THOUGHTS ON CYCLOMETRY

by Leonhard Euler

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SUMMARY

Various matters are exposed in this essay by its Most Illustrious Author concerning the construction of quadrable Lunules; it is known, however, that that question reduces to the problem of finding two angles which have to one another the duplicate ratio of their sines, that is, if one of them is called m, the other n, it must be the case that $m : n = \sin m^2 : \sin n^2$. It is generally assumed, indeed, in the solution of the problem of quadrable lunules, that m and n are commensurable; we do not find demonstrated, however, that in no other case can a Geometric solution be expected. From the very nature of the question, of course, it is clear that the ratio of sines must be algebraic, since otherwise it would not be possible to determine the ratio of the angles algebraically. Nevertheless, it does not appear to follow from this that that ratio could be determined only by integral numbers. Though indeed a Method has hitherto been lacking by means of which that problem could be settled also for incommensurable angles, the Most Illustrious Author has here set forth cases especially deserving consideration, in which the problem can be solved by straightedge and compasses, which cases number five. Thus if we let $m = \frac{1}{2}\mu\omega$ and $n = \frac{1}{2}\nu\omega$, the following cases admit a Geometrical construction by straightedge and compasses:

I. if $m = 45^\circ$ and $n = 90^\circ$, which is the well-known case of the Lunules of Hippocrates,
II. if $m = \frac{1}{2}\omega$ and $n = \frac{3}{2}\omega$,
III. if $m = \omega$ and $n = \frac{3}{2}\omega$,
IV. if $m = \frac{1}{2}\omega$ and $n = \frac{5}{2}\omega$, and finally
V. if $m = \frac{3}{2}\omega$ and $n = \frac{5}{2}\omega$.

All these cases indeed can be found quite easily, if we set $m = \mu z$ and $n = \nu z$, whence we will have

$$\mu : \nu = \sin \mu z^2 : \sin \nu z^2 \quad \text{or} \quad \sin \mu z : \sin \nu z = \sqrt{\mu} : \sqrt{\nu},$$

and since

$$\sin \mu z = \sin z(2^{\mu-1}\cos z^{\mu-1} - (\mu-2)2^{\mu-3}\cos z^{\mu-3} + \frac{(\mu-3)(\mu-4)}{1\cdot 2}2^{\mu-5}\cos z^{\mu-5} - \text{ etc.})$$
$$\sin \nu z = \sin z(2^{\nu-1}\cos z^{\nu-1} - (\nu-2)2^{\nu-3}\cos z^{\nu-3} + \frac{(\nu-3)(\nu-4)}{1\cdot 2}2^{\nu-5}\cos z^{\nu-5} - \text{ etc.}),$$

we will have

$$\sqrt{\nu}(2^{\mu-1}\cos z^{\mu-1} - (\mu-2)2^{\mu-3}\cos z^{\mu-3} + \frac{(\mu-3)(\mu-4)}{1\cdot 2}2^{\mu-5}\cos z^{\mu-5} - \text{ etc.}) =$$
$$\sqrt{\mu}(2^{\nu-1}\cos z^{\nu-1} - (\nu-2)2^{\nu-3}\cos z^{\nu-3} + \frac{(\nu-3)(\nu-4)}{1\cdot 2}2^{\nu-5}\cos z^{\nu-5} - \text{ etc.}).$$

Let us now take μ to be the smaller number, and substitute for μ the integers 1, 2, 3 etc., from which it will be clear that for $\mu = 1$, ν could have three values, in order that the equation be a quadratic, or easily depressible to a quadratic; indeed ν will either $= 2$, in which case $\cos z = \frac{1}{\sqrt{2}}$, or $\nu = 3$, whence arises the equation

$$\sqrt{3} = 4\cos z^2 - 1 \quad \text{or} \quad \cos z^2 = \frac{1+\sqrt{3}}{4},$$

or $\nu = 5$, whence

$$\sqrt{5} = 16\cos z^4 - 3\cdot 4\cos z^2 + 1,$$

a biquadratic equation, but which is easily depressed to a quadratic. If μ is taken $= 2$, ν will have only a single value, namely $= 3$, which produces the quadratic equation

$$2\sqrt{3}\cdot\cos z = 4\sqrt{2}\cdot\cos z^2 - \sqrt{2}.$$

Finally, to $\mu = 3$ corresponds $\nu = 5$, in order that the equation be a biquadratic, but depressible to a quadratic:

$$4\sqrt{5}\cdot\cos z^2 - \sqrt{5} = 16\sqrt{3}\cos z^4 - 3\cdot 4\sqrt{3}\cdot\cos z^2 + \sqrt{3}.$$

If some other ratio between μ and ν is taken, the construction of the lunules rises to higher Geometric loci, which nevertheless should always be considered to be Geometric, so long as μ and ν are mutually commensurable.

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1. The construction of quadrable lunules depends on finding two angles which have to one another the duplicate ratio of their sines. Clearly, therefore, it is necessary to look for two angles m and n, such that

$$m : n = \sin m^2 : \sin n^2 \quad\text{or}\quad \frac{m}{\sin m^2} = \frac{n}{\sin n^2},$$

which equation I have here undertaken to examine more accurately.

2. But given an arbitrary angle m, there is no difficulty in determining the value of the corresponding expression $\frac{m}{\sin m^2}$, indeed in two ways: either in degrees, if the angle m is given that way, or in parts of the radius $= 1$, if the arc which measures the angle is expressed in parts of the radius. Furthermore, it is easy to reduce one expression to the other.

3. Now as to the expression $\frac{m}{\sin m^2}$, I note first that, if the angle m vanishes, that expression becomes infinite. Then indeed it decreases up to a certain point, beyond which it increases once again until, if the angle m increases to two right angles, then since $\sin m = 0$, the expression again becomes infinite. To make this clearer, let r stand for a right angle, and since

$$\sin\tfrac{1}{3}r = \tfrac{1}{2},\quad \sin\tfrac{1}{2}r = \tfrac{1}{\sqrt{2}},\quad \sin\tfrac{2}{3}r = \tfrac{\sqrt{3}}{2},\quad\text{and}\quad \sin r = 1,$$

if we have

$$m \quad = 0r,\ \tfrac{1}{3}r,\ \tfrac{1}{2}r,\ \tfrac{2}{3}r,\ r,\ \tfrac{4}{3}r,\ \tfrac{3}{2}r,\ \tfrac{5}{3}r,\ \text{2r},$$

then

$$\frac{m}{\sin m^2} = \infty r,\ \tfrac{4}{3}r,\ r,\ \tfrac{8}{9}r,\ r,\ \tfrac{16}{9}r,\ 3r,\ \tfrac{20}{3}r,\ \infty r.$$

4. Consequently, before the angle m reaches ninety degrees, the formula $\frac{m}{\sin m^2}$ has a minimum. One sees, furthermore, that this will occur when $m = \frac{1}{2}\operatorname{tang} m$, in other words, when the arc which measures the angle becomes equal to half the tangent. This angle can only be determined approximately; by carrying out the calculation, it is found to be

$$= 66^\circ\, 46'\, 54\tfrac{1}{4}'',$$

whence the minimum value is

$$\frac{m}{\sin m^2} = 79{,}07102\,\text{degrees}.$$

5. The angle having this distinguishing property I will denote by the letter a, so that $a = 66^\circ\, 46'\, 54\frac{1}{4}''$ in degrees, while in parts of the radius $a = 1{,}165561$. Here it may be observed that this angle has a commensurable ratio neither with the circumference nor with the radius, but in respect to both must be held to be transcendental.

6. Nevertheless, for the present, it will be helpful to bear in mind the aforementioned properties of this angle; in addition, one finds:

$$\sin a = 0{,}9190096, \quad \cos a = 0{,}3942360, \quad \text{tang}\, a = 2{,}331122,$$

also

$$\sin 2a = 0{,}7246132 \quad \text{and} \quad \text{cosec}\, 2a = 1{,}380050;$$

I look at the double angle since, letting $a = \frac{1}{2}\,\text{tang}\, a$, our formula becomes

$$\frac{a}{\sin a^2} = \frac{1}{\sin 2a} = \text{cosec}\, 2a,$$

so that 1,380050 is the minimum value which it is possible for the formula $\frac{m}{\sin m^2}$ to attain. The same expressed in degrees gives as before 79°,07102.

7. Thus, in order for two angles m and n to take the duplicate ratio of their sines, in other words,

$$\frac{m}{\sin m^2} = \frac{n}{\sin n^2},$$

it must be that one of them m must be less than a, the other n indeed greater than a, and that however much the angle m is decreased below a, so much more the other n is increased, though not in the same ratio, until, taking $m = 0$, we have $n = 180°$, nor indeed does the angle a lie halfway between these bounds.

8. If therefore the angle m is very small, let it be $m = \mu$ in parts of the radius, and letting π denote the semicircumference, or the measure of two right angles, the angle n will be only slightly smaller than π. Therefore set $n = \pi - \nu$, and we will have

$$\mu \sin \nu^2 = (\pi - \nu) \sin \mu^2 \quad \text{or} \quad \mu(1 - \cos 2\nu) = (\pi - \nu)(1 - \cos 2\mu).$$

Now

$$1 - \cos \varphi = \tfrac{1}{2}\varphi^2 - \tfrac{1}{24}\varphi^4 + \tfrac{1}{720}\varphi^6 - \text{etc.}$$

Hence

$$\mu(2\nu\nu - \tfrac{2}{3}\nu^4 + \tfrac{4}{45}\nu^6 - \text{etc.}) = (\pi - \nu)(2\mu\mu - \tfrac{2}{3}\mu^4 + \tfrac{4}{45}\mu^6 - \text{etc.}),$$

whence we get approximately

$$\nu = \sqrt{\pi\mu} - \tfrac{1}{2}\mu + \frac{3 + 4\pi\pi}{24\sqrt{\pi\mu}}\mu\mu.$$

9. If on the other hand the angle m is made slightly less than the angle a, then n will be slightly greater. Therefore set $m = a - \mu$ and $n = a + \nu$; then the formula $\frac{m}{\sin m^2}$ becomes

$$\frac{2(a - \mu)}{1 - \cos 2(a - \mu)} = \frac{2(a - \mu)}{1 - \cos 2a \cos 2\mu - \sin 2a \sin 2\mu},$$

which, the angle μ being very small, can be expanded into the series:

$$A - B\mu + C\mu^2 - D\mu^3 + E\mu^4 - \text{etc.},$$

where the letters A, B, C, D etc. are defined in the following way

$$A \sin a^2 - a = 0 \quad \text{so that} \quad A = \frac{a}{\sin a^2} = \frac{1}{\sin 2a}$$

$$B \sin a^2 + A \sin 2a - 1 = 0 \quad \text{so that} \quad B = 0 \quad \text{since} \quad A = \frac{1}{\sin 2a}$$

$$C \sin a^2 + B \sin 2a + A \cos 2a = 0$$

$$D \sin a^2 + C \sin 2a + B \cos 2a - \tfrac{2}{3}A \sin 2a = 0$$

$$E \sin a^2 + D \sin 2a + C \cos 2a - \tfrac{2}{3}B \sin 2a - \tfrac{1}{3}A \cos 2a = 0$$

$$F \sin a^2 + E \sin 2a + D \cos 2a - \tfrac{2}{3}C \sin 2a - \tfrac{1}{3}B \cos 2a + \tfrac{2}{15}A \sin 2a = 0$$

etc.,

where the numerical coefficients are multiplied successively by the fractions $\frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}$ etc.

10. These letters having been determined, we will have

$$\frac{m}{\sin m^2} = A - B\mu + C\mu^2 - D\mu^3 + E\mu^4 - F\mu^5 + \text{ etc.}$$

and

$$\frac{n}{\sin n^2} = A + B\nu + C\nu^2 + D\nu^3 + E\nu^4 + F\nu^5 + \text{ etc.},$$

so that, since the series must be equal to one another, from the given angle μ the other ν is determined in such a way that, since $B = 0$, and letting

$$\nu = \mu + P\mu^2 + Q\mu^3 + R\mu^4 + S\mu^5 + \text{ etc.}$$

we will have

$$P = -\frac{D}{C},\ Q = \frac{DD}{CC},\ R = -\frac{2D^3}{C^3} + \frac{2DE}{CC} - \frac{F}{C} \text{ and } S = \frac{4D^4}{C^4} - \frac{6DDE}{C^3} + \frac{3DF}{CC}.$$

Carrying out the calculation, one finds(1)

$$\begin{array}{lll} A = 1{,}380050, & C = \quad 1{,}126090, & P = 0{,}265937, \\ B = 0, & D = -0{,}299469, & Q = 0{,}070722. \end{array}$$

11. But if the angle m is taken arbitrarily, the other angle n can be determined, very closely at least, using a known method of approximation. For example, if we take $m = 60°$, we find $n = 73°41'32''\frac{22}{100}$, while taking $m = 30°$ produces $n = 108°14'30''\frac{18}{100}$. Further, if $n = 135°$, then $m = 12°20'54''\frac{82}{100}$. On the other hand, if the angle m is taken so that $\operatorname{tang} m = \sqrt{2}$, in other words $m = 54°44'8\frac{1}{5}''$, we get $n = 79°14'23''\frac{36}{100}$. In none of these cases, however, is it possible to construct the second angle geometrically.

12. This kind of investigation is of no help, however, for the construction of quadrable lunules, since for that it would have to be possible to determine each of the angles m and n geometrically. This in turn would follow if one could find their sines or tangents geometrically. The great question which arises here, therefore, is in what way two geometrically determinable angles m and n could be found, such that

$$\frac{m}{\sin m^2} = \frac{n}{\sin n^2} \quad \text{or} \quad m(1 - \cos 2n) = n(1 - \cos 2m).$$

13. In solving this problem, it is commonly assumed that the two angles m and n must be mutually commensurable; up to now, however, it has not been demonstrated that a solution is not possible in any other case. To be sure, the sines of those angles must have an algebraic ratio, since otherwise the angles could not be determined geometrically, so that certainly the ratio of those angles themselves would have to be determined algebraically. It does not follow, however, that that ratio must be expressible by integers.

14. There are infinitely many ways, of course, in which angles can be assigned geometrically, yet not commensurable with one another, for example, angles whose sines are $\frac{1}{2}$ and $\frac{1}{3}$. Up to now, however, has it been shown satisfactorily that such angles could not have some particular irrational ratio? It would be remarkable to be able to decide whether angles whose ratio is, for example, $1 : \sqrt{2}$ could be exhibited geometrically.

15. Since the ratio between the angles m and n is to be $1 : \sqrt{2}$, solving the equation

$$\frac{m}{\sin m^2} = \frac{n}{\sin n^2}$$

(1) EULER's arithmetic is faulty here. The value of D should be -0.176788; correspondingly, $P = 0.156993$ and $Q = P^2 = 0.024647$. *Tr.*

gives the values

$$m = 55^\circ\, 28'\, 18'' \quad \text{and} \quad n = 78^\circ\, 27'\, 0'',$$

which angles, however, it does not seem possible to determine geometrically. Since, however, nothing can be concluded from this, we are compelled to admit that, up to now, the method is lacking, by means of which this problem could be resolved using incommensurable angles.

16. Let us see, however, if we take these angles to be commensurable with one another, in how many ways that question can be solved by straightedge and compasses, so that, in other words, the problem will be a plane problem: Therefore set $m = \frac{1}{2}\mu\omega$ and $n = \frac{1}{2}\nu\omega$, so that $m : n = \mu : \nu$. Then we must have

$$\mu(1 - \cos\nu\omega) = \nu(1 - \cos\mu\omega).$$

Let $\cos\omega = z$, so that:

$$\begin{aligned}
\cos 2\omega &= 2zz - 1\\
\cos 3\omega &= 4z^3 - 3z\\
\cos 4\omega &= 8z^4 - 8zz + 1\\
\cos 5\omega &= 16z^5 - 20z^3 + 5z\\
\cos 6\omega &= 32z^6 - 48z^4 + 18zz - 1\\
&\text{etc.}
\end{aligned}$$

17. Now let us run through the main cases of the resulting equation

$$\nu - \mu = \nu\cos\mu\omega - \mu\cos\nu\omega.$$

I. $\mu = 1$ and $\nu = 2$, whence $m = \frac{1}{2}\omega$ and $n = \omega$.
It follows that

$$1 = 2z - 2zz + 1 \quad \text{or} \quad zz - z = 0,$$

whence, dividing by $z - 1$, which can always be done, we have $z = 0$; hence $\omega = 90^\circ$, and $m = 45^\circ$ and $n = 90^\circ$, which is the case of the lunules of HIPPOCRATES.

II. $\mu = 1$ and $\nu = 3$, whence $m = \frac{1}{2}\omega$ and $n = \frac{3}{2}\omega$.
Consequently

$$2 = 3z - 4z^3 + 3z$$

or

$$1 - 3z + 2z^3 = 0,$$

which equation divided by $1 - z$ gives

$$1 - 2z - 2zz = 0$$

whence

$$z = \frac{-1 + \sqrt{3}}{2} = \cos\omega \quad \text{and} \quad \cos 2\omega = 1 - \sqrt{3};$$

a plane problem.

III. Let $\mu = 2$ and $\nu = 3$, whence $m = \omega$ and $n = \frac{3}{2}\omega$.
It will be the case therefore that

$$1 = 6zz - 3 - 8z^3 + 6z \quad \text{or} \quad 4 - 6z - 6zz + 8z^3 = 0,$$

which equation divided by $2(1 - z)$ gives

$$2 - z - 4zz = 0,$$

whence it follows that

$$z = \frac{-1 + \sqrt{33}}{8} = \cos\omega \quad \text{and} \quad \cos 2\omega = \frac{1 - \sqrt{33}}{16};$$

a plane problem.

IV. Let $\mu = 1$ and $\nu = 4$, whence $m = \frac{1}{2}\omega$ and $n = 2\omega$. Consequently,

$$3 = 4z - 8z^4 + 8zz - 1 \quad \text{or} \quad 1 - z - 2zz + 2z^4 = 0,$$

which equation divided by $1 - z$ gives

$$1 - 2zz - 2z^3 = 0,$$

a solid problem.

V. Let $\mu = 3$ and $\nu = 4$, whence $m = \frac{3}{2}\omega$ and $n = 2\omega$. Consequently,

$$1 = 16z^3 - 12z - 24z^4 + 24zz - 3$$

or

$$1 + 3z - 6zz - 4z^3 + 6z^4 = 0,$$

which divided by $1 - z$ gives

$$1 + 4z - 2zz - 6z^3 = 0,$$

a solid problem.

VI. Let $\mu = 1$ and $\nu = 5$, whence $m = \frac{1}{2}\omega$ and $n = \frac{5}{2}\omega$. Consequently,

$$4 = 5z - 16z^5 + 20z^3 - 5z$$

or

$$1 - 5z^3 + 4z^5 = 0,$$

which divided by $1 - z$ becomes

$$1 + z + zz - 4z^3 - 4z^4 = 0$$

and can be rearranged into this form:

$$(2zz + z - \tfrac{1}{2})^2 = \tfrac{5}{4} \quad \text{or} \quad 2zz + z - \tfrac{1}{2} = \frac{\sqrt{5}}{2},$$

whence

$$z = \frac{-1 + \sqrt{(5 + 4\sqrt{5})}}{4} = \cos\omega$$

and

$$\cos 2\omega = \frac{2\sqrt{5} - 1 - \sqrt{(5 + 4\sqrt{5})}}{4},$$

so that this produces a plane problem.

VII. Let $\mu = 2$ and $\nu = 5$, whence $m = \omega$ and $n = \frac{5}{2}\omega$. Consequently,

$$3 = 10zz - 5 - 32z^5 + 40z^3 - 10z$$

or

$$4 + 5z - 5zz - 20z^3 + 16z^5 = 0$$

and dividing by $1 - z$

$$4 + 9z + 4zz - 16z^3 - 16z^4 = 0$$

a solid problem.

VIII. Let $\mu = 3$ and $\nu = 5$, whence $m = \frac{3}{2}\omega$ and $n = \frac{5}{2}\omega$. Therefore

$$2 = 20z^3 - 15z - 48z^5 + 60z^3 - 15z$$

or

$$1 + 15z - 40z^3 + 24z^5 = 0$$

and dividing by $1 - z$

$$1 + 16z + 16zz - 24z^3 - 24z^4 = 0,$$

which reduces to the form

$$(1 + 8z + 6zz)^2 = 60(z + zz)^2,$$

whence it follows that $zz\sqrt{60} + z\sqrt{60} = 6zz + 8z + 1$ so that

$$z = \frac{4 - \sqrt{15} + \sqrt{(25 - 6\sqrt{15})}}{2\sqrt{15} - 6} \quad \text{or} \quad \frac{1}{z} = -4 + \sqrt{15} + \sqrt{(25 - 6\sqrt{15})}$$

that is

$$z = \frac{\sqrt{15} - 3 + \sqrt{(60 + 6\sqrt{15})}}{12} = \cos\omega \quad \text{and} \quad \cos 2\omega = \frac{1 + \sqrt{(25 - 6\sqrt{15})}}{6},$$

thus in this case also the problem is plane.

IX. Let $\mu = 4$ and $\nu = 5$, whence $m = 2\omega$ and $n = \frac{5}{2}\omega$.
Therefore

$$1 = 40z^4 - 40zz + 5 - 64z^5 + 80z^3 - 20z$$

or

$$1 - 5z - 10zz + 20z^3 + 10z^4 - 16z^5 = 0$$

and dividing by $1 - z$

$$1 - 4z - 14zz + 6z^3 + 16z^4 = 0,$$

but this equation admits only a solid construction.

18. We have therefore obtained five cases which exhibit quadrable lunules obtainable using straightedge and compasses, or in which the equation

$$\frac{m}{\sin m^2} = \frac{n}{\sin n^2}$$

can be constructed:

I. The case $m = 45°$ and $n = 90°$

II. The case $m = \frac{1}{2}\omega$ and $n = \frac{3}{2}\omega$ where
$$\cos\omega = \frac{\sqrt{3} - 1}{2} \quad \text{or} \quad \cos 2\omega = 1 - \sqrt{3}$$

III. The case $m = \omega$ and $n = \frac{3}{2}\omega$ where
$$\cos\omega = \frac{\sqrt{33} - 1}{8} \quad \text{or} \quad \cos 2\omega = \frac{1 - \sqrt{33}}{16}$$

IV. The case $m = \frac{1}{2}\omega$ and $n = \frac{5}{2}\omega$ where
$$\cos\omega = \frac{\sqrt{(5 + 4\sqrt{5})} - 1}{4}$$

V. The case $m = \frac{3}{2}\omega$ and $n = \frac{5}{2}\omega$ where
$$\cos\omega = \frac{\sqrt{15} - 3 + \sqrt{(60 + 6\sqrt{15})}}{12} \quad \text{or} \quad \cos 2\omega = \frac{1 + \sqrt{(25 - 6\sqrt{15})}}{6}.$$

19. The construction of lunules corresponding to other ratios between the arcs m and n will rise to higher geometric loci, though no less to be taken as geometric on that account. But whether except for these cases there could be any other lunules which could be determined geometrically, or not, appears to remain in doubt. Nor could the commonly adduced counterargument be valid, that from this the quadrature of the circle would follow; and yet even that, unless one has only indefinite quadrature in mind, appears to have too little force. Although it may indeed have been adequately established that the ratio of the circumference to the diameter could not be expressed by rational numbers, still less has it so far been possible to see

that even irrational numbers could not be used for this purpose. And indeed, even if it had been firmly demonstrated that this ratio could not be exhibited in any way by any surd number, nevertheless, one could hardly draw any conclusion from that about arcs incommensurable to the whole circumference. Thus, even if someone were able to show how to determine geometrically the arc of the circle whose sine, for example, is $= \frac{2}{3} = 0{,}6666666$, it would by no means be possible from that to compute the magnitude of the circumference; so that this kind of rectification hardly seems to tell against the impossibility of the quadrature of the circle; unless perhaps, in relation to that method by means of which someone arrived at the value of the arc whose sine is $= \frac{2}{3}$, we were to consider that since to each sine there correspond innumerable arcs, by the law of continuity the method would have to include all those arcs, and consequently could not be contained by an algebraic expression; not to mention that the difference or sum of these arcs would display the whole circumference. Hence, if this reasoning is ultimately confirmed, it will be possible to say that no circular arc whatever, which is geometrically constructible, could be a geometrically defined quantity.