

# On the infinity of infinities of orders of the infinitely large and infinitely small\*

Leonhard Euler

1.<sup>1 2</sup> If  $x$  denotes an infinitely large quantity, then the geometric progression

$$1, x, xx, x^3, x^4, x^5, \text{ etc.}$$

is thus constituted that each term is infinitely greater than the preceding, and indeed infinitely smaller than the following. Whence if we think of the power  $x^{1000}$  as the last term of this progression, it will be possible to put a thousand orders of different infinite magnitudes between it and the first term 1; here we refer to quantities having a finite ratio between themselves as being of the same order. Yet still the number 1000 does not furnish all the intermediate orders between 1 and  $x^{1000}$ ; it should be noted that when we speak of the particular number 1000 here, any other number, however large, can be put in its place.

2. As we just said, we are still far from representing in this progression all the different intermediate orders between 1 and  $x^{1000}$ . For if we put<sup>3</sup>  $x = y^{1000}$  so that

$$y = \sqrt[1000]{x},$$

since  $x$  is an infinite quantity even now  $y$  will be an infinite quantity; whence it follows that because again a thousand intermediate orders can be assigned between 1 and  $y^{1000}$ , each of which is also infinitely greater than the preceding and infinitely less than the following, then even between unity and  $x$  a thousand intermediate orders can be formed, even though before  $x$  had been the first order

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<sup>1</sup>Translator: E507 is cited in pp. 84–86 of H. J. M. Bos, *Differentials, higher-order differentials and the derivative in the Leibnizian calculus*, Arch. Hist. Exact Sci. **14** (1974), no. 1, 1–90. E507 is also mentioned on p. 700 of volume 4 of Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, 1894, and on p. 68 of Constantin Gutberlet, *Das unendliche: Metaphysisch und mathematisch betrachtet*, G. Faber, 1878

<sup>2</sup>Translator: Particularly useful references are G. H. Hardy, *Orders of infinity*, Cambridge Tracts in Mathematics, no. 12, Cambridge University Press, 1910; p. 402 of Hardy's *A course of pure mathematics*, seventh ed., Cambridge University Press, 1938; and p. 249 of Carl Boyer's *The history of the calculus and its conceptual development*, Dover, 1959.

<sup>3</sup>Translator: Original version has  $z = y^{1000}$ .

of infinity. Indeed, in a similar way a thousand intermediate orders can again be assigned between the preceding first order  $x$  and the second order  $xx$ . And it is thus between any two succeeding orders, which are all such that each is infinitely greater than the preceding and infinitely less than following.

3. But we still can't stop here. For since  $y$  is an infinitely large quantity, if we put

$$y = z^{1000},$$

then  $z$  will still be an infinitely large quantity; whence one sees that between 1 and  $z^{1000}$ , that is, between 1 and  $y$ , again a thousand orders of infinity can be constituted, and we may proceed as far along in this fashion as we please, so that the number of all the different orders can in fact be increased to infinity.

4. The same also holds for the infinitely small, in an inverted manner. For if  $x$  denotes an infinitely small quantity, then any term of the geometric progression

$$1, x, xx, x^3, \dots, x^{1000}$$

will be infinitely smaller than the preceding but infinitely greater than the following, and hence between 1 and  $x^{1000}$  we obtain a thousand intermediate orders of the infinitely small, all different; for each is infinitely less than the preceding, and infinitely greater than the following.

5. If now we again put  $x = y^{1000}$ , so that

$$y = \sqrt[1000]{x},$$

then  $y$  will still be an infinitely small quantity; whence it is clear that between 1 and  $y^{1000}$ , that is between 1 and  $x$ , a thousand intermediate orders of the infinitely small can again be constituted, which can also be done between  $x$  and  $xx$ , and similarly between  $xx$  and  $x^3$ , and so on in general between any two neighboring terms of the preceding series. Since by further putting  $y = z^{1000}$ ,  $z$  is even still an infinitely small quantity, the number of different orders once again exceeds one thousand, and this multiplication can be continued endlessly.

6. What has been deduced here from the consideration of powers is indeed common knowledge, and can even be considered to be part of common Algebra; but the higher Analysis also provides innumerable other orders of the infinitely large and infinitely small which cannot be put into any of the orders that we have just related, no matter how many times they are multiplied. Rather, they are found continually to be either infinitely greater or infinitely less than any of the preceding orders. Since I do not recall this being clearly explained so far anywhere, this worthwhile task will be carefully investigated here.<sup>4</sup>

7. Now, these quantities occurring in higher Analysis can be put into two classes, one of which is comprised of logarithms, and the other exponential

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<sup>4</sup>Translator: Euler also briefly discusses orders of infinity in §143, vol. 1 of his 1755 *Institutiones calculi differentialis* (E212).

quantities. Let us first deal with logarithms, and with  $x$  denoting an infinitely large number it follows that its logarithm will also be infinitely large. This is equally clear whatever base we choose for logarithms, be it the common or hyperbolic one, or any other type.<sup>5</sup>

8. When  $x$  is an infinitely large number, it is clear enough by itself that its logarithm, that is,

$$lx,$$

will indeed be infinite, but still infinitely less than the number  $x$ , and can be thus classed in a lower order. Since lower orders of  $x$  can be represented by  $x^{\frac{1}{n}}$ , namely with  $n$  denoting some sufficiently large number, it is hardly difficult to show that  $lx$  is always infinitely less than  $x^{\frac{1}{n}}$ , however large a number is chosen for  $n$ .

9. In fact, it possible to demonstrate in the following way that  $x^{\frac{1}{n}}$  is infinitely greater than  $lx$  when  $x = \infty$ , or in other words that the value of the fraction

$$\frac{x^{\frac{1}{n}}}{lx}$$

is infinitely large. For let us put this value =  $v$ , so that

$$v = \frac{x^{\frac{1}{n}}}{lx},$$

and set

$$p = \frac{1}{lx} \quad \text{and} \quad q = \frac{1}{x^{\frac{1}{n}}},$$

and so

$$v = \frac{p}{q};$$

both the numerator  $p$  and the denominator  $q$  of this fraction will be = 0 in the case  $x = \infty$ . Then, by the well known rule<sup>6</sup> it will also be

$$v = \frac{dp}{dq}.$$

Therefore, since

$$dp = -\frac{dx}{x(lx)^2} \quad \text{and} \quad dq = -\frac{dx}{nx^{\frac{1}{n}+1}},$$

we will have

$$v = \frac{nx^{\frac{1}{n}}}{(lx)^2},$$

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<sup>5</sup>Translator: The hyperbolic logarithm is the natural logarithm, the common logarithm is the logarithm with base 10.

<sup>6</sup>Translator: Namely L'Hospital's rule. A footnote in the *Opera omnia* refers to Euler's *Institutiones calculi differentialis*, latter part of chapter XV, and also cites L'Hospital, *Analyse des infiniment petits*, Paris, 1696, p. 145

which should be equal to the preceding value  $\frac{x^{\frac{1}{n}}}{lx}$ . Truly then, taking the square of this will give

$$vv = \frac{x^{\frac{2}{n}}}{(lx)^2},$$

and dividing this value by the previous yields<sup>7</sup>

$$v = nx^{\frac{1}{n}}.$$

Since this is clearly infinite, it is thus apparent that  $\frac{x^{\frac{1}{n}}}{lx}$  is an infinitely large quantity, that is, that  $lx$  is infinitely smaller than  $x^{\frac{1}{n}}$  no matter how large a number is taken for  $n$ .

10. It is therefore clear that if  $x = \infty$ , its logarithm  $lx$  cannot be grouped with any of the above orders of infinity, no matter how narrowly we divide these orders. Because of this, a new order of infinity, appropriate for the classification of the logarithm  $lx$ , must be constituted here, to which the power  $x^{\frac{1}{n}}$  approaches continually closer the greater the number that is put for  $n$ . Yet the case where  $n = \infty$  does not thereby satisfy this, for when  $\frac{1}{n} = 0$  it makes  $x^{\frac{1}{n}} = 1$ ,<sup>8</sup> while  $lx$  will be infinite; indeed it should be noted that the demonstration given above produced  $nx^{\frac{1}{n}}$ , which by taking  $n = \infty$  nonetheless yields  $v = n$ , which is thus infinite.<sup>9</sup>

11. Therefore, since  $lx$  constitutes as it were the lowest order of all the infinitely large quantities, it is evident that the number of orders that we established above, which was already seen to be infinite, can be further increased infinitely. For if one considers the order designated by the power  $x^\alpha$ , then the formula

$$x^\alpha lx$$

will be infinitely greater than  $x^\alpha$ ; yet when the exponent  $\alpha$  is increased by a fraction  $\frac{1}{n}$ , then the formula  $x^\alpha lx$  will certainly be infinitely smaller than  $x^{\alpha+\frac{1}{n}}$ , and so an order must be constituted between  $x^\alpha$  and  $x^{\alpha+\frac{1}{n}}$ .

12. Yet the multitude of all the different orders is by no means exhausted in this way. For  $(lx)^2$  will be infinitely greater than  $lx$  and thus ought to form its own order, which will still be less than the powers  $x^{\frac{1}{n}}$  however large the number  $n$  is taken to be. All the different powers of  $lx$  yield their own kind of infinity which has to be extended even to fractional powers. For, since  $(lx)^{\frac{\alpha}{\beta}}$  is certainly infinitely greater than  $(lx)^{\frac{\alpha}{\beta}-\frac{1}{n}}$  yet infinitely less than  $(lx)^{\frac{\alpha}{\beta}+\frac{1}{n}}$ , it should form its own order. As many new cases will arise if in addition we multiply by a

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<sup>7</sup>Translator: In fact  $\frac{x^{\frac{2}{n}}}{(lx)^2}$  divided by  $\frac{nx^{\frac{1}{n}}}{(lx)^2}$  is equal to  $\frac{x^{\frac{1}{n}}}{n}$ . But  $n$  is a fixed positive integer, so  $v$  is still infinite.

<sup>8</sup>Translator: As  $x \rightarrow +0$ ,  $x^x \rightarrow 0$ .

<sup>9</sup>Translator: As we noted earlier,  $v = \frac{x^{\frac{1}{n}}}{n}$  not  $nx^{\frac{1}{n}}$ . In fact, if  $x^{\frac{1}{n}} = 1$  then  $v = 0$ , not infinity.

power of  $x$ . Namely, the formula  $x^a(lx)^{\frac{\alpha}{\beta}}$  is infinitely greater than  $x^a(lx)^{\frac{\alpha}{\beta}-\frac{1}{n}}$ , but infinitely less than  $x^a(lx)^{\frac{\alpha}{\beta}+\frac{1}{n}}$ .

13. Yet still in this manner we have not enumerated all the orders of the infinite. For since  $lx$  is an infinitely large quantity, its own logarithm  $llx$  will still be infinite though infinitely less than  $lx$ ; whence one sees that again infinitely many new orders of the infinite have to be established from the formula

$$llx \quad \text{and its powers} \quad (llx)^{\frac{\alpha}{\beta}},$$

first if this formula is combined not only with powers of  $lx$  but also with powers of  $x$ ; and the same consideration can be extended further to formulas

$$lllx, \quad llllx, \quad \text{etc.}$$

14. This immense multitude of orders also occurs among the infinitely small, since of course they can be seen as reciprocals of infinite magnitudes. For if unity is divided by any infinity  $\infty$ , namely  $\frac{1}{\infty}$ , it should be considered to constitute a particular order of the infinitely small. Thus if  $x$  is an infinite quantity, not only will the series

$$\frac{1}{x}, \quad \frac{1}{xx}, \quad \frac{1}{x^3}, \quad \frac{1}{x^4}, \quad \text{etc.}$$

provide infinitely many orders of the infinitely small, but also the series

$$\frac{1}{lx}, \quad \frac{1}{(lx)^2}, \quad \frac{1}{(lx)^3}, \quad \frac{1}{(lx)^4}, \quad \text{etc.}$$

together with all the powers of all the terms will yield new orders of the infinitely small; then also, the series

$$\frac{1}{llx}, \quad \frac{1}{(llx)^2}, \quad \frac{1}{(llx)^3}, \quad \text{etc.}$$

and also all the further ones where more logarithms are taken such as  $lllx, llllx, \text{etc.}$ , and thus this multitude is augmented infinitely.

15. What has been propounded so far for logarithms can similarly be extended to exponential quantities, from which innumerable new orders of both the infinitely large and the infinitely small can be likewise constituted that will be entirely distinct from the preceding. For if, as has been the case so far,  $x$  denotes an infinite number, it is well known that the value of the power  $a^x$  will also be infinitely large exactly when  $a$  is a number greater than unity, while if  $a < 1$  then the power  $a^x$  will exhibit an infinitely small quantity. Now, let us first consider the infinitely large, by taking  $a > 1$ , and it's clear that the power  $a^x$  is not only infinitely greater than its exponent, but it can even be demonstrated that the quantity  $a^x$  is infinitely greater than the power  $x^n$  however large the exponent  $n$  may be. The demonstration can be obtained in the following way.

16. Let us put

$$\frac{a^x}{x^n} = v,$$

and let

$$p = \frac{1}{x^n} \quad \text{and} \quad q = \frac{1}{a^x},$$

so we have  $v = \frac{p}{q}$ . Both the numerator  $p$  and the denominator  $q$  of this fraction vanish in the case  $x = \infty$ , and so

$$v = \frac{dp}{dq}.$$

Now,

$$dp = -\frac{ndx}{x^{n+1}} \quad \text{and} \quad dq = -\frac{dxla}{a^x},$$

whence

$$v = \frac{na^x}{x^{n+1}la}.$$

Indeed this formula is much more complicated than the given  $v = \frac{a^x}{x^n}$ , so it might seem that nothing could be concluded. But from the comparison of these formulas the true value of  $v$  will be concluded. For, from the first

$$v^{n+1} = \frac{a^{x(n+1)}}{x^{n(n+1)}}$$

and from the latter

$$v^n = \frac{n^n}{a^{nx}} x^{n(n+1)} (la)^n,$$

and dividing the first value by the latter gives

$$v = \frac{a^x (la)^n}{n^n},$$

which is clearly an infinite value. Thus it has been demonstrated that the formula  $a^x$  is always infinitely greater than  $x^n$ , no matter how large the exponent  $n$  is taken, providing  $a > 1$ . It is apparent then that the exponential quantity  $a^x$  is infinitely greater than all the orders of the infinite arising from the power  $x^n$ . Hence, even though  $x$  is an infinite quantity, still all the fractions

$$\frac{a^x}{x}, \quad \frac{a^x}{xx}, \quad \frac{a^x}{x^3}, \quad \text{and in general} \quad \frac{a^x}{x^n}$$

forms an order of infinity that exceeds all orders of infinity from the first class. It is clear that this also happens for the formulas  $a^{\alpha x}$ , providing  $\alpha > 0$ , and it will even happen for the formulas  $a^{\alpha x^\beta}$ , as long as positive values are taken for the letters  $\alpha$  and  $\beta$ ; these are therefore infinitely many infinities that are higher than the powers of  $x$ , however large those may be.

17. Besides this, it should also be noted that even if the formula  $a^x$  belongs to an infinitely high order of the infinite, still, however little the value of the letter  $\alpha$  is increased, the value of this formula will rise to a higher infinity. For if  $b > a$  then the formula  $a^x$  will be to the formula  $b^x$  as 1 is to  $(\frac{b}{a})^x$ , that is, as 1 to infinity, an order of infinitesimals.

18. Whenever  $a > 1$ , then all the powers of  $a^x$  can be translated into powers of the fixed number  $e$ , whose hyperbolic logarithm is = 1, because

$$a^x = e^{x \ln a},$$

and thus all infinities of this type can be represented in the form  $e^{\alpha x^\beta}$ , supposing  $\alpha > 0$  and  $\beta > 0$ . Then the formula

$$\frac{e^{\alpha x^\beta}}{x^n}$$

will belong to an infinitely lower order of the infinite, but on the other hand this case be more than made up for if we write  $e^{\alpha x^\beta}$  in place of  $\alpha x^\beta$ , which leads to this form

$$e^{e^{\alpha x^\beta}}$$

and we can repeat this process arbitrarily many times.

19. This can all transferred in an inverted manner to the infinitely small, which we shall now consider in some detail. Thus let  $x$  denote an infinitely small quantity, whose powers

$$x^\alpha$$

therefore form innumerable orders of the infinitely small; if the exponent  $\alpha$  is increased by however small an amount they become infinitely smaller. However all of these orders can be included in the first class of the infinitely small, if the exponents  $\alpha$  are understood to take all positive values.

20. Let us refer to as the second class the infinitely small which arise from logarithms. For whenever  $l\frac{1}{x}$  is infinite, it's reciprocal

$$\frac{1}{l\frac{1}{x}}$$

will be infinitely small. Now, for convenience let us put

$$l\frac{1}{x} = u,$$

so that this form becomes  $\frac{1}{u}$ , which will be infinitely small such that it is infinitely greater than all the infinitely small in the first class. The formulas

$$\frac{1}{uu}, \quad \frac{1}{u^3}, \quad \frac{1}{u^4}, \quad \text{etc. and in general } \frac{1}{u^\alpha}$$

also belong to this class. The forms

$$\frac{x^\alpha}{u^\beta}$$

are the next to be considered, which are combinations from the first and second classes. Also indeed, since  $lu$  is infinitely large but infinitely less than  $u$ , its reciprocal

$$\frac{1}{lu}$$

will be infinitely small but infinitely greater than  $\frac{1}{u}$ . Likewise, the formulas

$$\frac{1}{llu} \quad \text{and} \quad \frac{1}{lllu}$$

will be continually infinitely larger than the preceding. By combining them with the previous, innumerable new orders of the infinitely small can be constituted which one can by no means enumerate.

21. In particular it ought to be noted that even though  $u = l\frac{1}{x}$  is infinitely large, yet all the products  $x^n u$  will be infinitely small, when  $n > 0$ . And even though this follows from the preceding, it can be succinctly demonstrated thus. Put

$$x^n u = v,$$

and let

$$x^n = p \quad \text{and} \quad \frac{1}{u} = q,$$

so that

$$v = \frac{p}{q}.$$

Both the numerator and the denominator of this fraction vanish when  $x = 0$ , whence we further have

$$v = \frac{dp}{dq}.$$

Indeed

$$dp = nx^{n-1} dx,$$

and because  $u = l\frac{1}{x}$ , i.e.  $u = -lx$ , it will be

$$du = -\frac{dx}{x},$$

and hence

$$dq = \frac{dx}{xuu},$$

from which we get  $v = nx^n uu$ . From the first value we have  $vv = x^{2n} uu$ , which divided by that just found gives

$$v = \frac{x^n}{n},$$

from which it is clear that the value of  $v$  is infinitely small, which also holds for the formula

$$x^n u^\alpha,$$

and hence not only when  $\alpha$  is a positive number but also when it is negative, since the formula  $\frac{x^n}{u^m}$  is by itself infinitely small.<sup>10</sup>

22. Besides these two classes of the infinitely small, the exponential quantities offer a third class to us. For when  $x = 0$ , the formula  $e^{\frac{1}{x}}$  exhibits as it were an infinite magnitude of the highest degree, whose reciprocal

$$\frac{1}{e^{\frac{1}{x}}} = e^{-\frac{1}{x}}$$

expresses the infinitely small of the highest degree, which namely will be infinitely smaller than any of the infinitely small from the first class, and in fact this should hold for the general form

$$\frac{1}{e^{\frac{\alpha}{x^\beta}}}.$$

Now, for the sake of brevity let us put

$$e^{\frac{\alpha}{x^\beta}} = v,$$

so that we can express the infinitely small counterpart in the form  $\frac{1}{v}$ . Then since  $lv = \frac{\alpha}{x^\beta}$ , by differentiating we'll have

$$\frac{dv}{v} = -\frac{\alpha\beta dx}{x^{\beta+1}}, \quad \text{and hence} \quad dv = -\frac{\alpha\beta v dx}{x^{\beta+1}}.$$

It should be noted here in particular that even if  $x$  is a vanishing quantity, still the formulas  $\frac{1}{x^n v}$  will express the infinitely small of the highest degree.<sup>11</sup>

23. With these classes now constituted, useful aids both for differentiating and for integrating such infinitely small quantities can be found. For instance, if for the first class we put

$$ax^\alpha = y,$$

it will be

$$\frac{dy}{dx} \alpha ax^{\alpha-1}$$

and

$$\int y dx = \frac{a}{\alpha + 1} x^{\alpha+1},$$

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<sup>10</sup>Translator: Since  $x$  is infinitely small and  $u$  is infinitely large,  $\frac{x^n}{u^m}$  is certainly infinitely small. I don't see why Euler writes  $u^m$  instead of  $u^\alpha$ .

<sup>11</sup>Translator: Perhaps because by L'Hospital's rule,  $\lim_{x \rightarrow 0} \frac{1}{x^n v} = \lim_{x \rightarrow 0} \frac{n}{\alpha\beta} x^{\beta-n} \cdot \frac{1}{v}$ . Doing this  $k$  times will yield  $\frac{n(n-\beta)(n-2\beta)\dots(n-(k-1)\beta)}{(\alpha\beta)^{k+1}} x^{n-k\beta} \cdot \frac{1}{v}$ , which for  $k = \lceil \frac{n}{\beta} \rceil$  has the limit 0 as  $x \rightarrow 0$ .

and it is clear that this integral will be infinitely less than  $y$ , providing that infinity is greater than the differential  $\frac{dy}{dx}$ ; indeed this can be made infinitely large if  $\alpha < 1$  and should for this purpose be thought of as belonging to the infinitely small of other classes.

24. Now let's consider the infinitely small of the second class, and put

$$l\frac{1}{x} = u,$$

so that

$$du = -\frac{dx}{x}.$$

Let us set

$$y = ax^\alpha u^m,$$

where  $\alpha > 0$ ,  $m$  indeed either a positive or negative number since in either case this formula is infinitely small. Then it will happen that

$$\frac{dy}{dx} = \alpha ax^{\alpha-1}u^m - amx^{\alpha-1}u^{m-1} = ax^{\alpha-1}u^{m-1}(\alpha u - m),$$

and because  $u$  is infinite, by neglecting the latter term  $-m$  it will follow that

$$\frac{dy}{dx} = \alpha ax^{\alpha-1}u^m.$$

Multiplying by  $dx$  and integrating gives

$$\int \alpha ax^{\alpha-1}u^m dx = y = ax^\alpha u^m,$$

from which arises this rather memorable integration:

$$\int x^{\alpha-1}u^m dx = \frac{1}{\alpha}x^\alpha u^m,$$

or, writing  $\beta$  in place of  $\alpha - 1$ , it will be

$$\int x^\beta u^m dx = \frac{1}{\beta+1}x^{\beta+1}u^m.$$

25. Then if take the curved line whose ordinate

$$y = ax^\beta u^m$$

corresponds to the abscissa  $x$ , where we let  $\beta > 1$  and the exponent  $m$  is either positive or negative, such that the initial ordinate of the curve, where  $x = 0$ , vanishes, the area of this curve corresponding to the infinitely small abscissa  $x$  will be

$$\int y dx = \frac{a}{\beta+1}x^{\beta+1}u^m = \frac{1}{\beta+1}xy;$$

namely it will be equal to a rectangle formed between the abscissa  $x$  and the ordinate  $y$ , divided by  $\beta + 1$ , which is all the more remarkable as the formula  $x^\beta u^m dx$  cannot be integrated except for that small portion of cases in which the exponent  $m$  is a positive integer.

26. Now let's consider likewise the infinitely small of the third class, and for the sake of brevity let us put as above

$$e^{\frac{\alpha}{x^\beta}} = v,$$

so that it is

$$dv = -\frac{\alpha\beta v dx}{x^{\beta+1}},$$

and, as we have seen, the formula

$$\frac{x^m}{v}$$

will always be an infinitely small quantity whether the exponent  $m$  is positive or negative.<sup>12</sup> And if one puts

$$\frac{x^m}{v} = z,$$

it will be

$$\frac{dz}{dx} = \frac{mx^{m-1} + \alpha\beta x^{m-\beta-1}}{v} = \frac{x^{m-\beta-1}}{v}(mx^\beta + \alpha\beta)$$

where, because  $mx^\beta$  vanishes before  $\alpha\beta$ , it will be

$$\frac{dz}{dx} = \frac{\alpha\beta x^{m-\beta-1}}{v}$$

whence by integrating in turn we'll have

$$z = \alpha\beta \int \frac{x^{m-\beta-1} dx}{v} = \frac{x^m}{v},$$

and then if we write  $n$  in place of  $m - \beta - 1$ , so  $n$  can thus be any positive or negative number, it will always be

$$\int \frac{x^n dx}{v} = \frac{1}{\alpha\beta} \frac{x^{n+\beta+1}}{v};$$

this integration is correct as long as  $x$  is infinitely small, though this differential expression altogether refuses integration.

27. Now if the curved line is taken whose ordinate corresponding to the abscissa  $x$  is

$$y = \frac{ax^n}{v},$$

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<sup>12</sup>Translator: See §22.

with

$$v = e^{\frac{\alpha}{x^\beta}},$$

where  $\alpha$  and  $\beta$  are positive numbers, and indeed the exponent  $n$  can be either positive or negative, the ordinate of this curve will vanish at the beginning, where  $x = 0$ . The area of this curve corresponding to an infinitely small abscissa  $x$  will be

$$\int ydx = \frac{a}{\alpha\beta} \frac{x^{n+\beta+1}}{v} = \frac{1}{\alpha\beta} x^{\beta+1} y,$$

and thus if

$$y = \frac{ax^n}{e^{\frac{1}{x}}},$$

where  $\alpha = 1$  and  $\beta = 1$ , it will be

$$\int ydx = xxy.$$

That is: the area of the curve will be equal with the rectangle formed from the square of the abscissa, and the ordinate.

28. Now, if we then look for a curve whose area in general will be

$$\int ydx = xxy,$$

this leads to the differential equation

$$ydx = 2xydx + xxdy,$$

whence we get

$$\frac{dy}{y} = \frac{dx(1-2x)}{xx}$$

and by then integrating

$$ly = -\frac{1}{x} - 2lx,$$

and writing this without logarithms,

$$y = \frac{a}{xxe^{\frac{1}{x}}},$$

which is contained in the given form if one takes  $n = -2$ , and indeed the above integration will work whenever  $x$  is infinitely small.

29. The last integration will even work if the infinitely small quantity also involves any member of the second class. For let

$$l\frac{1}{x} = u,$$

and let us put

$$z = \frac{ax^m u^n}{v}$$

(where the exponents  $m$  and  $n$  can be taken to be either positive or negative, since these quantities will always be infinitely small when  $v = e^{\frac{\alpha}{x^\beta}}$ ). So that we can more easily unravel the value  $\frac{dz}{dx}$ , let us take logarithms, and it will be

$$lz = la + mlx + nlu - lv$$

and hence

$$\frac{dz}{z} = \frac{mdx}{x} + \frac{ndu}{u} - \frac{dv}{v}.$$

Indeed because

$$du = -\frac{dx}{x} \quad \text{and} \quad dv = -\frac{\alpha\beta dx}{x^{\beta+1}}v,$$

substituting these two values into the above expression, it will take on the following form:

$$\frac{dz}{zdx} = \frac{m}{x} - \frac{n}{ux} + \frac{\alpha\beta}{x^{\beta+1}},$$

where since  $\beta + 1 > 1$ , both the prior terms vanish before the third, and thus it will neatly be

$$\frac{dz}{zdx} = \frac{\alpha\beta}{x^{\beta+1}}$$

and hence

$$dz = \frac{a\alpha\beta x^{n-\beta-1} u^m}{v} dx.$$

30. If we now write  $k$  in place of  $n - \beta - 1$ , so that  $k$  and  $m$  denote any positive or negative numbers, since  $n = k + \beta + 1$  it will always be

$$\frac{\int x^k u^m dx}{v} = \frac{1}{\alpha\beta} \cdot \frac{x^{k+\beta+1} u^m}{v}$$

and hence if  $\frac{x^k u^m}{v} = y$  and  $y$  is seen as the ordinate of a curve, its area will be

$$\int y dx = \frac{1}{\alpha\beta} \cdot y x^{\beta+1},$$

as long as  $x$  is infinitely small, which is all the more noteworthy as so far no way has been found for doing these integrations.