A Translation of Leonhard Euler's paper

Recherches sur le mouvement des rivières*

§1. There is little that Authors have written so far on the movement of rivers; and all that they have said is founded on nothing but arbitrary hypotheses and is often actually false. Because, although we have already been successful in applying mechanical principles to the movement of water, we have merely considered the cases where water flows through pipes that are not irregular; and on this account, we have even supposed that all the particles of the water in the same section perpendicular to the pipe move in an equal motion; so that the speed of the water in each section of the channel is reciprocally proportional to the amplitude.[1] And it is this rule that serves as the foundation for all the research which has been done until now on the movement of water. The profound speculations of Messrs. Bernoulli[2] and d'Alembert, to whom we are indebted for all that has been discovered until now in this science, are all established on this hypothesis: and it must be admitted that in all the cases where they have applied their theory, this hypothesis has found itself very much in agreement with the truth.

§2. But, when the motion of the water is such that its speed is not governed solely by the amplitude of the channel through which the water runs—so that the velocities in the same section of the channel differ—it is impossible to apply the principles of hydrodynamics, which were used with such good success in the research mentioned previously. This happens mainly when the channels through which the water passes are very large, for we then would deviate too significantly from the truth if we supposed that water flows with an equal movement for each portion of channel. Yet we understand easily that it is necessary to relate here the motion of rivers; since, in the same section that is perpendicular to the bed of the river, the water can have very different speeds; and it is evident that the particles of water which are found close to the bottom of the bed are driven by forces entirely different from the particles above: from whence there must necessarily result a very different movement in the particles which are located in the same section of the bed.

§3. So, to explore the movement of water in a river, it is necessary to abandon the hypotheses to which we had attached until now all hydraulic research, going back to the first principles of Mechanics, by which all the motions of solid and fluid bodies are determined. We must consider separately each particle of water, and search for all the forces to which it is subject, to determine the changes caused in its movement. This research being extremely difficult and enveloped in very perplexing calculations, I will limit myself to begin the explanation of

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[1] The original word used here is "l'amplitude," which is the distance from the middle of the river to the bank.

[2] Truesdell ([1], p. LVIII) thinks that Euler is speaking of both Johann and Daniel Bernoulli.

this theory with an example, which will serve as the basis for all the others, and which will not leave us to provide further clarifications in this matter. After the development of this example, it will not be very difficult to make the research more general, and to apply it to all the cases which can be encountered.

§4. I will therefore consider a vertical section made along a river which will represent to us an infinitely narrow river; and when I will have determined the movement of the water in this section, although I have not had consideration for the effect of the water located next to both sides, we will agree that this motion will not differ much from the truth regardless of how large the river is, provided that the form of the bed is not extremely irregular. So I suppose for the beginning that the pressures of the adjacent water are equal on both sides, so that all the particles of water located in this section always remain in the same section. Nevertheless the method I will use can also be readily applied to the case where water flows from such a vertical section to another, and when we have found a way to expedite the the calculations for the case that I am going to treat, we will arrive more easily at the end of the calculations which include the general solution.

§5. Therefore, let AC be the bed of such a section of river, or of an infinitely narrow river, which has the same infinitely small width throughout.[3] Now this bed AC is an arbitrarily curved line, which is supposed to be known; for this purpose I construct a horizontal line EF which serves as an axis in order to relate the line AC to the orthogonal coordinates EP and PQ. Now $ABCD$ is the river, which moves along this bed AC, and BD its upper surface; also I suppose that the river exists already in a permanent state or in equilibrium[4], so that its surface BD remains constantly the same, and at the same points, like M, the water particles which flow there always have the same speeds, and they are subjected to the same pressures.

§6. All the water that forms the river must have passed through the section AB, which I consider here to be the principal section, and against which I will determine both the location and the movement of each particle after an arbitrary period of time since it passed through the section AB. Let the height of this section $AB = a$, in which I shall consider an arbitrary point O, naming $AO = z$, and let OMG be the path that each particle of water which passes through the point O describes in being swept along by the current. Here we must first look at the motion with which each particle of water passes through the point O. This movement being broken down into the horizontal and vertical directions, let m be the speed in the horizontal direction, and n the speed in the vertical direction, which I suppose heads downward. So, naming $EA = b$ so that $EO = b + z$, after an infinitely small period of time $d\tau$, the point of water O will advance along the horizontal direction by the space $md\tau$, and along the vertical direction by the space $nd\tau$. Consequently, after the time $d\tau$, the point O will arrive at o, so that having drawn the vertical oe, $Ee = md\tau$ and $eo = b + z - nd\tau$. Now it is clear

[3] It is at this point that reference is made to "Fig. 1," though this figure is missing in the published version of this manuscript. The figure has been reconstructed using Euler's description of it, and is appended to the end of this translation.

[4] That is, a steady flow.

that m and n are functions of z.

§7. After an arbitrary period of time t, the point of water which passed through O has reached M, having described during this time t the line OM. We draw the perpendicular MP from M onto EF; naming $EP = x$ and $PM = y$, we see that x and y will be functions of the two variables t and z. Let, then, for expressing the dependence of these two variables,

$$dx = Pdt + Qdz, \ \ \& \ \ dy = Rdt + Sdz,$$ [5]

where we see that these differential formulas must be complete; in other words $\frac{dP}{dz} = \frac{dQ}{dt}$ and $\frac{dR}{dz} = \frac{dS}{dt}$[6], where $\frac{dP}{dz}$ marks the differential of P (in assuming only the z variable) divided by dz, and $\frac{dQ}{dt}$, the differential of Q (in assuming only the t variable) divided by dt, and similarly the others. We also see that these values of x and y, by setting z equal to 0, must express the form of the bed AC since the water that passes by the point A must slide over the bed itself. Now, if we set $z = a$, these same formulas for x and y will express the surface of the water BD, or the surface of the river.

§8. It is further noted that, when we set $t = 0$, the point M must revert to O. So, the functions of t and z which express the values of x and y must be such that if we set $t = 0$, then $x = 0$ and $y = b + z = EO$. Additionally, since $dx = Pdt + Qdz$ and $dy = Rdt + Sdz$: if we set $t = 0$, and we take z to be constant (so $dz = 0$, and $dt = d\tau$) these differentials will give the location of the point o, which the point O reaches in the time $d\tau$. It will therefore be, setting $t = 0$, in the quantities P and R, that $Ee = Pd\tau$ and $eo = b + z + Rd\tau$. Comparing these values with those which we have found above, we will have $P = m$ and $R = -n$, so that the functions P and R, setting $t = 0$, must give the horizontal and vertical speeds of the point O.

§9. In order to find the line OMG that all particles of water passing through the point O represent in the river—since it is necessary to view this point O as fixed in the section AB—we will have $dz = 0$; so the nature of the line OMG will be contained in these formulas:

$$dx = Pdt, \ \ \& \ \ dy = Rdt$$

which tells us that, in the element of time dt, the point M arrives at m, so that $Pp = Pdt$ and $pm = y + Rdt$. From there, we know the movement of the point M which is carried to m during the time dt. For, if we call the horizontal speed of point $M = v$, and its vertical speed $= u$, pointing downward, after the period of time dt it must be that $Pp = vdt$ and $pm = y - udt$ and therefore we will have $v = P$ and $u = -R$. Thus, knowing the functions P, Q, R, and S in the general formulas:

$$dx = Pdt + Qdz, \ \ \& \ \ dy = Rdt + Sdz,$$

[5]In modern notation, we would write $P = \frac{\partial x}{\partial t}$ and $Q = \frac{\partial x}{\partial z}$.
[6]This known today as the equality of mixed partials.

the functions P and R express simultaneously the speeds of the point M: the horizontal v and the vertical u.

§10. Let us now give the point O an infinitely small extension $OO' = dz$ in order to consider the entire stream[7] of water $OO'MM'GG'$ which passes through this opening $OO' = dz$, for it is necessary that this stream always remains continuous, without it introducing any gap. So the horizontal velocity of point O' will be $m + dm$, and the vertical velocity will be $n + dn$, these quantities m and n being functions of z. In the time $d\tau$ the point O' will reach o', so that $Ee' = (m + dm)d\tau$, and $e'o' = b + z + dz - (n + dn)d\tau$. Consequently, in this same time interval $d\tau$, the mass of the water $OO'oo'$ passes by the opening $OO' = dz$, whose volume can be found in this manner:[8]

$$\begin{aligned}
\text{The area of the trapezoid } EO'o'e' \text{ is} &= \frac{1}{2}Ee'(EO' + e'o'),\\
\text{... of the trapezoid } EOoe \text{ ...} &= \frac{1}{2}Ee(EO + eo),\\
\text{... of the trapezoid } eoo'e' \text{ ...} &= \frac{1}{2}ee'(eo + e'o'),
\end{aligned}$$

from which we find the area

$$\begin{aligned}
OO'o'o &= \frac{1}{2}Ee'(EO' + e'o') - \frac{1}{2}Ee(EO + eo) - \frac{1}{2}ee'(eo + e'o')\\
&= \frac{1}{2}Ee(OO' + e'o' - eo) + \frac{1}{2}ee'(EO' - eo),
\end{aligned}$$

and therefore it will be $= mdzd\tau + \frac{1}{2}dmdzd\tau - \frac{1}{2}mdnd\tau^2 + \frac{1}{2}ndmd\tau^2$.

§11. After a period time t, with the point O coming to M so that $EP = x$ and $PM = y$, the point O' will reach M', so that $EP' = x + Qdz$ and $P'M' = y + Sdz$, and again after an infinitely small period of time $d\tau$, these points M and M' will be transported to m and m' so that $Ep = x + Pd\tau$, $pm = y + Rd\tau$, $Ep' = x + Pd\tau + Qdz$ and $p'm' = y + Rd\tau + Sdz$. Therefore the mass $OO'o'o = mdzd\tau$ will reach $MM'm'm$ after the period of time t, where it will fill this space: this is why it is necessary that the area $MM'm'm$ is equal to $OO'o'o = mdzd\tau$. However, to find this area, we need only find these four trapezoids.

$$\begin{aligned}
PMmp &= \frac{1}{2}Pp(PM + pm) = \frac{1}{2}Pd\tau(2y + Rd\tau),\\
P'M'm'p' &= \frac{1}{2}P'p'(P'M' + p'm') = \frac{1}{2}Pd\tau(2y + Rd\tau + 2Sdz),\\
PMM'P' &= \frac{1}{2}PP'(PM + P'M') = \frac{1}{2}Qdz(2y + Sdz),\\
pmm'p' &= \frac{1}{2}pp'(pm + p'm') = \frac{1}{2}Qdz(2y + 2Rd\tau + Sdz),
\end{aligned}$$

[7] The French word used here is “filet.” Truesdell ([1], p. LIX) prefers “fillet” as a translation. Euler uses the word in other papers, like E200, with a different meaning.

[8] In the original, the first trapezoid is $EO'o'e$, which is either a typo or means Euler constructed his figure differently than we understand it.

and since $MM'm'm = P'M'm'p' + PMM'P' - pmm'p' - PMmp$, we will have $MM'm'm = PSdzd\tau - QRdzd\tau$. So it must be that $PS - QR = m$, and this is the first condition which must be satisfied.

§12. This condition that we have just discovered contains the continuity of the fluid, whereby it is necessary that $PS - QR$ is equal to m, that is, to a function of z, where the time t does not matter. Considering the initial state in AB, we discover still other properties that the functions P, Q, R, and S must have. Because, by varying the point O as z, that the time t, to reach the point o', we will have $Ee' = md\tau$ and $e'o' = b + z - nd\tau$, where $d\tau$ marks the element of time t, which is itself in this case $= 0$. So if $t = 0$, it must be that $dx = md\tau + odz$, and $dy = -nd\tau + dz$. Now, having generally supposed that $dx = Pd\tau + Qdz$, and $dy = Rd\tau + Sdz$, it is required that setting $t = 0$, it becomes:

$$P = m;\ Q = 0;\ R = -n;\ \&S = 1.$$

These conditions, joined with the fact that $PS - QR = m$, and that the differential formulas $Pd\tau + Qdz$ and $Rd\tau + Sdz$ must be complete, or integrable, determine already in part the nature of these functions; and besides this, it is necessary that, setting $z = 0$, the coordinates x and y express the nature of the line of the bed AC.

§13. Now, to find the acceleration of the element of water $MM'm'm$, whose mass is $= mdzd\tau$, regard must be had to the forces which are acting there. These forces are firstly the weight of this element, which I will express by its volume $mdzd\tau$, and by this force the element is pushed downward. Next, this same element is subjected to the pressures of the particles of water by which it is surrounded, and these pressures are expressed most conveniently by the height of a column of water that would exert the same pressure. So let p be the height that expresses the pressure at point M, and p will be a specific function of the coordinates x and y, or else of the variables t and z; and to represent this dependence, let $dp = Mdt + Ndz$. From there we will know the pressures at points M', m, and m'; for we have the pressure at $M' = p + Ndz$, at $m = p + Mdt$,[9] and at $m' = p + Ndz + Md\tau$, using $d\tau$ for dt, like we have done previously in considering these points.

§14. So, on the side MM' acts a force $= MM'(p + \frac{1}{2}Ndz)$, on the side Mm a force $= Mm(p + \frac{1}{2}Md\tau)$, on the side $M'm'$ a force $= M'm'(p + Ndz + \frac{1}{2}Md\tau)$, and on the side mm' a force $= mm'(p + Md\tau + \frac{1}{2}Ndz)$. Let us break down these forces according to the directions of the coordinates EP[10] and EA[11], because, since these forces act perpendicularly to the sides, the resolution will provide:

gives the forces

[9]It is likely that Euler intended to use $Md\tau$ here, but we kept Mdt, as originally printed.
[10]That is, the horizontal direction.
[11]The vertical direction.

The force on:	by EP	by EA
$MM' = MM'(p + \frac{1}{2}Ndz)$;	$+Sdz(p + \frac{1}{2}Ndz)$;	$-Qdz(p + \frac{1}{2}Ndz)$,
$Mm = Mm(p + \frac{1}{2}Md\tau)$;	$-Rd\tau(p + \frac{1}{2}Md\tau)$;	$+Pd\tau(p + \frac{1}{2}Md\tau)$,
$mm' = mm'(p + Md\tau + \frac{1}{2}Ndz)$;	$-Sdz(p + Md\tau + \frac{1}{2}Ndz)$;	$+Qdz(p + Md\tau + \frac{1}{2}Ndz)$,
$M'm' = M'm'(p + Ndz + \frac{1}{2}Md\tau)$;	$+Rd\tau(p + Ndz + \frac{1}{2}Md\tau)$;	$-Pd\tau(p + Ndz + \frac{1}{2}Md\tau)$.

So, taking all these forces together, the element $MM'm'm$ will be pushed in the horizontal direction EP by the force $= -MSdzd\tau + NRdzd\tau$ and in the vertical direction EA by the force $= MQdzd\tau - NPdzd\tau$; and from this it is necessary to subtract the force of gravity of this element which is $= PSdzd\tau - QRdzd\tau = mdzd\tau$.

§15. The mass $MM'm'm$, being acted upon by two forces, one along the horizontal EP, which is $= (NR - MS)dzd\tau$, and the other along the vertical EA which is $= (MQ - NP)dzd\tau - mdzd\tau$,

$$\text{the accelerating force in the direction } EP \text{ will be } = \frac{NR - MS}{m}, \text{ and}$$

$$\text{the accelerating force in the direction } EA \text{ will be } = \frac{MQ - NP}{m} - 1.$$

Consequently, the point M will be accelerated by these two accelerating forces. So, taking z constant, and the element of time dt equally constant, according to the principles of acceleration, we will have:

$$\frac{NR - MS}{m} = \frac{2ddx}{dt^2}, \quad \& \quad \frac{MQ - NP}{m} - 1 = \frac{2ddy}{dt^2}.$$ [12]

However, having $dx = Pdt + Qdz$, and $dy = Rdt + Sdt$, if we set $dP = \mathfrak{P}dt + \mathfrak{Q}dz$, and $dR = \mathfrak{R}dt + \mathfrak{S}dz$, it will be that $ddx = \mathfrak{P}dt^2$, and $ddy = \mathfrak{R}dt^2$, from which we finally derive these two equations

$$NR - MS = 2m\mathfrak{P}, \quad \& \quad MQ - NP - m = 2m\mathfrak{R}.$$

§16. So, to determine the movement of the river $ABCD$, which is formed by water flowing continuously past the section $AB = a$, on the bed AC whose form is given, having taken the horizontal line EF for an axis, and naming $AE = b$, let a particle of water arrive, which passes by O, setting $AO = z$, to M after an elapsed period of time t, and let the coordinates be named $EP = x$ and $PM = y$, these quantities x and y will be some functions of t and z. So let

$$dx = Pdt + Qdz, \quad \& \quad dy = Rdt + Sdz,$$

where P, Q, R and S are such functions of t and z that these differential equations are integrable. Now, for the functions P and R, also let:

$$dP = \mathfrak{P}dt + \mathfrak{Q}dz, \quad \& \quad dR = \mathfrak{R}dt + \mathfrak{S}dz.$$

[12] Truesdell explains the appearance of the number two in these equations by saying that Euler used special units to "take the ratio of the units of length and time in such a way that $g = \frac{1}{2}$" ([1], p. XLIII - XLIV). Euler himself made this change of units—starting in E177 and continuing into later papers—with little in the way of explanation until a later paper (§20 of E222).

Finally, let the water pressure at the point M expressed by height $= p$, which is likewise a function of the variables t and z, be

$$dp = Mdt + Ndz.$$

§17. The issue is thus to find the functions P, Q, R, S, M, and N; for this the following conditions must be satisfied:

1) The coordinates x and y must be such functions of t and z that when we set $z = 0$, they express the figure of the bed AC; thus, setting $z = 0$, we will have for the form of the bed AC the formulas $dx = Pdt$ and $dy = Rdt$. However, when we set $t = 0$, it must become $x = 0$ and $y = b + z$.

2) The movement of the water flowing through the point O is assumed to be such that its speed in the horizontal direction is $= m$, and its speed in the vertical direction pointed downward is $= n$; it must be, setting $t = 0$, that

$$P = m;\ Q = 0;\ R = -n;\ S = 1;$$

where m and n will be functions of the single variable z, without containing the other, t.

3) The third condition requires that in general $PS - QR = m$, where $PS - QR$ must be a function of the single variable z, without the variable t entering.

4) The consideration of acceleration provided us with these equations which must be satisfied:

$$NR - MS = 2m\mathfrak{P}, \ \ \&\ \ MQ - NP = 2m\mathfrak{R} + m.$$

5) Finally, it is evident that the pressure p must be such a function of t and z that when we set $z = a$—in which case the pressure will refer to the surface BD—the value of p vanishes. Then it must be that $M = 0$, if we set $z = a$.

§18. The equations of the fourth condition, since they are by virtue of the third $m = PS - QR$, will give

$$\begin{aligned} M &= -2P\mathfrak{P} - 2R\mathfrak{R} - R, \\ N &= -2Q\mathfrak{P} - 2S\mathfrak{R} - S, \end{aligned}$$

and from there we obtain:

$$\begin{aligned} dp = &-2P\mathfrak{P}dt - 2R\mathfrak{R}dt - Rdt, \\ &-2Q\mathfrak{P}dz - 2S\mathfrak{R}dz - Sdz. \end{aligned}$$

But, having $Pdt + Qdz = dx$ and $Rdt + Sdz = dy$, it will be

$$dp = -2\mathfrak{P}dx - 2\mathfrak{R}dy - dy.$$

Then, since this equation must be integrable, it is necessary that $\mathfrak{P}dx + \mathfrak{R}dy$ is a complete differential formula.

§19. From there we can also find the following condition: since $\mathfrak{P}dt = dP - \mathfrak{Q}dz$ and $\mathfrak{R}dt = dR - \mathfrak{S}dz$, the substitution of these formulas will give:

$$\begin{aligned} dp &= -2PdP + 2P\mathfrak{Q}dz + 2R\mathfrak{S}dz \\ &\quad -2RdR - 2Q\mathfrak{P}dz - 2S\mathfrak{R}dz - dy, \end{aligned}$$

whose integral, as it can be taken, will be:

$$p = C - y - PP - RR + 2\int dz(P\mathfrak{Q} - Q\mathfrak{P} - R\mathfrak{S} - S\mathfrak{R}).^{13}$$

Then it is necessary that the equation $P\mathfrak{Q} - Q\mathfrak{P} + R\mathfrak{S} - S\mathfrak{R}$ is a function of the single variable z, since otherwise the integration could not take place.

§20. So let $P\mathfrak{Q} - Q\mathfrak{P} + R\mathfrak{S} - S\mathfrak{R} = w$ so that w represents a function of the single variable z, and the pressure at M will be

$$p = C - y - PP - RR + 2\int wdz,$$

and $\int wdz$ will similarly be a function of z. However, since the expression of p must vanish if we set $z = a$, it is necessary that this position $z = a$ must make every t vanish in the equation $y + PP + RR$, so that it will become a constant. And then we will need only to determine C, so that p vanishes in this case $z = 0$.[14]

§21. Since we also have $PS - QR = m$, where m is likewise a function of only z, these two equations:

$$PS - QR = m,$$

$$P\mathfrak{Q} + Q\mathfrak{P} + R\mathfrak{S} - S\mathfrak{R} = w,^{15}$$

can serve to eliminate Q and S: and we find

$$\begin{aligned} Q &= \frac{PP\mathfrak{Q} + PR\mathfrak{S} - m\mathfrak{R} - wP}{P\mathfrak{P} + R\mathfrak{R}}, \\ S &= \frac{PR\mathfrak{Q} + RR\mathfrak{S} + m\mathfrak{P} - wR}{P\mathfrak{P} + R\mathfrak{R}}, \end{aligned}$$

and these values must make integrable the equations

$$dx = Pdt + Qdz, \ \ \& \ \ dy = Rdt + Sdz.$$

[13] This appears to be a typo; see the representation of the same expression on the next line.
[14] This also seems to be a typo, since the case Euler has been examining is $z = a$, not $z = 0$.
[15] Again, a typo in this equation. It should be $P\mathfrak{Q} - Q\mathfrak{P} + R\mathfrak{S} - S\mathfrak{R}$.

Now substituting these values found by introducing the equations $dP = \mathfrak{P}dt + \mathfrak{Q}dz$, and $dR = \mathfrak{R}dt + \mathfrak{S}dz$, we will have:

$$
\begin{aligned}
dx &= \frac{PPdP + PRdR - m\mathfrak{R}dz - wPdz}{P\mathfrak{P} + R\mathfrak{R}}, \\
dy &= \frac{PRdP + RRdR + m\mathfrak{P}dz - wRdz}{P\mathfrak{P} + R\mathfrak{R}},
\end{aligned}
$$

and both of these equations must be integrable.

§22. All these questions are then reduced to the research of the nature of these two functions P and R, on which depend the functions $\mathfrak{P}$ and $\mathfrak{R}$, so that these two equations

$$
\begin{aligned}
dx &= \frac{P(PdP + RdR) - m\mathfrak{R}dz - wPdz}{P\mathfrak{P} + R\mathfrak{R}}, \\
dy &= \frac{R(PdP + RdR) + m\mathfrak{P}dz - wRdz}{P\mathfrak{P} + R\mathfrak{R}},
\end{aligned}
$$

become integrable. And once we have found a method to resolve this problem in general, it will not be more difficult to determine these functions so that they satisfy the other conditions. However, this problem is so difficult that, although it depends only on the analysis, we can almost never hope to find the general solution, which could be used to determine the movement of all kinds of rivers.

§23. These difficulties compel me to dwell on individual cases, whose development can at the same time serve to show us how we must go about seeking the general solution.[16] Since, then, setting $t = 0$ must give $x = 0$ and $y = b + z$, I will assume

$$x = Vt + Att, \ \ \& \ \ y = b + z + Zt + Btt,$$

where V and Z represent functions of only the variable z. We then will have:

$$
\begin{aligned}
P &= V + 2At; & R &= Z + 2Bt, \\
Q &= \frac{tdV}{dz}; & S &= 1 + \frac{tdZ}{dz}, \\
\mathfrak{P} &= 2A; & \mathfrak{R} &= 2B, \\
\mathfrak{Q} &= \frac{dV}{dz}; & \mathfrak{S} &= \frac{dZ}{dz},
\end{aligned}
$$

[16] A brief historical note: this paper was published in 1767 but written in 1751. This large gap between dates, along with the missing figure and now the fact that Euler resorts to examining specific cases, suggests that this paper was a draft. Truesdell suggests that the Berlin Academy, being deprived of a large quantity of papers it would normally have published, was looking for extra papers to print and selected this work from Euler's unpublished manuscripts.

and by the second condition we will have $m = V$ and $n = -Z$. Now the third condition gives

$$PS - QR = V + 2At + \frac{VtdZ}{dz} + \frac{2AttdZ}{dz} - \frac{ZtdV}{dz} - \frac{2BttdV}{dz},$$

and this expression must be $= m = V$, from which we derive these two equations:

$$2Adz + VdZ - ZdV = 0, \ \ \& \ \ AdZ = BdV,$$

where the latter gives $Z = \frac{BV}{A} + C$, which being put in the first produces $2Adz - CdV = 0$, and thus $V = \frac{2Az}{C} + D$, so $Z = \frac{2Bz}{C} + \frac{BD}{A} + C$.

§24. Let us change these constants, and let $D = \frac{2Ac}{C}$; $A = \frac{1}{2}\alpha C$; $B = \frac{1}{2}\zeta C$, to have $V = \alpha(z+c)$; and $Z = C + \zeta(z+c)$, and our equations will become:

$$\begin{array}{ll}
x = \alpha(z+c)t + \frac{1}{2}\alpha Ctt; & y = b + z + Ct + \zeta(z+c)t + \frac{1}{2}\zeta Ctt, \\
P = \alpha(z+c) + \alpha Ct; & R = C + \zeta(z+c) + \zeta Ct, \\
Q = \alpha t; & S = 1 + \zeta t, \\
\mathfrak{P} = \alpha C; & \mathfrak{R} = \zeta C, \\
\mathfrak{Q} = \alpha; & \mathfrak{S} = \zeta,
\end{array}$$

From there we obtain the function of z, which was named $w = P\mathfrak{Q} - Q\mathfrak{P} + R\mathfrak{S} - S\mathfrak{R}$, which produces

$$\begin{array}{rcl}
w & = & \alpha\alpha(z+c) \ + \alpha\alpha Ct \ + \ \zeta C \ + \zeta\zeta(z+c) \ + \zeta\zeta Ct, \\
& & \qquad\qquad\quad - \alpha\alpha Ct \ - \ \zeta C \qquad\qquad\qquad - \zeta\zeta Ct, \\
\text{or } w & = & (\alpha\alpha + \zeta\zeta)(z+c).
\end{array}$$

Since the terms containing t eliminated themselves, we do not need further reductions.

§25. From there we will then have $2\int wdz = (\alpha\alpha + \zeta\zeta)(zz + 2cz + cc)$, and the pressure at the point M will be

$$\begin{array}{rcl}
p & = & \text{Const.} - b - z - Ct - \zeta(z+c)t - \dfrac{1}{2}\zeta Ctt + \alpha\alpha(z+c)^2 \\
& & -\alpha\alpha(z+c)^2 - 2\alpha\alpha Ct(z+c) - \alpha\alpha CCtt + \zeta\zeta(z+c)^2 \\
& & -\zeta\zeta(z+c)^2 - 2\zeta CCt - \zeta\zeta CCtt \\
& & -2C\zeta(z+c) - 2\zeta\zeta Ct(z+c) \\
& & -CC
\end{array}$$

or

$$\begin{array}{rcl}
p & = & \text{Const.} - b - z - 2C\zeta(z+c) - CC - (1 + 2\zeta C)Ct \\
& & -(\zeta + 2\alpha\alpha C + 2\zeta\zeta C)(z+c)t - C\left(\dfrac{1}{2}\zeta + \alpha\alpha C + \zeta\zeta C\right)tt.
\end{array}$$

Now this expression needing to disappear, setting $z = a$, whichever value obtains the variable t, we will derive three equations

$$\text{Const.} = b + a + 2C\zeta(a+c) + CC.$$
$$(1 + 2\zeta C)C + (\zeta + 2\alpha\alpha C + 2\zeta\zeta C)(a+c) = 0.$$
$$C\left(\frac{1}{2}\zeta + \alpha\alpha C + \zeta\zeta C\right) = 0,$$

of which the last provides us with two solutions which I will develop separately.

§26. Now, for the first solution, let $C = 0$; and the second [equation] gives $c = -a$; and the first [equation] Const $= b + a$. Then our equations for the movement of water will be:

$$x = \alpha(z-a)t, \;\; \& \;\; y = b + z + \zeta(z-a)t,$$

and the pressure $p = a - z - \zeta(z-a)t$.

Since z cannot become bigger than a, it is necessary for α to be negative, so that

$$x = \alpha(a-z)t, \;\; \& \;\; y = b + z - \zeta(a-z)t,$$

and the pressure $p = a - z + \zeta(a-z)t$.

Setting $z = 0$, the form of the bed AC in this case will be expressed by the equations $x = \alpha a t$ and $y = b - \zeta a t$, from which we know that the bed AC will be a straight line inclined to the horizon at an angle whose tangent $= \frac{\zeta}{\alpha}$.

§27. For the second case, we will have $C = \frac{-\zeta}{2(\alpha\alpha+\zeta\zeta)}$, from which the second equation becomes $(1 + 2\zeta C)C = 0$, or $\alpha\alpha\zeta = 0$,[17] so it will be α or $\zeta = 0$: if it was that $\zeta = 0$, it would be $C = 0$, like in the previous case, so let $\alpha = 0$; and it will be that $C = -\frac{1}{2\zeta}$; and Const. $= b + a - a - c + \frac{1}{4\zeta\zeta} = b - c + \frac{1}{4\zeta\zeta}$. From which our equations for the movement of water will be:

$$x = \alpha(z+c)t, \;\; \& \;\; y = b + z - \frac{t}{2\zeta} + \zeta(z+c)t - \frac{1}{4}tt,$$

& $p = 0$. Now α being $= 0$, since $x = 0$, the water flowing through AB will never leave by the perpendicular BAE.

§28. In this case then, the water will flow perpendicularly from top to bottom. But, if we take $c = \infty$, so that αc obtains a finite value $= f$, and if we also set $\zeta = 0$ so $\frac{1}{2\zeta} - \zeta c = g$, the following case will result:

$$x = ft, \;\; \& \;\; y = b + z - gt - \frac{1}{4}tt,$$

and the pressure will consequently be $= 0$. Then, the bed being under no pressure, this case describes the motion where water falls freely and in an oblique arbitrary direction. Now it is also clear that the water must pass by the height AB with the same speed and in the same

[17] This is incorrect, although it does not affect the analysis that follows. It should be that $\alpha\alpha = 0$.

direction so that each particle of water describes a parabola, as if it was separated from the rest, since it does not suffer from any pressure. Also, we see from our equations that the form of the bed, or the path of particles that pass by A, is a parabola included in these formulas

$$x = ft, \;\; \& \;\; y = b - gt - \frac{1}{4}tt.$$

§29. In the first case, having set $C = 0$ to satisfy the second equation, instead of setting $c = -a$, we can also set $\zeta = 0$; and the first [equation] will be Const. $= b + a$. In this case, we will have:

$$x = \alpha(z + c)t, \;\; \& \;\; y = b + z,$$

and $p = a - z$, where in place of $x = \alpha(z + c)t$, we can set $x = (\alpha z + \zeta)t$. Here we see that the bed AC becomes a horizontal line like the surface BD: for each particle of water will be moved uniformly in a horizontal direction, and $\alpha z + \zeta$ marks the speed with which the water passes by the point O. Also, the various parts of water will act on one another only by virtue of their weight; hence the pressure $p = a - z$ is the same throughout as if the water was at rest. In this case, then, the surface of the river will be perfectly horizontal and the movement of all the parts will be horizontal and uniform.

§30. Having found $x = (\alpha z + \zeta)t$, the speed of the water at point O would be $= \alpha z + \zeta$; but we easily understand that this same case must exist in some manner that varies the speeds for various points O. Also, we see that these values

$$x = Zt, \;\; \& \;\; y = b + z,$$

where Z marks an arbitrary function of z, satisfy all the required conditions as well. For, having

$$P = Z; \quad Q = \frac{tdZ}{dz}; \quad R = 0; \quad S = 1, \quad \text{\& from there}$$

$$\mathfrak{P} = 0; \quad \mathfrak{Q} = \frac{dZ}{dz}; \quad \mathfrak{R} = 0; \quad \mathfrak{S} = 0, \quad \text{it will be}$$

$$m = PS - QR = Z = \text{a function of } z,$$

$$w = P\mathfrak{Q} - Q\mathfrak{P} + R\mathfrak{S} - S\mathfrak{R} = \frac{ZdZ}{dz} = \text{a function of } z,$$

from which we derive the pressure at an arbitrary point M,

$$p \;=\; \text{Const.} - b - z - ZZ + 2\int ZdZ = a - z.$$

Thus a river can exist on a horizontal bed when all the particles of water are moved uniformly in horizontal directions and the upper surface remains horizontal. Additionally, the water pressure throughout will be the same as if all the water was at rest.

§31. Here are three different values of coordinates x and y that satisfy the conditions required for representing the movement of a river.

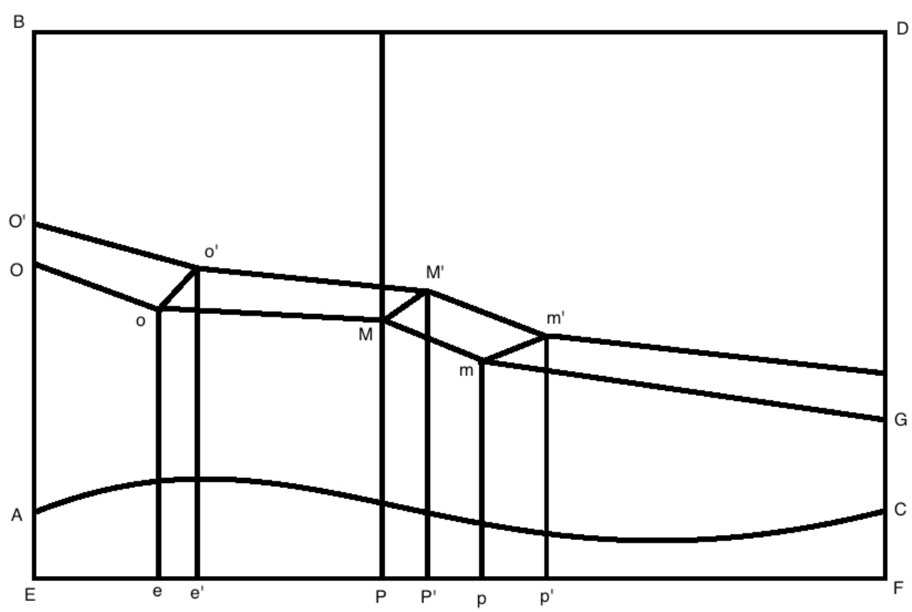
$$\begin{array}{lll} \text{I.} & x = ft; & y = b + z - gt - \frac{1}{4}tt, \\ \text{II.} & x = \alpha(a-z)t; & y = b + z + \zeta(a-z)t, \\ \text{III.} & x = Zt; & y = b + z, \end{array}$$

where Z represents an arbitrary function of z.

Now the second case can yet be made more general, by setting:

$$x = (a-z)Zt, \ \ \& \ \ y = b + z + \zeta(a-z)Zt;$$

and this will be from the consideration of these particular cases that we can expect the general solution.



A reconstruction of Figure 1.

References

[1] Clifford Truesdell, *Editor's Introduction, Leonhardi Euleri Opera Omnia II 12*, Orell Füssli Turici, Lausanne, 1964.