E314: Conjecture sur la raison de quelques dissonances generalement recues dans la musique

(Conjecture on the Reason for some Dissonances Generally Accepted in Music)

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Introduction to the translation:

In this paper, Euler refers to studying tones by "numbers." This could be ambiguous unless it is made known that the tuning he works with is known as "just intonation" in a diatonic scale. This type of tuning forces the frequencies of the tones to occur in ratios of small whole numbers. The specific form of just intonation that Euler employs takes C as the lowest frequency, so it can be thought of as 1. Furthermore, he takes F-A-C, C-E-G, and G-B-D to be "just major triads." In a just major triad, the ratio from the second note's frequency to that of the first is 5:4 (a major third), and from the third to the first 3:2 (a perfect fifth). When the numbers are worked out, the first two octaves of the notes expressed as ratios to the lowest C (ignoring sharps and flats) are the following:

C	*D*	*E*	*F*	*G*	*A*	*B*	*c*	*d*	*e*	*f*	*g*	*a*	*b*
1	*9/8*	*5/4*	*4/3*	*3/2*	*5/3*	*15/8*	*2*	*9/4*	*5/2*	*8/3*	*3*	*10/3*	*15/4*

Presumably to avoid dealing with fractions, he goes one step further and multiplies all of these ratios by 24 in order to obtain whole numbers:

C	*D*	*E*	*F*	*G*	*A*	*B*	*c*	*d*	*e*	*f*	*g*	*a*	*b*
24	*27*	*30*	*32*	*36*	*40*	*45*	*48*	*54*	*60*	*64*	*72*	*80*	*90*

There is also a type of tuning that he compares to just intonation called "12 tone equal temperament." In this tuning, all 12 notes in an octave are evenly spaced apart. One octave would appear like so:

C	*C#*	*D*	*D#*	*E*	*F*	*F#*	*G*	*G#*	*A*	*A#*	*B*	*C*
1	$\sqrt[12]{2}$	$\sqrt[12]{2^2}$	$\sqrt[12]{2^3}$	$\sqrt[12]{2^4}$	$\sqrt[12]{2^5}$	$\sqrt[12]{2^6}$	$\sqrt[12]{2^7}$	$\sqrt[12]{2^8}$	$\sqrt[12]{2^9}$	$\sqrt[12]{2^{10}}$	$\sqrt[12]{2^{11}}$	*2*

In this type of tuning, all intervals are technically slightly out of tune, but they closely approximate what the intervals should really be and appear to the ear as such. This tuning has the advantage of being able to modulate into different keys without losing the precise sound of the music. Imperfections in just intonations (such as the unavoidable minor third problem that Euler mentions) are eliminated. Most western instruments today are tuned this way, including those of orchestral groups and household pianos.

It is these ratios that Euler considers when discussing tones. In this paper, he focuses his attention on the octave starting with the G of ratio 36 up to the next g of ratio 72. Before reading, it is also important to know that in Euler's writing, when ratios are expressed as fractions, the lower divides the higher frequency, but when they are written out the smaller number is written first. For example "9/4" is equivalent to "4:9" or "4 to 9."

In translating this short paper on musical dissonance, I have tried to preserve the original tone (pun not intended, but thoroughly enjoyed) of Euler at the same time as making it comprehensible in modern language. Any ambiguities, unexplained assumptions, or terms and phrases requiring a musical background to understand are cleared up in italicized footnotes at the end of each section. Enjoy!

1. The seventh chord and the chord resulting from the sixth joined to the fifth are used in music with such success that we do not question their harmony or agreement. Though it is true that we defer them to the class of dissonance, we must note that consonance only differs from dissonance because consonance is contained in simpler proportions that are easier on our ears. Dissonance results from complicated proportions, making it more difficult for the ear to understand them. It is thus only by degree[1] that dissonant sounds differ from consonant ones, and these different tones must be discernible by ear. Several tones, having no perceptible ratio among them, would be a confused noise absolutely intolerable in music. With that in mind, it is certain that the dissonances that I plan to discuss contain perceptible ratios; otherwise, they would not be allowed in music.

   [1] *That is, steps of tone*

2. Now, expressing the tones that form the seventh chord or the sixth with the fifth in numbers, we arrive at proportions so complicated that it seems nearly impossible for the ear to grasp them. There are in fact much less complicated chords that are left out of music because the mind would not be able to identify the ratios. Here is a seventh chord expressed in numbers[1]:

G	H	d	f
36	45	54	64

   The smallest number divisible by all of these is 8640, or by factors $2^6 * 3^3 * 5$, a number that I call the exponent of this chord. This number is what we can use to judge the ease with which the ear can understand the chord. The other chord[2] is represented like so:

H	d	f	g
45	54	64	72

   which has the same exponent.

   [1] *In this text, Euler uses a German notation for notes. He uses "H" for our "B" and "B" for our "B^b" ("B-flat").*

   [2] *That is, the chord with the fifth and sixth interval together, known as a "triad add sixth." Specifically Euler gives a B-minor triad with an added sixth.*

3. It is difficult to believe that the ear can pick out the proportions between these large numbers, but the dissonance does not even seem so strong as to require such a high degree of skill. Indeed, if the ear can perceive such an exponent, then adding more tones made up of the same exponent should not

make the perception any more difficult. Now, without leaving this octave, the exponent $2^6 * 3^3 * 5$ also contains the factors 40, 48 and 60, which correspond to the notes *A*, *c*, and *e*, so that we have this chord:

G	A	H	c	d	e	f
36	40	45	48	54	60	64

which should be just as agreeable to the ear as the first chord. All musicians concur that this dissonance would be unbearable. Logically, we must then pass the same judgment on the first dissonance proposed, and if not we must say that it does not follow the rules of harmony established in music theory.

4. It is the tone *f* that troubles these chords by making their exponent so complicated, thereby revealing their dissonance to the musician's ear. We have only to omit this tone, and the numbers of the others become divisible by 9. The chord

G	H	d
4	5	6

creates an agreeable and perfect consonance, known as the harmonic triad[1], which has the exponent $2^2 * 3 * 5 = 60$ - a number 144 times smaller than the 8640 of before. From this, it seems that the addition of the note *f* ruins the harmony of this consonance too much for it to have a place in music. However, to the ear's judgment, this dissonance is at worst unpleasant, and has been used in music with great success. It even seems that musical composition acquires a certain force from it, and without it would be too smooth and dull. Here we have a paradox - the theory seems to be in contradiction with the practice – of which I will try to give an explanation.

1 *More commonly known as a major triad, in this case G Major*

5. Mr. Alembert[1], in his treatise on musical composition, appears to have the same sentiment in regards to this dissonance. To him it seems too rough sounding on its own, and contradictory to the principals of harmony. But he believes that a different peculiar circumstance allows it to be tolerated in music. He observes that this chord *G, H, d, f,* is only used when the composition is written in the key of *C*, and he believes that the *f* is added to keep listeners' attention fixed on *C.* This way the listeners do no think that the composition has passed to the key of *G,* since it is in that key that the chord *G, H, d*, is the main consonance. Following this explanation, it is not by some principle of harmony that we use the dissonance *G, H, d, f,* but only to indicate to the listeners that the piece being played must be in the key of *C*. Without this guidance, we could be mistaken and believe that the harmony must be part of the key of *G*. For the same reason he says that when using the

chord *F, A, c,* the tone *d* is added (which is the *sixth* of *F*), so that listeners do not think that the piece has turned into the key of *F*.

[1] *This is the French mathematician and physicist Jean le Rond d'Alembert*

6. I highly doubt that everybody will agree with this explanation, for it seems to me that it is too arbitrary and removed from real harmony principles. If it were absolutely necessary that each chord represented the entire system of notes of the key being played in, we should be able to use them all at once, but that would create without question a terrible effect in music. However, the uncertainty of Mr. Alembert's conclusion truly begins with the following observation: The chord *G, H, d, f*, being listened to alone, without being tied to other tones, does not shock the ears very much. It would seem that it must do so because of the large ratios that it contains[1]. It is certain that the majority of ears are not capable of appreciating intervals so complicated, yet we see that almost everyone finds this chord somewhat agreeable. The goal is therefore to discover the physical cause of this paradoxical phenomenon.

[1] *These ratios create the large exponent 8640*

7. To start, I will remark that we must distinguish the proportions that our ears actually hear from those that tones expressed in numbers contain. Nothing happens more often in music than when the ear senses a proportion much different than the one that actually exists among the tones. In equal temperament where all 12 intervals of an octave are equal, there are no exact consonances except for the octaves[1]. The fifth here is expressed by the irrational ratio of 1 to $\sqrt[12]{2^7}$, which hardly differs from the proportion of 2 to 3. However, whatever the instrument tuned by this rule, the ear is not bothered by this irrational proportion[2], and in hearing the interval *C:G* still thinks the ratio is 2 to 3. If it were possible for the ear to sense the true tonal proportion, it would be much more shocked than when it hears the strongest dissonance, like that of the false fifth[3].

[1] *That is to say, only every octave's ratio is a whole number. When Euler writes "equal temperament," he is discussing what is known as "12 tone equal temperament" (12 TET), the most common tuning of western music today, rather than the "just intonation" of before.*

[2] *Irrational in the mathematical sense only! There is actually a very rational explanation for these strange ratios in 12 TET.*

[3] *In modern language, the "tritone," an extremely dissonant interval obtained by taking the principal tone and adding a flattened fifth to it.*

8. When tuning in harmonic temperament[1], where the numbers already explained express the tones of an octave, some fifths are not perfect although the ear hears them as such. Thus, the interval[2] from *B* to *f* being contained in the proportion 675 to 1024 surpasses the proportions of the true fifth of 2 to 3, by an interval[3] of $\frac{2048}{2025}$. The ear, however, hardly distinguishes this from an exact fifth. In the same way, the interval from *A* to *d* contains the proportion 20 to 27, which the ear confuses as that of 3 to 4 even though the

difference is that of a *Comma*[4], expressed by the proportion 80:81. We also take the interval[5] of G_s to *c*, of which the ratio is 25:32, for a major third that has a proportion of 4:5, despite the difference between these ratios being 125 to 128. And I strongly doubt that in hearing the chord *d:f*, one would sense the proportion of 27 to 32 instead of 5 to 6, which is of course more simple.

Here we have the ordinary system[6]:

F	2^9	= 512
F_S	$2^2 * 3^3 * 5$	= 540
G	$2^6 * 3^2$	= 576
G_S	$2^3 * 3 * 5^2$	= 600
A	$2^7 * 5$	= 640
B	$3^3 * 5^2$	= 675
H	$2^4 * 3^2 * 5$	= 720
c	$2^8 * 3$	= 768
c_S	$2^9 * 5^2$	= 800
d	$2^5 * 3^3$	= 864
d_S	$2^2 * 3^2 * 5^2$	= 900
e	$2^6 * 3 * 5$	= 960
f	2^{10}	= 1024

[1] *"Harmonic temperament" seems to be a synonym for "just temperament."*

[2] *Remember, Euler's B actually B^b*

[3] *These small proportions (almost equal to 1) are just the ratios between the proportion of the notes actually played and that which the ear perceives. Here the expression is 1024/675 divided by 3/2 = 2048/2025. Because this ratio is nearly equal to one, there is hardly a perceptible difference in the two tones.*

[4] *Euler refers here to the comma known as a "syntonic comma," which is actually a perceptible interval with a ratio of 81/80.*

[5] *In Euler's writing G_s is equivalent to present-day $G^\#$, pronounced "G-sharp."*

[6] *This "system" is the just intonation frequencies of notes for Euler's time. With the lower C not listed having a frequency of 384, you can relate these frequencies to the fractions of the preface.*

9. It is sufficiently proven that the proportion perceived by the senses is often different than that which truly occurs in tones. Each time that this happens, the perceived proportion is simpler than the real, and the difference is so small that it escapes notice. Our hearing is accustomed to taking all proportions that differ very little from uncomplicated ratios as such. The simpler the proportion, the more sensitive our hearing is to noticing small aberrations. This is the reason why we can hardly stand any deviation in octaves; we demand that all octaves be exact and that they do not differ at all from doubling. Despite this fact, in a concert some octaves could be about a hundredth of a tone[1] too high or too low. I strongly doubt that the most sensitive ear would notice; it seems that one could suffer an even greater aberration without the ears being bothered.

[1] *It seems that a "hundredth of tone" refers to how much the ratio between the true proportion and the perceived proportion deviates from 1. If the deviation is 0.01, meaning the ratio is 1.01, then the difference between the two is a hundredth of a tone.*

10. In fifths we can bear an even stronger deviation; musicians agree the shift equal temperament contains is absolutely imperceptible. The error approaches about a hundredth of a tone[1]. In harmonic temperament, there are fifths that differ by a *comma* from the correct value, and the *comma* is worth about a tenth of the tone[2] expressed by the ratio of 8 to 9. This difference is detectable, and to seems to have caused the majority of musicians to embrace equal temperament where the error is 10 times smaller. Perhaps a half or a third of a *comma* would also be unbearable in the fifths. In major thirds, for which just intonation would have a ratio of 4 to 5, equal temperament deviates two-thirds of a *comma*. In minor thirds a difference of an entire *comma* is often allowed; for in harmonic temperament there are two types of minor thirds, one is expressed by a ratio of 5 to 6, and the other by 27 to 32. These two types are ordinarily taken as the same even though the difference is a *comma*.

[1] *The calculation here is 1024/675 divided by 3/2 = 2048/2025 = 1.011358… Note that they are off by approximately 0.01 or one hundredth of a tone.*

[2] *The calculation here is 9/8 divided by 81/80 = 1.11111… So they are off by about 0.1, or a tenth of a tone.*

11. However, we cannot know from these numbers to fix limits on tone deviations. Such limits would depend on the sensitivity of the listener, and fine and delicate ears certainly distinguish smaller differences than average ears. If men had judgment so exact that they could pick out the smallest aberrations, it would be the end of music. Where would we find musicians capable of executing tones so exact that there are no aberrations? Almost all chords would appear to these men as atrocious dissonances, while less delicate ears would fine them perfectly harmonious. It is therefore a great advantage for practical music that hearing is not so perfect and that it generously pardons the smallest errors in execution. Also, it is certain that the more exquisite the musical taste of the listeners, the more the execution needs to be exact; listeners with less delicate taste settle for a less elegant execution.

12. When the proportion between tones that we listen to is simple enough, such as 2:3, 3:4 or 4:5 etc., the perceived proportion is the same for all ears. But when the proportion is more complicated, even though it closely approaches a simple one, the ear will hear the simple proportion without noticing the small deviation. Thus two tones in the ratio of 1000 to 2001 would be taken for an octave, the perceived proportion being 1 to 2 exactly. In the same manner, two tones with a ratio of 200 to 301, or 200 to 299 will evoke the feeling of a perfect fifth. More generally, given a complicated ratio that describes the tones, the ear will substitute a close approximation that is

simpler. Thus the proportions heard are different than the true, and it is from them that we must judge the true harmony - not from the actual numbers.

13. Therefore, when we hear this chord *G, H, d, F,* expressed by the numbers 36, 45, 54, 64, a perfect ear will well understand the proportions contained in these numbers. But ears less perfect, to which the perception of these intervals is too difficult, will try to substitute other numbers that give simpler proportions. They will change nothing of the first three tones *G, H, d,* since they contain a perfect consonance, but I am inclined to believe that will substitute a 63 in place of the 64, so that all the numbers become divisible by 9. The numbers 4, 5, 6, 7, which are certainly easier on perception, now express the ratios of our four tones. Indeed, if we were presented with these two chords, one containing 36, 45, 54, 64 and the other 36, 45, 54, 63, we would have to have quite an ear to distinguish them, unless they were played at the same time. Outside of that case, these two chords will certainly have the same impression.

14. I therefore believe that in listening to the tones 36, 45, 54, 64, we imagine hearing 36, 45, 54, 63, or even 4, 5, 6, 7, seeing as the auditory effect is absolutely the same[1]. I am not sure if the following reasoning is sufficient to prove my point: If the ear hears the first numbers[2], the chord should not be changed even if other sounds were added contained in the same exponent, such as 40, 48, and 60. Now it is certain that by this addition the chord would change in nature and would become intolerable. From this, I conclude that the ear effectively hears the sound expressed by these small numbers 4, 5, 6, 7, whose exponent permits no other interpolation of tones. Thus, when the seventh chord given by *G, H, d, f,* is heard, in place of the tone *f,* we substitute another that is slightly flatter; whose ratio to the true is 63 to 64. It is true that this interval is a little bigger than a *comma*, but larger errors are often neglected, especially in chords with so many notes.

[1] *By this I assume he means the effect of 36, 45, 54, 63, is the same as 4, 5, 6, 7.*

[2] *The "first numbers" he refers to are 36, 45, 54, 64, which are the numbers of the true tones.*

15. It seems then that a chord such as *G, H, d, f,* is only allowed in music as if it corresponds to the numbers 4, 5, 6, 7, and that the ear substitutes in place of the note *f* another slightly lower one with a ratio of 64 to 63 in comparison to the *f*. It is one's mental judgment that attributes to this sound a value that it actually does not have. If in a musical instrument this note *f* was a little lower than the rules of harmony allow, I suspect that this same chord would produce a better sound. But the other chords that precede or follow it would assume the natural value of *f*, and it would be as if two different tones were being used that correspond to the numbers 64 and 63. Even though the *f* is in reality only one tone, the senses report differently[1]. Perhaps it is here that the rules of the preparation and resolution of dissonances are founded, to almost alert the listeners that it is the same sound even though it serves as

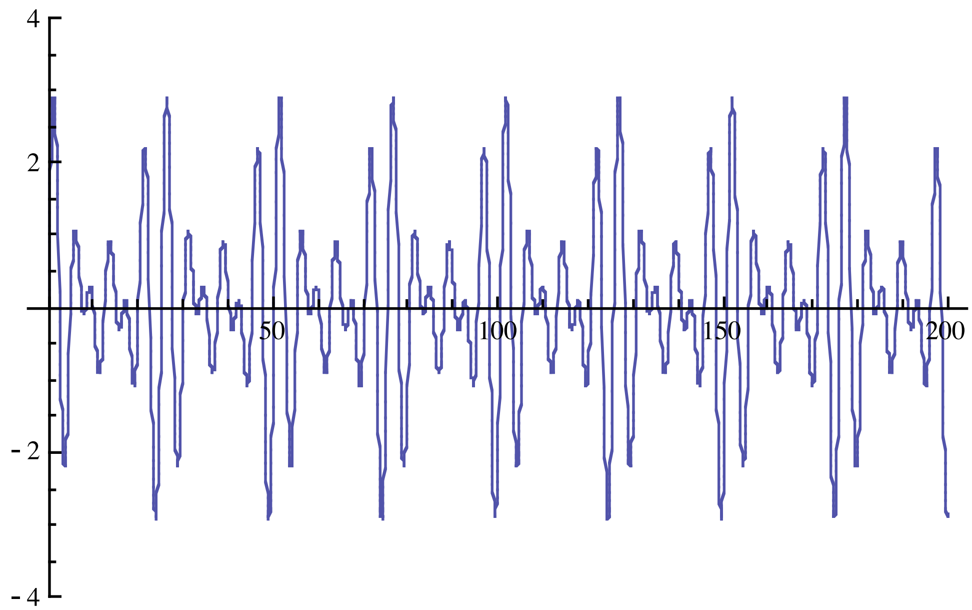
two different ones, so that they do not think that a new tone has been introduced.

[1] *He means that the ears would assume the natural value of f, 64, in all the other chords except for the seventh chord, even though his hypothetical tuning would actually contain 63 to make the seventh chord sound better.*

16. It is universally accepted that in music we do not use the proportions composed of 2, 3, and 5[1]. The great Leibnitz has already observed that in music, we have not yet learned how to count past 5. This is incontestably true in the instruments tuned according to the principles of harmony. But, if my conjecture is true, we could say that in composition we have already counted to 7, and that the ear is already accustomed to doing so. This is a new genre of music that has already begun and that was unknown to the ancients. In this kind of music, the chord 4, 5, 6, 7, is the most complete harmony, since it contains the numbers 2, 3, 5, and 7. But it is also more complicated than the perfect chord in the common genre that only contains the number 2, 3 and 5. If it is perfect in composition, then we might try to hold the instruments to the same level.

[1] *A strange statement that I spent a great deal of time puzzling over. I'm not quite sure what he means, but if anybody has any suggestions please contact me. I believe he means that intervals are not exact in tuning.*

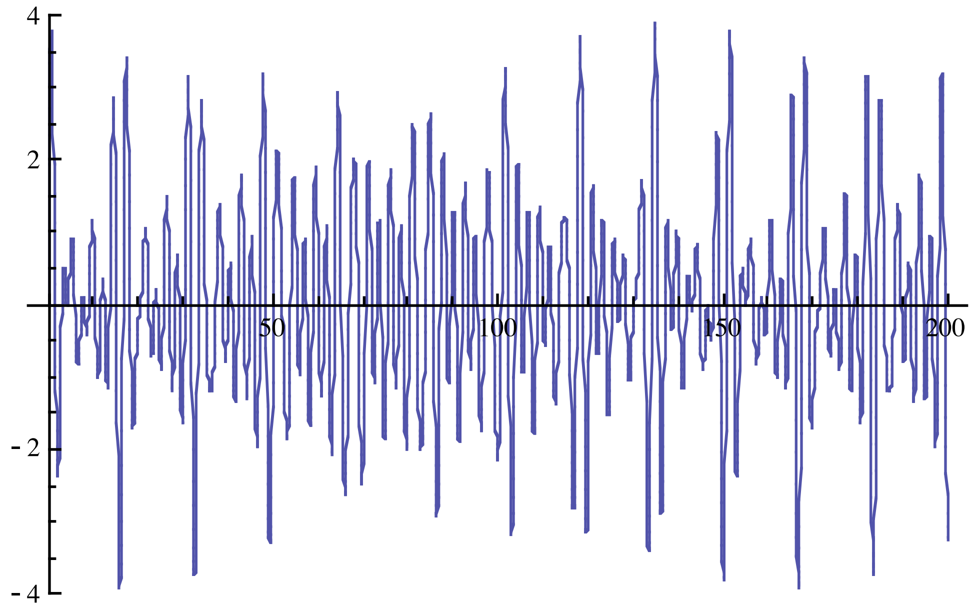
Translator's Afterword: I entered in the equations of sound waves corresponding to the chords that Euler has discussed into Mathematica in order to illustrate the complexities of proportions that occur in dissonant tones. A sound wave (a pressure wave propagating through a medium) can be characterized by the equation $y(x,t) = A\sin(kx - \omega t)$*, where* A *is the pressure amplitude,* x *is position,* k *is the wave number,* ω *is the angular frequency and* t *is time. We add together the sine waves of different frequencies to have an idea of what hits our ears when a chord is played. For example, a major triad would look like so:*



This is the graph of the equation $y(x=0,t) = \sin(t) + \sin\left(\frac{5}{4}t\right) + \sin\left(\frac{3}{2}t\right)$*. Note that there is a nice periodic result. According to Euler, it is this simplicity that is related to consonance.*

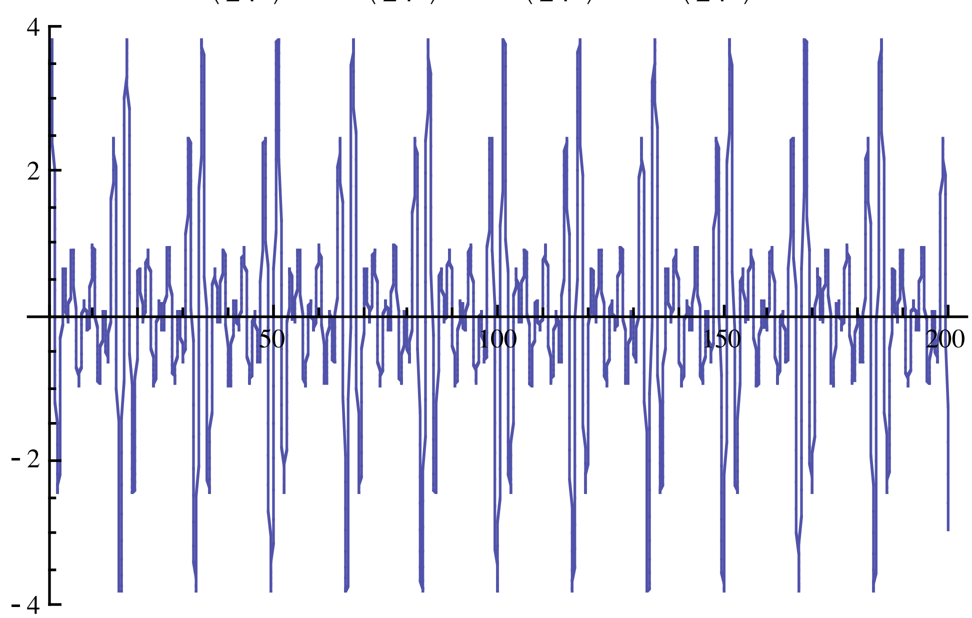
A seventh chord in just intonation could have an equation such as

$y(x=0,t) = \sin\left(\frac{36}{24}t\right) + \sin\left(\frac{45}{24}t\right) + \sin\left(\frac{54}{24}t\right) + \sin\left(\frac{64}{24}t\right)$*:*



The division by 24 is only for the purpose of graphing and does not affect the relationship between the tones that make up the chord. Note that it is not so elegant in its periodicity. However, the deviation is so slight that Euler claims our ear changes the final tone (corresponding to the f with the 64) so that the final note has a value of 63. If our ear were sensitive enough, the true chord would sound horrible, yet he claims that the ear hears the equation

$$y(x=0,t)=\sin\left(\frac{36}{24}t\right)+\sin\left(\frac{45}{24}t\right)+\sin\left(\frac{54}{24}t\right)+\sin\left(\frac{63}{24}t\right):$$



This would most surely have a more pleasant sound. Similar graphs can be constructed for all kinds of chords, and can further demonstrate the qualitative "look" of consonance vs. dissonance. If equal temperament were graphed, slight aberrations from simple periodicity would be observed, but the ear generally cannot hear them.