

On de Lisle's Geographic Projection and its Use in the General Map of the Russian Empire *

1. Upon lengthy consideration of which projection should be used in the general map of the Russian Empire, the Stereographic projection, ¹ normally used to represent both terrestrial hemispheres, the upper and lower, came to mind, since this method not only displays all circles of Parallel intersecting Meridians at right angles, but also exhibits all small proportions on the map in similitude to the features they represent on the surface of the sphere. And indeed, this type of projection is used by the most excellent geographer and professor at Wittenberg, Hasius ², in producing his map of the Empire.
2. However, in this projection we observe two inconvenient characteristics, which are greatly opposed to the purpose. First, on the central meridian, the gradations of latitude are excessively unequal; near the equator they are only half as large as near the poles, whence arises an enormous difficulty, since for the regions situated near the edges of the map, the scale is much larger than in the middle. For example, the Province of Kamchatka appears four times larger than a province of the same size in the middle of the map. But in the construction of this kind of map it is above all necessary that regions of the same magnitude be drawn as equally sized, wheresoever on the map they be located.
3. The second inconvenience is, that in this projection, as we move from the middle towards the edges, the Meridians are more and more curved, and indeed at the extremes are represented as semicircles. Thus, for example, in the Province of Kamchatka, all meridians are quite noticeably curved, and if someone were to extract this portion out of the overall map, in order to obtain a particular map of this Province, the result would be most incongruous and contrary to the rules which are usually observed in

*Translation of Euler's "de Projectione Geographica de Lisliana in Mappa Generali Imperii Russici Usitata". Translation date April 2007. (*Opera Omnia*, ser. 1, vol. 28, pp. 288 ff.) Translator's name and email address: George W. Heine, <gheine@mathnmaps.com>

¹Euler apparently refers to the stereographic projection in equatorial aspect, i.e., the origin is a point on the equator. This was a commonly used projection for world maps in the XVIIIth century. An example can be viewed in the Euler Archives at <<http://www.math.dartmouth.edu/~euler/atlas/map02.jpg>>

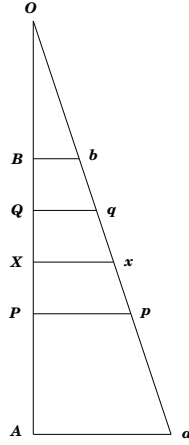
²Johann Matthias Hasius, 1684–1742, Professor of Mathematics at Wittenberg from 1720.

constructing geographic charts. It is indeed very desirable, that from a general map one be able to cut out a map of any particular region and use it in that form with out any further reduction.

4. Having rejected, therefore, this method of projection, it was considered whether to substitute that method which is commonly used to represent the polar Hemispheres. ³ All Meridians are represented by straight lines concurrent at the poles, and thus one of the above problems is certainly avoided. Nevertheless, in every Meridian the gradations of latitude are excessively unequal, being half as small near the poles as near the equator. This method of projection must also be rejected, since it was originally stipulated that scale be the same everywhere on the map, and that it be possible to deduce the true magnitude of every Province by looking at their appearance on the geographic chart.
5. One must therefore conceive another method of projection, in which, first, all Meridians are displayed as straight lines, and all degrees of latitude remain the same size, and next, all Parallels meet Meridians at right angles. Since, evidently, this cannot be achieved when every degree on the Parallels holds the same ratio to the degrees on the Meridians, as they do on the surface of the sphere, it seemed better not to fill this latter condition exactly, in order to retain the previous two advantages. Thus the question becomes: how can the Meridians and Parallels be set up, so that the deviation from the true ratios which the degrees of latitude and longitude hold among themselves on the Sphere, can be as small as possible throughout the extent of the map? The deviation, though necessary, must be so small that in choosing a projection with the above advantages, the error is hardly noticed.
6. DELISLE ⁴, the most celebrated Astronomer and Geographer of the time, to whom the care of such a map was first entrusted, in trying to fulfill these conditions, made the relationship between latitude and longitude exact at two noteworthy Parallels. He was of the opinion that if the named circles of Parallel were at the same distance from the middle Parallel of the map as from its outermost edges, the deviation could nowhere be significant. Now the question is asked, which two circles of parallel ought to be chosen, so that the maximum error over the entire map be minimized.
7. Let AB be a part of any Meridian through which the Russian Empire

³Euler refers to the north polar aspect of the stereographic projection, where the origin is at the south pole. An example is in the Euler Archives, at <http://www.math.dartmouth.edu/~euler/atlas/map03.jpg>.

⁴Joseph Nicolas Delisle, 1688–1768, Euler's colleague and leading astronomer and geographer at the St. Petersburg Academy from 1726–1747. For an account of the difficult relationship between Delisle and Euler, see Calinger, Ronald. "Leonhard Euler: The First St. Petersburg Years (1727–1741)." *Historia Mathematica* 23 (1996): 121–166.



passes, where A is the southernmost, and B the northernmost endpoint. Put the latitude at $A = a$, and the latitude at $B = b$; these will be near to $a = 40^\circ$ and $b = 70^\circ$. Furthermore, designate by δ the length of one degree on all meridians. Finally, let P and Q be the points, at which the degrees of latitude and longitude hold their true relationship, and let p be the latitude of P and q the latitude of Q . Since the degrees on any Parallel on the sphere have to the degrees on the meridians a ratio equal to the cosine of the latitude to 1, the length of a degree of longitude at P must be $Pp = \delta \cos p$, while the length of a degree of longitude at Q must be $Qq = \delta \cos q$, and the small lines Pp , Qq , although they really are circular arcs, can be looked at as lines perpendicular to the meridian AB .

8. Now trace a straight line pqO , which passes through the points p and q and cuts the principal Meridian AB at O . This straight line represents the next Meridian, at a distance of one degree of longitude from the principal Meridian. Continuing in such a manner we can easily trace the remaining Meridians.

To determine the intersection O , we have the proportion

$$(Pp - Qq) : PQ = Pp : PO,$$

that is,

$$\delta(\cos p - \cos q) : q - p = \delta \cos p : PO,$$

or

$$PO = \frac{(q - p) \cos p}{\cos p - \cos q}.$$

If we take $p = 50^\circ$, $q = 60^\circ$, then we find the distance $PO = 45^\circ 1'$. Now because the point P is 50 degrees from the equator, the distance of the point O amounts to $95^\circ 1'$; O lies therefore on the far side of the pole at a distance of $5^\circ 1'$ from the latter.

9. Now because that point O , in which all Meridians of the map intersect, is different from the true pole on the earth, from which all the Meridians on the Sphere proceed, an entirely wrong picture of regions near the pole emerges. However, no places beyond the 70th degree of latitude need be represented on the general map of the Russian empire. If, therefore, the error is not very large at this latitude, the deviation will be tolerable. Having found the point O , one describes about it a circle of radius OP , whose circumference is divided into parts $= \delta \cos p$, that is, parts of the same length of an actual degree on this Parallel, and then tracing from O toward the unique marked points, one forms the Meridians on the map. Moreover, one gets the circles of parallel on the map, in the circles which one describes around O , whose radii are each different by one degree. Then

evidently for the two latitudes p and q the ratio of degrees of latitude to degrees of longitude is correct. Proceeding thus, one can easily construct the network of degrees over the entire map, and then to set down single places or Provinces gives no further difficulty.

10. Now we wish above all to see how far this representation deviates from reality at the extreme points A and B of the map. Let Aa be one degree on the Parallel that goes through A ; and Bb be one degree on the Parallel that passes through B . These lines must in fact have lengths $\delta \cos a$ and $\delta \cos b$. In order to find the actual size which they have on the map, we want first the angle POp , to which a degree of longitude corresponds, which is

$$= \frac{Pp}{PO} = \frac{\delta(\cos p - \cos q)}{q - p} = \frac{(\cos p - \cos q)}{q - p}, \quad \text{when } \delta = 1^\circ.$$

For the sake of brevity, we put this quantity $= \omega$, thus

$$\omega = \delta(\cos p - \cos q)/(q - p).$$

As above, we take $p = 50^\circ$, $q = 60^\circ$, so that angle POp becomes $\omega = 49'6''$. In calculation of the same, the difference $q - p$ must be expressed not in degrees, but in parts of the radius; it is noted that the quantity of one degree is 0.01745329. The angle ω which represents a single degree of longitude at O , is therefore strictly smaller than one degree.

11. In order to treat the question generally, we designate that angle which corresponds to one degree $= \omega$, so that

$$\omega = \frac{\delta(\cos p - \cos q)}{q - p},$$

where it is noted, that if p and q are expressed in degrees, the distance $q - p$ must be multiplied by 0.017452329, which for the sake of brevity we write as α , so that

$$\omega = \frac{\delta(\cos p - \cos q)}{\alpha(q - p)},$$

and here δ can be replaced by 1° , if we want the angle ω in degrees. Moreover, we put the distance of the point O from the pole, on the far side of which it lies, $= z$ degrees. Since the distance of the point P from the pole is $90^\circ - p$, its distance from O is $90^\circ - p + z$, whose value in parts of a radius will be $\alpha(90^\circ - p + z)$. Previously it was found that this same distance is

$$= PO = \frac{(q - p) \cos p}{\cos p - \cos q},$$

which, expressed in degrees, must be the angle $90^\circ - p + z$, so that

$$z = \frac{(q - p) \cos p}{\cos p - \cos q} - 90^\circ + p.$$

12. Furthermore, if the distance of point A from the pole $= 90^\circ - a$, the distance $AO = 90^\circ - a + z$, and in parts of a radius, $= \alpha(90^\circ - a + z)$, which quantity, on multiplication by ω , gives the length of the degree Aa ; whose magnitude therefore is

$$\frac{\delta(90^\circ - a + z)(\cos p - \cos q)}{q - p},$$

while the length to which it corresponds must be $= \delta \cos a$; the difference between the two values indicates the error of the map at the extreme point A . In the same manner, at the other extreme B , one degree on the Parallel will be

$$\frac{\delta(90^\circ - b + z)(\cos p - \cos q)}{q - p},$$

while the length to which it corresponds is $= \delta \cos b$. The difference between these two values expresses the error of the map at the extreme point B .

13. Next it is determined how to to appropriately situate the two internal points P and Q , so that the errors at the extreme ends A and B are equal to one another. This gives the equation

$$\frac{90^\circ - a + z)(\cos p - \cos q)}{q - p} - \cos a = \frac{90^\circ - b + z)(\cos p - \cos q)}{q - p} - \cos b,$$

which is reduced to the form

$$(a - b)(\cos p - \cos q) + (q - p)(\cos a - \cos b) = 0.$$

14. To facilitate our investigation, we introduced into the calculation, in place of p and q , the distance z , measured in degrees, which the point O lies beyond the pole; moreover, the angle ω , that represents a single degree of longitude at O ; in other words, the angle that separates two neighboring Meridians, which on the earth on one degree apart, from each other on the map. We suppose that this angle ω is expressed in degrees or in the usual parts of a degree. Therefore we take δ as unity. Thus, one degree of a Parallel through the endpoint A is $= \alpha(90^\circ - a + z)\omega$, and through the endpoint $B = \alpha(90^\circ - b + z)\omega$. Since the lengths on the sphere to which these distances correspond are $\cos a$ and $\cos b$, the condition that the errors in A and B are equal to one another yields the equation:

$$\alpha(90^\circ - a + z)\omega = \alpha(90^\circ - b + z)\omega,$$

which is reduced to

$$\alpha(b - a)\omega = \cos a - \cos b;$$

whence we conclude

$$\omega = \frac{\cos a - \cos b}{\alpha(b - a)},$$

where the value is expressed in parts of a degree.

15. Now that we have made the errors in the projection equal at the extreme points A and B , we want to arrange that these errors are equal to the largest error which can appear within the segment AB . Now the largest error appears at the midpoint X , whose latitude $= (a + b)/2$, where the error will be

$$\alpha \left(90^\circ - \frac{a+b}{2} + z \right) \omega - \cos \frac{a+b}{2},$$

but this error has the opposing sense to the corresponding errors at A and B , therefore we must reverse its sign, so it becomes

$$\cos \frac{a+b}{2} - \alpha \left(90^\circ - \frac{a+b}{2} + z \right) \omega;$$

and setting this equal to the errors at A and B , the following two equations arise:

$$\alpha (90^\circ - a + z) \omega - \cos a = \cos \frac{a+b}{2} - \alpha \left(90^\circ - \frac{a+b}{2} + z \right) \omega$$

and

$$\alpha (90^\circ - b + z) \omega - \cos b = \cos \frac{a+b}{2} - \alpha \left(90^\circ - \frac{a+b}{2} + z \right) \omega.$$

16. The equality of the errors in A and B has already given us the equation

$$\omega = \frac{\cos a - \cos b}{\alpha(b - a)},$$

and substituting this into the first of the two previous equations yields this equation:

$$\frac{(180^\circ - \frac{3}{2}a - \frac{1}{2}b + 2z)(\cos a - \cos b)}{b - a} = \cos a + \cos \frac{a+b}{2},$$

which is reduced to this form:

$$180^\circ - \frac{3}{2}a - \frac{1}{2}b + 2z = \frac{b - a}{\cos a - \cos b} \left(\cos a + \cos \frac{a+b}{2} \right),$$

and from this the distance z is easily computed.

17. Now we apply this result to the case of the map of the Russian Empire, where $a = 40^\circ$, $b = 70^\circ$, thus $\frac{a+b}{2} = 55^\circ$. Therefore, first of all, for the angle ω arises this equation:

$$\omega = \frac{\cos 40^\circ - \cos 70^\circ}{30\alpha} = \frac{0.4240243}{0.5235987} \quad ,$$

whence it is found that $\omega = 48'44''$. Substituting this value into the next-to-the-last equation of §15 yields

$$\alpha(85^\circ + 2z)\omega = \cos 40^\circ + \cos 55^\circ = 1.33962;$$

since $\alpha\omega = \frac{1.33962}{30} = 0.01410$, we have

$$85^\circ + 2z = \frac{1.33962}{0.01410} = 95^\circ, \quad \text{that is,} \quad z = 5^\circ.$$

18. We had previously assumed that the largest error falls about in the middle of the the meridian AB ; however, this might not be the exact location. We want to determine that point X at which the error is greatest. Denote, therefore, by x the latitude of that place; the error there is

$$\alpha(90^\circ - x + z)\omega - \cos x,$$

and we equate the differential of this expression to zero. However, we must take guard that we do not, as usual, write $d \cos x = -\sin x dx$, for here x is expressed in degrees, so that the differential of the corresponding arc, which is αx , must be multiplied by $\sin x$. Accordingly,

$$d. \cos x = -\alpha dx \sin x,$$

and the condition that the differential of the above-mentioned expression vanishes, then yields $-\alpha\omega dx + \alpha dx \sin x = 0$, or

$$\sin x = \omega,$$

where ω was found above to be the fraction

$$= \frac{\cos a - \cos b}{\alpha(b - a)},$$

which in our case is

$$\frac{4.4240243}{0.5235987} = \sin x,$$

which makes $x = 54^\circ 4'$. Therefore the point X does not lie exactly at the midpoint of AB .

19. After determining the value of x the error at this place is found to be

$$\alpha(90^\circ - x + z)\omega - \cos x,$$

whose negative must be equal to the errors at A and B . This gives the equation

$$\alpha(180^\circ - a - x + 2z)\omega = \cos a + \cos x;$$

and from this the value of z is to be calculated. Since $x = 54\frac{1}{15}^\circ$, it follows that

$$85\frac{14}{15}^\circ + 2z = \frac{\cos a + \cos x}{\alpha\omega} = 95^\circ 56' \quad ,$$

whence $2z = 10^\circ$, $z = 5^\circ$ so that $\omega = 0.8098270$ degrees, that is $\omega = 48'44''$.

20. We now see how large is that maximum error at the places A, B , and X . To this purpose we compute the error in A ; this is

$$\alpha\omega(90^\circ - a + z) - \cos a = 55^\circ\alpha\omega - 0.7660444 = 0.00946$$

since $\alpha\omega = 0.01410$; evidently, while one degree on the Parallel through A should = 0.76604, in this projection it is somewhat larger, namely 0.77550. This number represents the error expressed in parts of a degree of longitude, and since such a degree spans 15 miles, the error equal 0.14190 miles, which is approximately the seventh part of a mile, or one Ruthenian Werst. At the point B , that is, at 70° latitude, where a degree of the Parallel has a length of 0.34202, the error amounts to only the thirty-eighth part of the whole; in that region, such an error is allowable.

21. For the construction of the map of the Russian Empire, it is most appropriate to set up the point O , which lies on the Meridian BA at a distance of 5 degrees on the far side of the pole. Then on AB single degrees of latitude are easily marked off at equal distances, and circles centered at O are described through these; the circles represent the Parallels of latitude on the map, while the Meridians are straight lines passing through O , which form, at O , an angle of $48'45''$ with each other. Since $OA = 55^\circ$, the length of one degree on the Parallel drawn through A is $55\alpha\omega = 0.77550$; that is, such a degree is to a degree on the Meridian as 0.77550:1. This division can be carried out readily enough.
22. Since in this projection all Meridians are represented by straight lines, other great circles which one is able to draw on the map, do not deviate considerably from straight lines. The Equator is represented by a circle with center O and radius = 95° ; the single degrees of the equator therefore have length of $95\alpha\omega = 1.33950$, instead of being equal to the degrees on the Meridian. However since the Equator lies outside our map, this error settles nothing. Let us see, therefore, by how much a great circle drawn on the map differs from a straight line.

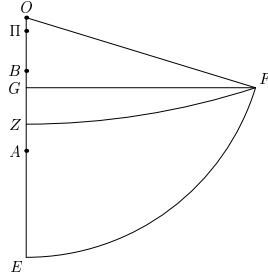
23. To facilitate this investigation, let our middle Meridian AB be produced on the one hand up to the point O , and on the other hand down to the Equator, meeting the latter at E , so that

$$AE = 40^\circ, \quad AB = 30^\circ, \text{ and } BO = 25^\circ.$$

Furthermore, the pole lies at Π , so that $\Pi O = 5^\circ$. The circle described around O through E represents the Equator, which we do not require on our map. On this circle is taken an arc EF , which spans a distance of 90° , and whose central angle therefore

$$\text{angle } EOF = 90^\circ \cdot \omega = 72^\circ 53',$$

while the span $OF = 95^\circ$. This point F is the common intersection of all great circles, which can be drawn perpendicular to our meridian AB .



24. We now wish to construct a great circle, perpendicular to AB at an arbitrary point Z between A and B , and passing through F . The true shape of the curve corresponding to this circle will depend on a transcendental equation; nevertheless, this curve will not noticeably differ from a circular arc which passes through the points Z and F and is perpendicular to the straight line AB . To determine the curvature of this circular arc ZF , a perpendicular FG is drawn at OE , so that

$$OG = 95^\circ \cos 72^\circ 53' = 27.96024$$

$$FG = 95^\circ \sin 72^\circ 53' = 90.79221.$$

From this one sees that the straight line FG itself represents a quadrant of a great circle normal to AB , and that FG contains very nearly ninety degrees, so that along it the map will hardly have any distortion. If on the other hand one draws a perpendicular great circle through the endpoint A , then the arc AF will be somewhat larger than the straight line FG . However the error can easily be tolerated. The radius of such a circle will amount to 165.9477° ; this is so large, that its curvature on the map will hardly be perceptible. And for that reason, any great circle drawn in this projection will hardly differ from a straight line.

25. What is said here regarding the great circles normal to the Meridian AB , is true for the same reason for all great circles, which are normal to other

Meridians. In this projection is obtained the extraordinary advantage, that straight lines, which go from any point to any other point, correspond rather exactly to great circles and therefore the distances between any places on the map can be measured by using a compass without considerable error. Because of these important characteristics the projection discussed was preferred before all others for a general map of the Russian Empire, even though, under rigorous examination, it differs not a little from the truth.