	1^8	2^4
7^7	8^6	9^2	1^4	2^3	3^8	4^1	5^9	6^5
8^8	9^7	7^3	2^5	3^4	1^9	5^2	6^1	4^6
9^9	7^8	8^4	3^6	1^5	2^1	6^3	4^2	5^7

118. Here, we have benefited from the beautiful connection which is found among the nine formulas shown above; but in using any other guiding formula, it is not difficult to discover all the other formulas that have the same property with regards to the divisibility by 3. Let us take for example the following guide chosen at random

1 3 8 6 7 9 2 5 4

We write under each term, in the form of an exponent, the value

$$u = x - t + w$$

as well as the others of the same type, like this:

t	=	1	2	3	4	5	6	7	8	9
x	=	1	3	8	6	7	9	2	5	4
u	=	1	2	9	3	6	4	5	7	8
		1^1	3^2	8^9	6^3	7^6	9^4	2^5	5^7	4^8
		1^1	6^5	2^3	3^9	4^3		5^8		7^2
			9^8	5^6	9^6		3^7	8^2	2^4	

and presently everything comes back to taking simple formulas from it where not only all the terms themselves but also their exponents are different. For example

1^1 6^5 2^3 3^9 7^6 9^4 8^2 5^7 4^8
1^1 6^5 8^9 9^6 4^3 3^7 5^8 2^4 7^2

from which we can derive new formulas of the same type which, being joined to the proposed one, can serve to construct 27 new complete squares.

119. Before finishing this section, I will add a proof of the first rule of inversion, which has been assumed true thus far. This proof is all the more necessary since there are a number of Latin squares where the inversion is really incapable of giving guides. It is a question of showing that when the number u, which is equal to $x - t + w$, goes through all its values, while t and x undergo the appropriate variations, the formula $t - x + w$, which I will name v, will also receive all the different values. For this, it is necessary to consider all the different types that the two numbers t and x can include, as was shown in the proof of the previous theorem (§114 and 115) relative to the guides that correspond to the exponents 2 and 3, and as this table explains:

t	$=$	$3\beta +$	1	2	3	1	2	3	1	2	3
x	$=$	$3\alpha +$	1	2	3	2	3	1	3	1	2
u	$=$	$3\gamma +$	1	1	1	2	2	2	3	3	3
v	$=$	$3\gamma +$	1	1	1	3	3	3	2	2	2

From this it is clear that when u is of the form $3\gamma + 1$, v will be of the form $-3\gamma + 1$ and thus, the sum will be equal to 2, that is, in the case $u = 3\gamma + 1$, the number v will be the complement of u to 2 (or $n + 2$), n being the root of the square in question. In the two other cases, $u = 3\gamma + 2$ or $u = 3\gamma + 3$, we will have $v = -3\gamma + 3$ or $v = -3\gamma + 2$, and thus in either $u + v = 5$ (or $= n + 5$), that is to say, in these two cases, v is the complement of u to 5 (or $n + 5$). Thus it is shown that in varying u, the number v will also go through all the values. For the case where $n = 9$, let us write the us in their natural order, that is

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

The vs, in accordance with the rules will be

$$1 \quad 3 \quad 2 \quad 7 \quad 9 \quad 8 \quad 4 \quad 6 \quad 5,$$

from which we see more clearly how all the values of v change according to the variations of the letter u.

End of the Third Section

FOURTH SECTION

GENERAL FORM OF QUADRUPLE STEP LATIN SQUARES

1	2	3	4	5	6	7	8	etc.
2	–	–	–	6	–	–	–	
3	–	–	–	7	–	–	–	
4	–	–	–	8	–	–	–	
5	6	7	8	9	10	11	12	etc.
6	–	–	–	10	–	–	–	
7	–	–	–	11	–	–	–	
8	–	–	–	12	–	–	–	
etc.				etc.				etc.

120. Since, as is obvious by the general form, this section can only consider squares whose root n is divisible by 4, we will set $n = 4m$. m will be the number of members of which the square is composed, which will contain four in each row and column, or sixteen terms in all. So if we represent, in the manner introduced at the beginning of the previous section, members by the letters A, B, C, etc. so that

$$A = \begin{cases} 1 & 2 & 3 & 4 \\ 2 & - & - & - \\ 3 & - & - & - \\ 4 & - & - & - \end{cases} \qquad B = \begin{cases} 5 & 6 & 7 & 8 \\ 6 & - & - & - \\ 7 & - & - & - \\ 8 & - & - & - \end{cases} \qquad C = \begin{cases} 9 & 10 & 11 & 12 \\ 10 & - & - & - \\ 11 & - & - & - \\ 12 & - & - & - \end{cases}$$

the different cases that we will have to consider will be understood in the following forms:

$$A \qquad \begin{matrix} A & B \\ B & A \end{matrix} \qquad \begin{matrix} A & B & C \\ B & C & A \\ C & A & B \end{matrix} \qquad \begin{matrix} A & B & C & D \\ B & C & D & A \\ C & D & A & B \\ D & A & B & C \end{matrix} \qquad \text{etc.}$$

121. If we wanted to treat these squares along the same lines as in the previous sections, we would fall into some very cumbersome calculations. Thus, it will be necessary to employ another method, which can be used when the proposed squares are of other steps beyond the quadruple. This is why I will propose here a method that will facilitate considerably these investigations and by which all the objects will be represented in a manner that is as clear as it is easy.

122. First, by considering any term of the proposed square, which we will indicate by the letters x, we must uncover the relationship that the term has with the indices, the vertical index being t, the horizontal being u, where it is clear that we must take into account four terms whose formulas will be $4\lambda+1, 4\lambda+2, 4\lambda+3, 4\lambda+4$. In accordance with these four types, we will always set

$t = 4p + f, u = 4q + g, x = 4s + h,$

where the numbers p, q, s will always be less than m and the other letters, f, g, h will always indicate one of the four numbers 1, 2, 3, 4. Additionally, in considering the proposed square, we easily see that we will always have

$s = p + q,$

by observing that when the number x becomes larger than $n = 4m$, we must remove the number n, and the excess will show the correct value of the letters.

123. We have already noted above that in the case of quadruple step squares, the first member A can receive four different forms (seen in §16) that it would be good to have here in front of us

I.				II.				III.				IV.			
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	3	4	1	2	1	4	3	2	1	4	3	2	4	1	3
3	4	1	2	3	4	1	2	3	4	2	1	3	1	4	2
4	1	2	3	4	3	2	1	4	3	1	2	4	3	2	1

From these terms for the first member, it will be easy to extract those for the following members B, C, D, etc., by increasing all the terms: for the second, B, by 4; for the third, C, by 8; [for the fourth, D,] by 12; and so on.

124. We begin with the first form whose first row represents the values of f for the form $t = 4p + f$, while the first column gives the values of g for the form $u = 4q + g$. The terms themselves of this form denote the values of the letter h for the form $x = 4s + h$, by observing that $s = p + q$. This interpretation can be represented in this way

g \ f	1	2	3	4
1	1	2	3	4
2	2	3	4	1
3	3	4	1	2
4	4	1	2	3

where the terms of the square denote the numbers h for all the values of f and g.

125. From there we can easily construct another square that will represent the values of the letter g which corresponds to the values of the letter f and h, and a third for the values of f that corresponds to the values of g and h.

h \ f	1	2	3	4
1	1	4	3	2
2	2	1	4	3
3	3	2	1	4
4	4	3	2	1

h \ g	1	2	3	4
1	1	4	3	2
2	2	1	4	3
3	3	2	1	4
4	4	3	2	1

These diagrams can be very conveniently applied to test the guiding formulas whose properties demand that all the values of $t = 4p + f$ correspond to as many different values of the letter $x = 4s + h$ and also that the values of the index $u = 4q + g$ are also all different.

126. Having expressed the values of the numbers t, ux by two members, it would be good to name them, for the convenience of the following explanations. The first we call the Characteristic, which denotes the closest multiple of 4 that is less than the given letter, and the other is the Mantissa, which indicates the form of a proposed number with respect to its divisibility by 4. Thus, for the numbers of the first member A, which are 1, 2, 3, 4, the characteristic will be 0; for those of the second member, B, which are 5, 6, 7, 8, the characteristic will be 4; for those of the third, C, that is 9, 10, 11, 12, it will be 8; and so on. For the rest, it is obvious that the characteristic of x is always equal to the sum of the characteristics of t and u, so that with regards only to the characteristics, we will always have $x = t + u$ and thus

$u = x - t.$

127. Thus, since in all the cases the characteristics are not subject to any difficulty, we can do without it entirely, and thus, we only have to look at the mantissas, f, g, h, that create the forms of t, u, x, or what remains after the division by 4. For this reason, we can also do without the letters f, g, h, in place of which we will use only t, u, and x, as we have done in the previous sections, which will facilitate our research considerably. However, we will add to these three letters t, u, x, a fourth, v, which is related in the same manner to the letters x and t as u relates to the letters t and x, so that in looking at the characteristics, we will have

$v = t - x,$

where we once had $u = x - t$. From this we see that the characteristic of v will always be the negative of that of u, in other words, its complement to the number $n = 4m$. The sum of the characteristics of these two letters will always be 0 or n.

128. Now it will be easy to represent by the appropriate tables how each of the four letters is determined by two others. By assuming that the letters t and u are known, the form of the number x will be determined by the first table, from which we will easily form the second for the values of u, when t and x are known.

u \ t	1	2	3	4
1	1	2	3	4
2	2	3	4	1
3	3	4	1	2
4	4	1	2	3

x \ t	1	2	3	4
1	1	4	3	2
2	2	1	4	3
3	3	2	1	4
4	4	3	2	1

From this table, we then easily extract the third for the values of v using t and x, since we only have to change the indices t and x; in other words, in leaving these, we only have to change the rows and columns, as we can see by the diagram.

3rd table for the values of v

x \ t	1	2	3	4
1	1	2	3	4
2	4	1	2	3
3	3	4	1	2
4	2	3	4	1

129. From the first of these three tables, it is clear that in transposing the letters t and u, the table stays the same. Thus, when we have found any guiding formula in which the vertical index t corresponds to the term x, we can derive another in which, by letting X be the term that corresponds to the vertical index T, we only have to take $T = u$ and $X = x$, and in naming U the horizontal index of this new formula, we will have $U = t$. It is clear that while the two letters T and X go through all the values, the letter U will go through all the same variations. Thus, it is only a matter of arranging the different values of $u = T$ in their natural order.

130. On the other hand, it is not difficult to demonstrate that having found a guiding formula between the letters t and x, we can always derive another between T and X, by taking

$$T = x \text{ and } X = t$$

.

We see by the third table that the horizontal index u will be in this case equal to v. Thus, it is only a matter of demonstrating that, while the values of u go through all the numbers from 1 to 4, those of v are also subjected to the same changes. For this, let us take a new table, where we denote the sum of the two letters u and v by the given t and x.

x \ t	1	2	3	4
1	2	6	6	6
2	6	2	6	6
3	6	6	2	6
4	6	6	6	2

from which it is clear that since the characteristics of u and of v cancel each other out we will always have $u + v = 2$, or $u + v = 6$, the first of which will take place everytime that $u = 1$ or $u = 4\lambda + 1$; in all other cases, we will have $u + v = 6$ or $u + v = n + 6$.

131. Let us develop these different cases. First, by taking $u = 4\lambda + 1$, we will have $v = -4\lambda + 1$, or, by adding n to it, we will have $v = 4(m - \lambda) + 1$. From this we see that while the letter u receives all the values of the form 1, the letter v will also receive all the values. On the other hand, by taking $u = 4\lambda + 2$, we will have $v = 4(m - \lambda) + 4$; in other words, v will be the complement of u to 6 (or to $n + 6$). Thus, while u goes through all the values of the form 2, the letter v will go through those of the form 4. If u goes through those of the form 3, v will receive the values of the same form. Finally, while u takes all the values of the form 4, v will take those of the form 2. Thus, it is clear that in general, if u goes through all the variations, $v = U$ will do the same; consequently, the inversion of the guiding formulas takes place in all these cases without the slightest restriction.

132. From this double transformation of each guiding formula, we can derive several others. Having arrived at the values

$$T = x, X = t, U = v,$$

by changing the letters T and U following the first transformation, we will have this new, transformed one

$$T = v, X = t, U = x$$

.

From there, by changing the letters T and X following the other transformation, we will have this new one

$$T = t, X = v, U = u$$

,

which corresponds to what we found in the previous sections by our second rule.

133. Even though we have found still other transformations, it will suffice to use the two that correspond to those from the other sections, since having seen that by the combination of these two rules, we can derive up to twelve new guiding formulas from each proposed one. That is why we will write them down here: Having any guide in which the term x corresponds to the vertical index t, by setting the index T and the term that corresponds to it for the new guide equal to X, we will always have

by the first rule $T = x$ and $X = t$ by the second rule $T = t$ and $X = v$,

where the number v must be determined by the third table given above and that we have shown again here, since all the transformations that we want to do depend only on this form.

x \ t	1	2	3	4
1	1	2	3	4
2	4	1	2	3
3	3	4	1	2
4	2	3	4	1

134. After having found all the guides for the exponent 1, or at least a good portion of them, it is clear that in adding to each term of such a guide either 4 or 8 or 12, etc., we will have the guides for the exponents 5, 9, 13 and so on. Thus, all that remains is to show how to find guides for the exponents 2, 3, 4, 6, 7, etc., so that we can extract a whole system of guides, after which, as we have seen up until now, it is no longer difficult to construct the complete square.

135. For the term x in the guide for the exponent 1, let the horizontal index equal u. In the guide for the exponent 2, let the horizontal index of the term x equal u; in that for the exponent 3, let it equal the term x and the index equal u; and so on for the others, x and u, x, and u, etc. This said, the first member, A, shows us the following relations between the different values of x:

$$
\begin{aligned}
x &= 1,\quad 2,\quad 3,\quad 4,\\
x' &= 2,\quad 3,\quad 4,\quad 1,\\
x'' &= 3,\quad 4,\quad 1,\quad 2,\\
x''' &= 4,\quad 1,\quad 2,\quad 3.
\end{aligned}
$$

It will be necessary to demonstrate that while the letter u goes through to all the values, the letters u, u, u will also be subjected to the same variations.

136. For this let us consider the tables taken from the second one given above, which expresses the values of u by t and x; they will be represented in the following manner:

x \ t	1	2	3	4
1	1	4	3	2
2	2	1	4	3
3	3	2	1	4
4	4	3	2	1

x' \ t	1	2	3	4
1	2	1	4	3
2	3	2	1	4
3	4	3	2	1
4	1	4	3	2

x''	t: 1	2	3	4
1	3	2	1	4
2	4	3	2	1
3	1	4	3	2
4	2	1	4	3

x'''	t: 1	2	3	4
1	4	3	2	1
2	1	4	3	2
3	2	1	4	3
4	3	2	1	4

In comparing the second of these tables with the first, we see that we always have $u = u + 1$ or $u = u - 3$, the latter of which occurs when $u = 4$; in all the other cases, we have $u = u + 1$. Next, in comparing the third table with the first, we will have either $u = u + 2$ or $u = u - 2$, the latter case taking place when $u = 3$ or 4. Finally, the comparison of the fourth table tells us that we will have $u = u - 1$ in all cases except those where $u = 1$, for which it becomes $u = u + 3$. It is shown that in giving all the appropriate values to u, the letters u, u, u will go through the same variations.

137. Thus, we clearly see in what manner for any guide for the exponent 1, we can form a complete system of guides and a complete square. But from what we have said in the previous sections, we understand just as easily that in order to form the guides for the exponents 2, 3, 4, we can employ different guides for the exponent 1, provided that their terms follow the same order with regards to the divisibility by 4. This is a very fertile source that multiplies considerably the number of complete squares, with regards to all the different guides found for the exponent 1.

138. After these general investigations for all the squares divisible by 4, we are going to consider some particular cases. First, when the proposed square only contains one member, A, which is a single step square, we have shown in the previous section that it will not permit guides. This is why we limit ourselves to reexamining the case of $n = 8$, where the square includes two members, A and B, whose form is

1	2	3	4	5	6	7	8
2	3	4	1	6	7	8	5
3	4	1	2	7	8	5	6
4	1	2	3	8	5	6	7
5	6	7	8	1	2	3	4
6	7	8	5	2	3	4	1
7	8	5	6	3	4	1	2
8	5	6	7	4	1	2	3

It is apparent that this square will furnish 48 guides for the exponent 1, by examining it following the rules given above; I am going to reexamine those that I have found by the first method, taught in §11 and following, which are

1	3	7	5	8	4	2	6
1	3	7	5	4	8	6	2
1	3	8	6	4	2	5	7
1	3	8	6	7	5	2	4
1	3	5	7	8	4	2	6
1	3	5	7	2	8	6	4
1	3	6	8	7	5	4	2
1	3	6	8	2	4	5	7

1	4	7	6	8	4	2	5
1	4	7	6	2	5	8	3
1	4	8	7	3	5	6	2
1	4	8	7	3	2	6	5
1	4	5	8	6	3	2	7
1	4	5	8	2	7	6	3
1	4	6	5	8	7	3	2
1	4	6	5	3	2	8	3

from which we can easily find the rest by applying to it the rules that we have repeated so many times.

139. We will not stop to develop the magic squares that this case can give, since all the principles have been sufficiently explained and demonstrated. The three other cases of the form of the first member, A, no longer being the least bit difficult, and treating it in the same way as the first form, it would be superfluous to push the research any further. Thus, we will finish this section with the remark that the case that we have just examined cannot occur when the number of members A, B, C, etc., is 3 or 5 or perhaps any other odd number.

End of the Fourth Section

FIFTH SECTION

THE TRANSFORMATION OF THE SIMPLE AND COMPLETE SQUARES

140. Having seen that all methods that we have shown so far do not give any magic squares for the case of $n = 6$ and that the same conclusion seems to the oddly even number n, we could believe that if such squares are possible, the Latin squares that serve as their base, not following any order that we have just considered, would be completely irregular. Thus, it would be necessary to examine all the possible cases of such Latin squares for the case of $n = 6$, the number of which us undoubtedly extremely large. As the formation of irregular squares is not so easy, I am going to reexamine a method by whose means we can easily transform, in several different forms, all the regular squares and then examine whether they permit guides.

141. This method follows this principle: that if, in a proposed Latin square, two numbers a and b are located at the angles of a rectangular parallelogram, in the manner that this figure represents

$$\begin{array}{ccccccc} a & . & . & . & . & . & b \\ \vdots & & & & & & \vdots \\ b & . & . & . & . & . & a \end{array}$$

we can switch these two letters, writing a instead of b and b instead of a, the reason for which is obvious. We see that notwithstanding this transposition, all the rows and columns still include the same numbers. Thus, it is obvious that by this principal we will be able to transform each proposed square into several other different forms that will have, with regards to the guiding formulas, very particular properties.

142. Let us consider, for example, the following single step Latin square with 36 entries:

1	2	3	4	5	6
2	$\underline{3}$	4	5	$\underline{6}$	1
3	4	5	6	1	2
4	5	6	1	2	3
5	$\underline{6}$	1	2	$\underline{3}$	4
6	1	2	3	4	5

which, as we have demonstrated in section I, §20, does not permit any guides. Let us transpose, in the manner described, the two numbers marked, 3 and 6, arranged in a parallelogram. We will obtain the following square:

1	2	3	4	5	6
2	6	4	5	3	1
3	4	5	6	1	2
4	5	6	1	2	3
5	3	1	2	6	4
6	1	2	3	4	5

which, despite the apparent similarity, differs so essentially from the proposed square that we can derive a great number of guides from it for all six exponents, though the other does not permit any. Here they are:

1	6	5	2	4	3	4	3	2	5	1	6
1	6	5	3	2	4	4	3	2	6	5	1
1	4	6	2	3	5	4	1	3	5	6	2
1	4	2	5	6	3	4	1	5	2	3	6
1	5	4	3	6	2	4	2	1	6	3	5
1	5	2	3	6	4	4	2	5	3	6	1
1	3	4	6	2	5	4	6	1	3	5	2
1	3	6	5	4	2	4	6	3	2	1	5

2	4	3	1	6	5	5	7	6	4	3	2
2	3	5	1	4	6	5	6	2	4	1	3
2	3	6	4	1	5	5	6	3	1	4	2
2	1	5	4	6	3	5	4	2	1	3	6

3	2	4	1	6	5	6	5	1	4	3	2
3	6	2	1	5	4	6	3	5	4	2	1
3	6	1	4	2	5	6	3	4	1	5	2
3	5	2	4	6	1	6	2	5	1	3	4

143. After having found all these guides, all that remains is to find out if we can form a complete system from them, with which we can complete the proposed simple (Latin) square. In carefully considering the guides for the exponents 2, 3, 5, 6, we will see that no matter how we combine them, they will give only 1 and 4 in the fourth column, so that these two numbers will necessarily be located twice in the same line of the complete system, the impossibility of which leaps to our attention. Thus, we can boldly say that the proposed simple square will not give a solution to the problem.

144. I have examined a great number of squares similarly transformed with this method without encountering a single one that didnt have the same problem of not giving a single system of guides whose columns didnt contain a number twice. Thus, I have not hesitated to conclude from this that we cannot produce a complete square with thirty-six entries, and that the same impossibility extends to the cases of $n = 10, n = 14$ and in general to all the oddly even numbers. Since we have found a method for transforming any magic square into several different forms (as many as 24), if we found a single complete square for the case of $n = 6$, there would certainly be several others whose fundamental Latin squares would all be different. So, having

examined a considerable number of such squares, it seems impossible that all the aforementioned cases have escaped me.

145. This reasoning can be taken to a much higher degree of certainty by the general transformation that we are going to reveal, with which each proposed Latin square can be transformed into several others that all have the same property with regards to guides. Thus, if the proposed square doesnt permit any guides at all, all the transformed squares will also be of the same nature. In the event that the proposed magic square gives a complete system, all those that have been derived from it will also give complete magic squares.

146. For this *general transformation*, we only have to change the definition of the numbers of which the Latin square is composed by substituting other numbers in any order and by then rewriting the new square in the order that we have observed thus far, that is, that the numbers of the first row and column are in their natural order. In this way, we will always get a new square possessing the same properties with regards to guides, because all we have to do is to apply the same changes to the guides of the proposed square. From here we see that this method must be as fertile in the production of new squares as the number n is large. For the cases of $n = 2, 3, 4$, there is no expected change. For the case of $n = 5$, the variation could increase up to three and for the case of $n = 6$, the number must be even more considerable because the order of six numbers can receive up to 720 variations, though many of them will give the same form.

147. To clarify the manner and the usage of these transformations, we are going to take for example the last square of 6 that had been so fertile in guides; thus, by changing the numbers in any manner, for example by writing

4 6 1 3 2 5

instead of

1 2 3 4 5 6

we will get the following square

4	6	1	3	2	5
6	5	3	2	1	4
1	3	2	5	4	6
3	2	5	4	6	1
2	1	4	6	5	3
5	4	6	1	3	2

which, by putting the rows and columns in order, will become this ordinary form:

1	2	3	4	5	6
2	4	1	5	6	3
3	5	2	6	4	1
4	1	6	2	3	5
5	6	4	3	1	2
6	3	5	1	2	4

If we treated in the same manner all the Latin squares with 36 entries, be they single, double, or triple step, that, as we have shown, do not permit any guides, we would get a great number of similar squares that would not be more permissive of guides; so that it will suffice to have examined a single one to come to a conclusion on the nature of all the others.

148. From here it is clear that if there existed a single complete magic square with 36 entries, we could derive several others using these transformations that would also satisfy the conditions of the problem. But,

having examined a great number of such squares without having encountered a single one, it is most likely that there arent any. For the number of Latin squares cannot be so enormous that the number of those that I have examined does not give one that permits guides, if there are any. Given that the case of $n = 2$ and $n = 3$ only give one, the case of $n = 4$ four, the case of $n = 5$ fifty-six, according to an exact count. Thus, we see that the number of variations for the case of $n = 6$ cannot be so prodigious that the fifty or sixty that I have examined were but a small part. I observe further here that the exact count of all the possible cases of similar variations would be an object worthy of the attention of Geometers even more so because all the known principles in the doctrine of combinatorics do not lend the least bit of help.

149. In examining several such squares formed at random, I have noticed a surprising difference with regards to guides; sometimes I encountered some that gave one, and other times those didnt give any for two exponents, but two for each of the others. Among others, I have also come across a square that seems to merit particular attention, since it gave me four guides for each exponent, and even some that seemed to promise a complete system; this is why I am going to report here the square that gave them.

Square.

1	2	3	4	5	6
2	1	5	6	3	4
3	4	1	2	6	5
4	5	6	1	2	3
5	6	4	3	1	2
6	3	2	5	4	1

Guides.

1	4	6	5	3	2	3	2	6	5	1	4	5	1	2	4	6	3
1	5	2	3	6	4	3	1	4	5	2	6	5	2	1	6	4	3
1	6	5	2	4	3	3	6	5	4	2	1	5	4	2	1	3	6
1	3	4	6	2	5	3	6	2	1	5	4	5	4	3	6	2	1
2	4	6	3	5	1	4	1	3	5	6	2	6	1	4	2	5	3
2	5	1	3	4	6	4	3	1	6	5	2	6	5	1	4	3	2
2	6	3	1	4	5	4	2	5	3	6	1	6	2	4	1	3	5
2	3	6	4	1	5	4	3	5	2	1	6	6	5	3	2	1	4

All these guides have the beautiful property that each of them has its inverse among the others. But, to form a complete system from them, we only have to combine four of them in the two following ways

1	5	2	3	6	4	1	3	4	6	2	5
2	6	3	1	4	5	2	5	1	3	4	6
3	1	4	5	2	6	3	6	2	1	5	4
4	3	1	6	5	2	4	1	3	5	6	2

It is clear that the guides for the next exponents, 5 and 6, do not complete the system.

150. We could apply similar transformations to the true magic or complete squares; but it would be superfluous to construct others by switching the numbers. On the other hand, there is another type of transformation that is particular to them, since in all magic squares the Latin and Greek numbers can be switched with each other, from which we always obtain a new and entirely different square. Thus, by taking for example

the following complete square with 25 entries

$$\begin{array}{ccccc}
1^1 & 2^5 & 3^4 & 4^3 & 5^2 \\
2^2 & 3^1 & 4^5 & 5^4 & 1^3 \\
3^3 & 4^2 & 5^1 & 1^5 & 2^4 \\
4^4 & 5^3 & 1^2 & 2^1 & 3^5 \\
5^5 & 1^4 & 2^3 & 3^2 & 4^1
\end{array}$$

we will extract, by the aforementioned switching of numbers, the following square

$$\begin{array}{ccccc}
1^1 & 5^2 & 4^3 & 3^4 & 2^5 \\
2^2 & 1^3 & 5^4 & 4^5 & 3^1 \\
3^3 & 2^4 & 1^5 & 5^1 & 4^2 \\
4^4 & 3^5 & 2^1 & 1^2 & 5^3 \\
5^5 & 4^1 & 3^2 & 2^3 & 1^4
\end{array}$$

which, being put in order, will take its primitive form; but this change is also only a very particular case of the general transformation that we are going to propose.

151. Let us note that as each term of a complete square contains two numbers, one of which has been named *the Latin number* and the other *the Greek number*, the entry that the term occupies also is determined by two numbers, one of which is the horizontal index and the other vertical. Each term with the entry that it occupies is thus determined by four numbers, a, b, c, d, the first of which a, we let be the horizontal index, b the vertical index, c the Latin number and d the Greek number. All of the four numbers a, b, c, d will be permutable. In this way, the terms of the last square of 25 entries could be represented in the following manner:

$$\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 1 & 2 & 2 \\
3 & 1 & 3 & 3 \\
4 & 1 & 4 & 4 \\
5 & 1 & 5 & 5
\end{array}
\qquad
\begin{array}{cccc}
1 & 2 & 2 & 5 \\
2 & 2 & 3 & 1 \\
3 & 2 & 4 & 2 \\
4 & 2 & 5 & 3 \\
5 & 2 & 1 & 4
\end{array}
\qquad
\begin{array}{cccc}
1 & 3 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 3 & 5 & 1 \\
4 & 3 & 1 & 2 \\
5 & 3 & 2 & 3
\end{array}
\qquad
\begin{array}{cccc}
1 & 4 & 4 & 3 \\
2 & 4 & 5 & 4 \\
3 & 4 & 1 & 5 \\
4 & 4 & 2 & 1 \\
5 & 4 & 3 & 2
\end{array}
\qquad
\begin{array}{cccc}
1 & 5 & 5 & 2 \\
2 & 5 & 1 & 3 \\
3 & 5 & 2 & 4 \\
4 & 5 & 3 & 5 \\
5 & 5 & 4 & 1
\end{array}$$

If we reflect upon these blocks of four, we will easily see that all four numbers can be switched in all the possible manners, and I do not need to add that the number of variations is 24, which admittedly will not produce all the new squares, but a good number, and even more as the number n grows larger.

152. I have observed above §148 that an exact count of all the possible variations of Latin squares would be a very important question, but which to me seems extremely difficult and almost impossible when the number n surpasses 5. To approach this enumeration, it would be necessary to start with this question: *In how many different ways, the first row being given, can one vary the second row for each proposed number n?*

The solution is contained in the following table:

n	nombre des variations				
1	0				
2	1				
3	1	$=$	$1\cdot 1$	$+$	$0\cdot 0$
4	3	$=$	$2\cdot 1$	$+$	$1\cdot 1$
5	11	$=$	$3\cdot 3$	$+$	$2\cdot 1$
6	53	$=$	$4\cdot 11$	$+$	$3\cdot 3$
7	309	$=$	$5\cdot 53$	$+$	$4\cdot 11$
8	2119	$=$	$6\cdot 309$	$+$	$5\cdot 53$
9	16687	$=$	$7\cdot 2119$	$+$	$6\cdot 309$
10	148329	$=$	$8\cdot 16687$	$+$	$7\cdot 2119$
etc.			etc.		

From this it is clear that the numbers constitute a progression or type of recurring series, each term of which is determined by the two preceding ones, but where the scale of relation is variable. Thus, if we let the letters P, Q, R, S be the number of variations that correspond to the numbers $n, n+1, n+2, n+3$, we will always have

$$R = nQ + (n-1)P$$

and
$S = (n+1)R + nQ.$
We can find from this a formula independent of n, by which each term S can be expressed by the three preceding ones, P, Q, R. The penultimate equation giving

$$R - Q = (n-1)(Q+P),$$

there will be

$$n - 1 = \frac{R-Q}{P+Q};$$

thus, we see that $R - Q$ is always divisible by $P + Q$. In the same manner we will have

$$S - R = n(Q+R)$$

and thus

$$n = \frac{S-R}{Q+R}$$

. Then by subtracting the previous equation from this one, we will have

$$1 = \frac{S-R}{Q+R} - \frac{R-Q}{P+Q}$$

, from which we take

$$PS - PR + QS - QR - RR = PQ + PR + QR$$

and thus

$$S = \frac{PQ + 2PR + 2QR + RR}{P+Q}$$

or

$$S = 2R + Q + \frac{RR + PQ}{P+Q} - Q = 2R + Q + \frac{(R+Q)(R-Q)}{P+Q}$$

. Thus, by taking

$$P = 53, Q = 309, R = 2119$$

, we will have

$$2R + Q = 4547, R - Q = 1810, R + Q = 2428, P + Q = 362;$$

from there

$$\frac{R-Q}{P+Q}=5,$$

and thus

$$S=4547+5\cdot 2428=16687$$

Or, by taking

$$P=309, Q=2119, R=16687,$$

there will be

$$2R+Q=35493, R-Q=14568, R+Q=18806, P+Q=2428;$$

from there

$$\frac{R-Q}{P+Q}=6$$

and thus

$$S=35493+6\cdot 18806=148329$$

The series of the numbers of variation has yet another very beautiful property, the truth of which is nothing less than obvious: it is that we can determine each term by the single previous one. Thus, when the number of variations for the number of terms of the second row, n, is equal to P and the number $n+1=Q$, we will always have

$$Q=nP+\frac{-P\pm 1}{n},$$

where the top sign holds when n is an odd number, and the bottom if it is even. Moreover, taking R for the number of variations of the case $n+2$, since we have found

$$R=nQ+(n-1)P,$$

if we put in place of Q the calculated value

$$Q=nP+\frac{-P\pm 1}{n}$$

, we will have a formula that determines the term R only by the term two previous, P, that is,

$$R=nnP-P+1+(n-1)P=(n-1)(n+2)P\pm 1.$$

Thus, by taking $n=6$ and $P=53$, we will have

$$R=5\cdot 8\cdot 53-1=2119$$

and taking $n=7$, where $P=309$, we will get

$$R=6\cdot 9\cdot 309+1=16687$$

But I must admit that I have not found the property that determines each number only by the previous one except by pure induction, and I do not see how to derive it very well from the nature of the series. However, there is a method of deriving it immediately from the series; at least the following reflections bring us closer to the truth of the assertion that

$$Q=nP+\frac{-P\pm 1}{n}$$

. If Q is the number of variations for any case of n, be it odd or even, and R the number of variations for the next case, where the number of terms is $n+1$, we will get, in accordance with the given expression,

$$nQ=(nn-1)P\pm 1$$

and

$$(n+1)R=(nn+2n)Q\mp 1,$$

where the top sign is used if n is an odd number, the bottom if it is even. The sum of the two expressions gives this equation

$$(n+1)R+nQ=(nn+2n)Q+(nn-1)P,$$

which reduces to

$$(n+1)R = (nn+n)Q + (nn-1)P,$$

from which we extract, by dividing by $n+1$, the value of

$$R = nQ + (n-1)P,$$

which perfectly matches the one we have derived above from the nature of the series. Here we have what I believe I have added with regards to the counting of the variations that can take place in the simple fundamental squares, leaving it to the Geometers to see if there are methods of achieving the enumeration of all the possible cases which seems to provide a vast field for some new and interesting research. Here, I bring mine to an end on a question that, although is of little use itself, has led us to some observations as important for the doctrine of combinatorics as for the general theory of magic squares.