DE RESOLUTIONE AEQUATIONUM CUIUSVIS GRADUS

ON THE SOLUTION OF EQUATIONS OF ANY DEGREE

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Novi commentarii academiae scientiarum Petropolitanae **9** (1762/3), 1764, pp. 70-98 [E282]. The Summary appears on pp. 13-16 of the same issue. [1]

Summary

This investigation concerns algebraic equations, the degree of which is defined as the highest power of the unknown quantity whose value is thereby to be determined; thus, after equations of this type have been brought to their proper form, they can in general be represented according to degree in the following manner:

degree	*equations*
I.	$x + A = 0$
II.	$x^2 + Ax + B = 0$
III.	$x^3 + Ax^2 + Bx + C = 0$
IV.	$x^4 + Ax^3 + Bx^2 + Cx + D = 0$
V.	$x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$
VI.	$x^6 + Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F = 0$

Now it is known that the solution of these equations in general has so far not been investigated beyond the fourth degree, which is the more surprising because, although the second degree already had been explained by the most ancient Greek and Arab Geometers, and indeed the third and fourth degrees were explained long ago by Scipio Ferro and Bombelli in almost the very infancy of Analysis, as it were, from that time onwards, after Analysis was refined with great zeal, no progress has yet been permitted beyond these limits. Since, moreover, it is agreed that the solution of each degree depends on all lower degrees and the number of values taken by the unknown quantity is the same as the degree of the equation, the Eminent Author of this dissertation previously proposed a conjecture [2] that for any degree, such as the fifth

$$x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

an equation may be given, lower by one degree, such as

$$y^4 + \alpha y^3 + \beta y^2 + \gamma y + \delta = 0,$$

[1]The translators have incorporated the corrections and some of the footnotes made by Ferdinand Rudio in the version that appears in Vol. I.6 of Euler's *Opera Omnia*. We are grateful to William Dunham for his helpful comments and suggestions about this translation.

[2]Rudio points the reader to E30.

which he calls its resolvent, such that, if the roots of the latter are

$$p,\ q,\ r,\ s,$$

the root of the former will have the form

$$x = f + \sqrt[5]{p} + \sqrt[5]{q} + \sqrt[5]{r} + \sqrt[5]{s},$$

where it is, in fact, clear that $f = -\frac{1}{5}A$; which conjecture seems all the less unreasonable because it not only agrees excellently with the known solution of equations of the second, third and fourth degree, but also it encompasses the particular solvable cases of other degrees that were previously discovered by de Moivre. [3]

Now, however, the Eminent Author observes that the form of this very conjecture, whereby for example, the root of an equation of the fifth degree is expressed thus

$$x = f + \sqrt[5]{p} + \sqrt[5]{q} + \sqrt[5]{r} + \sqrt[5]{s},$$

is not yet sufficiently limited. For since these individual radical formulas by their nature involve five different values, one easily realizes that not all combinations of these can occur, because otherwise the number of different values of x would increase enormously, which nevertheless certainly cannot exceed five. Therefore he now restricts that form, which ranges too widely, such that he states that the root of an equation of the fifth degree in general is expressed as follows

$$x = f + \mathfrak{A}\sqrt[5]{p} + \mathfrak{B}\sqrt[5]{p^2} + \mathfrak{C}\sqrt[5]{p^3} + \mathfrak{D}\sqrt[5]{p^4}$$

and similarly for the remaining degrees; where now it is clear that there cannot be more than five different values for x. For as soon as the meaning of the part $\sqrt[5]{p}$ is defined, which can occur in five ways, the remaining parts are determined at the same time. Then it is furthermore evident that the expression for the case under consideration can encompass no more than five parts, since the further formulas $\sqrt[5]{p^5}$, $\sqrt[5]{p^6}$ etc. would automatically return to the preceding ones and would not involve a new irrational quantity.

He shows then how beautifully this new conjecture agrees with the solutions already known, and although, from this wellspring, one may not in the least effect a general solution of equations beyond the fourth degree, nevertheless from this he has deduced other additional solvable cases for higher degrees beyond those of de Moivre, whence not a little light seems to flood into this very obscure part of Analysis.

[3] Rudio cites A. de Moivre's *Aequationum Quarundam Potestatis Tertiae, Quintae, Septimae, Nonae, et Superiorum, ad Infinitum Usque Pergendo, in Terminis Finitis, ad Instar Regularum pro Cubicis Quae Vocantur Cardani, Resolutio Analytica*, Philosophical Transactions (London) **25**, 1707, pp. 2368-2371.

On the Solution of Equations of Any Degree

1. That which so far has been handed down concerning the solution of equations, if we look at general rules, extends only to equations which do not exceed the fourth degree, nor have rules thus far been discovered, with the aid of which equations of the fifth or any higher degree can be solved, so that all of Algebra is confined to equations of the first four orders. Moreover, this must be grasped about general rules that have been applied to all equations of the same degree; for in any degree, there are given infinitely many equations which can be solved through division into two or more equations of lesser degrees, whose roots, therefore, taken together, provide all the roots of those equations of higher degrees. Then indeed certain special equations in any degree have been observed by the celebrated de Moivre; which, even if they cannot be solved through division into factors, nevertheless one may write down their roots.

2. Moreover, from the known general solution of equations of the first, second, third, and fourth degree it is certainly agreed that equations of the first degree can be solved without any extraction of a root; but the solution of equations of the second degree already requires the extraction of a square root. The solution of equations of the third degree, however, involves the extraction of the square root as well as the cube root and the solution of the fourth degree further demands the extraction of the biquadratic root. [4] From this, moreover, one may safely conclude the solution of an equation of the fifth degree requires the general extraction of the fifth root besides all lesser roots; [5] and in general the root of an equation of any degree n will be expressed through a form that is composed of all the radical signs of degree n as well as lesser degrees.

3. Weighing carefully these matters at one point in *Comment. Acad. Imper. Petrop. Tomo* VI, [6] I ventured to propose a conjecture about the forms of the roots of every possible equation. For having proposed an equation of any degree

$$x^n + Ax^{n-2} + Bx^{n-3} + Cx^{n-4} + \text{etc.} = 0,$$

in which I assumed that the second term was absent, which indeed one may always assume, I conjectured an equation to be given, lower by one degree, such as

$$y^{n-1} + \mathfrak{A}y^{n-2} + \mathfrak{B}y^{n-3} + \mathfrak{C}y^{n-4} + \text{etc.} = 0,$$

which I called its resolvent, such that, if all the roots of this equation should be determined, which are

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon \quad \text{etc.},$$

whose number is $n-1$, from these the root of the original equation may be thus expressed, so that it would be

$$x = \sqrt[n]{\alpha} + \sqrt[n]{\beta} + \sqrt[n]{\gamma} + \sqrt[n]{\delta} + \text{etc.}$$

I confirmed this conjecture by showing that the solution of lower degree equations in fact follows from this general form; nor even now do I doubt that this conjecture is consistent with the truth.

[4] Translators: That is, the fourth root.

[5] Translators: Here and elsewhere, Euler uses the Latin term "radix surdesolida" to describe the fifth root.

[6] Rudio identifies this work as E30.

4. Aside from the fact that the discovery of the resolvent equation becomes very difficult, if the proposed equation goes beyond the fourth degree, and seems thus far in general to be beyond our capacity, just as the solution itself of the proposed equation does, so that, except for special forms similar to the cases of de Moivre it avails us nothing at all, I have additionally observed other difficulties in that form, which have led me to think that perhaps another form is given, not at all unlike the former one, which would not be subject to these difficulties and, accordingly, would give us more hope of at last penetrating further into this difficult task of Algebra. It will be no small advantage in this matter to have examined more carefully the true form of the roots of each equation.

5. In the form arrived at through the conjecture above, I note the flaw, first of all, that all the roots of the proposed equation are expressed with insufficient distinction. For although any radical sign $\sqrt[n]{\alpha}$ includes as many different values as the number n contains unities, so that if

$$\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d},\ \mathfrak{e}\quad \text{etc.}$$

denote all the values of formula $\sqrt[n]{1}$, one may write for $\sqrt[n]{\alpha}$ any of the formulas

$$\mathfrak{a}\sqrt[n]{\alpha},\ \ \mathfrak{b}\sqrt[n]{\alpha},\ \ \mathfrak{c}\sqrt[n]{\alpha},\ \ \mathfrak{d}\sqrt[n]{\alpha}\ \text{ etc.},$$

nevertheless it is clear that this variation in the single terms $\sqrt[n]{\alpha}$, $\sqrt[n]{\beta}$, $\sqrt[n]{\gamma}$, $\sqrt[n]{\delta}$ etc. cannot be established at will. For if the combination of these terms with the letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$, etc. were left to our discretion, then many more combinations would result than the equation contains roots, of which the number is $= n$.

6. Thus in order that the form of the root x displayed above may include all the roots together of the equation, it is necessary to restrict in some way the combinations of the terms $\sqrt[n]{\alpha}$, $\sqrt[n]{\beta}$, $\sqrt[n]{\gamma}$, $\sqrt[n]{\delta}$ etc. with the letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, etc., and to exclude combinations that are unsuitable for representing the roots of the equation. Indeed, from the solution of equations of the third and fourth degree we have seen that among the roots of unity of the same name $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ etc. a certain order ought to be established, according to which the combinations ought to be made as well. To this end, however, a similar order in the very members of the root $\sqrt[n]{\alpha}$, $\sqrt[n]{\beta}$, $\sqrt[n]{\gamma}$, $\sqrt[n]{\delta}$ etc. will have to be adhered to, so that the combination may be arranged. Indeed, because it is not obvious how in the roots of higher powers such an order ought to be established, this is doubtless a significant difficulty, with which the form resting on my conjecture struggles, which therefore I have proposed to eliminate in this dissertation. [7]

7. First, however, it will be fitting to establish a certain order in the roots of unity of any power, [8] whereby the very great variability of the combinations may be for the most part restricted. To which end I observe that, if any other value except unity of this $\sqrt[n]{1}$ is $= \mathfrak{a}$, then also $\mathfrak{a}^2$, $\mathfrak{a}^3$, $\mathfrak{a}^4$ etc. exhibit suitable values of this $\sqrt[n]{1}$; for if $\mathfrak{a}^n = 1$, then it will also be the case that $(\mathfrak{a}^2)^n = 1$, $(\mathfrak{a}^3)^n = 1$, $(\mathfrak{a}^4)^n = 1$ etc. Hence if the remaining roots are written as $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ etc., since in them are found $\mathfrak{a}^2$, $\mathfrak{a}^3$, $\mathfrak{a}^4$ etc., then a certain order is perceived whereby these letters ought to be distributed among themselves. Thus if after

[7] Translators: That is, Euler's conjecture that the root has the form $x = \sqrt[n]{\alpha} + \sqrt[n]{\beta} + \sqrt[n]{\gamma} + \sqrt[n]{\delta} + \text{etc.}$ encounters the difficulty that it is not clear how to combine $\sqrt[n]{\alpha}$, $\sqrt[n]{\beta}$, $\sqrt[n]{\gamma}$, $\sqrt[n]{\delta}$ etc. with $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ etc.; he will attempt to eliminate this difficulty by proposing, in Sec. 8, a different form for the root.

[8] Translators: That is, the nth roots of 1, for any n.

unity, which must be agreed to hold the first position, we begin from letter $\mathfrak{a}$, the values of the formula $\sqrt[n]{1}$ will be

$$1,\ \mathfrak{a},\ \mathfrak{a}^2,\ \mathfrak{a}^3,\ \mathfrak{a}^4,\ \cdots\ \mathfrak{a}^{n-1},$$

whose number is n; for more cannot occur, since $\mathfrak{a}^n = 1$, $\mathfrak{a}^{n+1} = \mathfrak{a}$, $\mathfrak{a}^{n+2} = \mathfrak{a}^2$ etc.; and a similar situation will obtain if, after unity, we should begin from any other letter $\mathfrak{b}$ or $\mathfrak{c}$ or $\mathfrak{d}$ etc. [9]

8. Hence therefore I justifiably suppose that such an order also exists in the terms expressing the root x of the equation, or the individual radical members have been arranged so that, with respect to each one of them, the rest are powers of it; [10] however, it will now be necessary to assign indefinite coefficients to the individual members. Therefore if the equation is deprived of its second term

$$x^n + Ax^{n-2} + Bx^{n-3} + Cx^{n-4} + Dx^{n-5} + \text{etc.} = 0,$$

it seems most probable [11] that any root of this equation is expressed in such a way that

$$x = \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \mathfrak{D}\sqrt[n]{v^4} + \cdots + \mathfrak{O}\sqrt[n]{v^{n-1}},$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. are either rational quantities or at least do not involve the radical sign $\sqrt[n]{\ }$, which certainly affects only the quantity v and its powers; much less would the quantity v itself involve such a sign.

9. From this form first it is clear that it cannot contain more than $n-1$ members; for even if we continue that series further according to its nature, the following terms will already be discovered contained in the preceding ones. For it will be the case that

$$\sqrt[n]{v^{n+1}} = v\sqrt[n]{v},\quad \sqrt[n]{v^{n+2}} = v\sqrt[n]{v^2}\ \text{etc.},$$

so that the irrationality involving the radical sign $\sqrt[n]{\ }$ does not allow more than $n-1$ different types. Therefore, even if that series is continued without bound, nevertheless, by joining terms of the same type by means of the theory of irrationality, all are reduced to terms that are $n-1$ in number. Therefore since we have already seen that more terms do not enter into the expression for the root, hence no slight evidence is obtained that this new form is quite consistent with the truth; its truth, moreover, will be much more confirmed by the following arguments.

10. This expression also automatically extends to equations in which the second term is not absent, while the previous expression requires the removal of the second term, from which very fact this new more natural expression is to be valued. For the continuation of irrational terms $\sqrt[n]{v}$, $\sqrt[n]{v^2}$, $\sqrt[n]{v^3}$ etc. also involves the rational terms $\sqrt[n]{v^0}$, $\sqrt[n]{v^n}$, which ought to be added on account of the second term of the equation. Hence more generally we shall be able to declare that, if a complete equation of any order n is proposed

$$x^n + \Delta x^{n-1} + Ax^{n-2} + Bx^{n-3} + Cx^{n-4} + \text{etc.} = 0,$$

[9] Translators: It is not clear whether Euler intends the values $1, \mathfrak{a}, \ldots, \mathfrak{a}^n$ to be distinct; if so, then $\mathfrak{a}$ must be a *primitive* nth root of 1.

[10] Translators: Euler's meaning here is unclear. In the formula for x that he introduces later in this paragraph, every radical term $\sqrt[n]{v^j}$ is a power of $\sqrt[n]{v}$, but not necessarily of *each* radical term $\sqrt[n]{v^i}$.

[11] Rudio: Euler made this conjecture in a letter to Christian Goldbach in 1752.

its root is expressed in a form of this type

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \mathfrak{D}\sqrt[n]{v^4} + \cdots + \mathfrak{O}\sqrt[n]{v^{n-1}},$$

where ω displays the rational part of the root, which is agreed to be $= -\frac{1}{n}\Delta$. The remaining terms, moreover, contain irrational parts involving the root of the power n, the number of which, inasmuch as they are different, cannot exceed $n-1$, entirely as it is understood in the earlier form. [12]

11. Hence we see further that if v is a quantity of the type, so that the root of the power n can be actually extracted from it, whether $\sqrt[n]{v}$ can be expressed either rationally or through radical signs of lower powers, then the irrationality of degree n is removed directly from the form of the root. Moreover this should happen inevitably as often as the proposed equation is resolvable into factors; for then no root will contain the radical sign $\sqrt[n]{\ }$. Wherefore since the nature of the matter demands that in these cases all radical signs $\sqrt[n]{\ }$ should disappear and be reduced to simpler signs, but from the prior form it may not be clear how, with the disappearance of one sign $\sqrt[n]{\alpha}$ of this kind, the remaining signs $\sqrt[n]{\beta}$, $\sqrt[n]{\gamma}$ etc. should disappear, this expression for this reason must be judged much more suitable to the nature of equations.

12. Additionally, this form, upon which the entire matter hinges, also shows all the roots of the equation without any ambiguity; nor do we linger further on the question of how, with all the radical signs $\sqrt[n]{\ }$, the equally numerous values of the root $\sqrt[n]{1}$ are to be combined. For if all the roots of unity of power n are 1, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ etc. and we combine $\sqrt[n]{v}$ with any one $\mathfrak{a}$ of them, since $\sqrt[n]{v}$ is at any rate $\mathfrak{a}\sqrt[n]{v}$, then for $\sqrt[n]{v^2}$, $\sqrt[n]{v^3}$, $\sqrt[n]{v^4}$ etc. should be written $\mathfrak{a}^2\sqrt[n]{v^2}$, $\mathfrak{a}^3\sqrt[n]{v^3}$, $\mathfrak{a}^4\sqrt[n]{v^4}$ etc. Moreover the constant term ω, because it represents the form $\omega\sqrt[n]{v^0}$, will change into $\mathfrak{a}^0\omega\sqrt[n]{v^0} = 1\cdot\omega$ because $\mathfrak{a}^0 = 1$ and therefore in all roots undergoes no change, as the remaining members do. Since this is self evident from the solution of all equations, we have hereby a new and sufficiently well established criterion of the truth of this new form, which seems to embrace the roots of all equations.

13. Hence moreover it is further evident how, if one root of any equation be known, all the remaining roots can be displayed; for this purpose, one must only know all the roots of unity of the same power or all values of $\sqrt[n]{1}$, whose number $= n$. And if these roots of unity be 1, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ etc. and one root of the equation has been found

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \cdots + \mathfrak{O}\sqrt[n]{v^{n-1}},$$

the remaining roots will be

$$x = \omega + \mathfrak{A}\mathfrak{a}\sqrt[n]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[n]{v^2} + \mathfrak{C}\mathfrak{a}^3\sqrt[n]{v^3} + \cdots + \mathfrak{O}\mathfrak{a}^{n-1}\sqrt[n]{v^{n-1}},$$

$$x = \omega + \mathfrak{A}\mathfrak{b}\sqrt[n]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[n]{v^2} + \mathfrak{C}\mathfrak{b}^3\sqrt[n]{v^3} + \cdots + \mathfrak{O}\mathfrak{b}^{n-1}\sqrt[n]{v^{n-1}},$$

$$x = \omega + \mathfrak{A}\mathfrak{c}\sqrt[n]{v} + \mathfrak{B}\mathfrak{c}^2\sqrt[n]{v^2} + \mathfrak{C}\mathfrak{c}^3\sqrt[n]{v^3} + \cdots + \mathfrak{O}\mathfrak{c}^{n-1}\sqrt[n]{v^{n-1}}$$

etc.

[12] Translators: A "root of the power n" is an nth root

and thus always as many roots are obtained as the exponent n, which designates the degree of the equation, contains unities.

14. Therefore by these arguments this new form of the roots now has been raised to the height of probability; and nothing else is required to demonstrate complete certitude but that a rule be discovered, through the aid of which for any proposed equation the form can be defined and the coefficients $\mathfrak{A}$,$\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. can be designated, along with the quantity v; if we could achieve this, we would doubtless have the general solution of all equations that has been sought through the heretofore futile efforts of all Geometers. Nor then do I give myself so much credit as to believe that I can discover this rule, but shall be content to have fully demonstrated that the roots of all equations are definitely contained in this form. Moreover, this without a doubt lends much illumination to the solution of equations, since, once the true form of the roots is known, the path of the investigation is rendered significantly easier, which one may not even embark upon as long as the form of the roots is unknown.

15. Although from the proposed equation itself we may not yet designate its root or the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. along with the quantity v, nonetheless the demonstration of its truth will equally succeed if in turn we should infer, from the assumed root, that equation of which it is a root. Moreover, this equation should be free of radical signs $\sqrt[n]{}$, since the equations whose roots are under investigation are generally assumed to consist of rational terms. The question then is reduced to this, that an equation of this type

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \cdots + \mathfrak{O}\sqrt[n]{v^{n-1}}$$

may be freed from irrationality or radical signs $\sqrt[n]{}$ and thence a rational equation may be derived, about which we will then certainly be able to assert that its root is the assumed expression itself; likewise we shall thence be able to write down the remaining roots which equally conform to the same equation. In this way then we shall at least be able to exhibit infinitely many equations whose roots will be known to us, and if these equations include among themselves general equations of all degrees, the solution of these will also be within our grasp.

16. Indeed it will seem we have produced little, if we present only more equations, whose roots can be designated, since it is agreed from first principles how an equation of any degree, which has given roots, should be formed. For if any number of formulas of the type $x - a$, $x - b$, $x - c$ etc. should be multiplied on themselves in turn, surely an equation will be obtained whose roots are going to be $x = a$, $x = b$, $x = c$ etc.; but the formation of such an equation produces little of profit for the solution of equations. First however I observe that other equations are not produced in this way except those which are going to have factors; but the solution of equations that can be resolved into factors is afflicted with no difficulty. Of no greater importance in this matter are equations that are produced from the multiplication of two or more equations of lower degree, the solution of which clearly produces nothing towards achieving a general solution.

17. But if from our form

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \text{etc.}$$

we should arrive at a rational equation, it surely will have factors that are not rational; for if it were to have such a factor, its roots, which would likewise be the roots of equations of lesser degrees, would not involve the radical sign $\sqrt[n]{}$. He must be thought exceedingly distinguished who wrote down the roots of an equation of some higher degree that cannot be resolved into factors. Also for this reason the celebrated de Moivre deserves great thanks because from particular degrees of equations he has displayed one that is irresolvable into factors whose roots can be written down; and if his formulas were extended more widely, they would doubtless have a much greater utility, while on the other hand clearly no result can be attributed in this matter to equations resolvable into factors.

18. But yet let us return to that form to be freed from the irrationality of the sign $\sqrt[n]{}$, and if we should consider the customary methods of eliminating radical signs, the resulting equation would often seem to go up to very many dimensions. For if there were a single radical sign present, for example

$$x = \omega + \mathfrak{A}\sqrt[n]{v},$$

then a rational equation could go up to n dimensions of the same x, whence it seems to be going to increase to many more dimensions if more radical signs of this sort should be present; which doubtless ought to be the result if those radical signs were not wholly mutually dependent on each another. But because all are powers of the first, I shall show that perfect rationality can be obtained without ascending beyond the power of the exponent n. Surely thus I shall demonstrate that the form

$$x = \omega + \mathfrak{A}\sqrt[n]{v} + \mathfrak{B}\sqrt[n]{v^2} + \mathfrak{C}\sqrt[n]{v^3} + \cdots + \mathfrak{O}\sqrt[n]{v^{n-1}}$$

can so be freed from irrationality in such a way that the rational equation thereby resulting does not surpass the power x^n. Therefore there will result an equation of this form

$$x^n + \Delta x^{n-1} + Ax^{n-2} + Bx^{n-3} + \text{etc.} = 0,$$

whose root will be that assumed form; and because the number of roots of this equation is $= n$, from the same form we shall be able to write down all the roots of this equation.

19. Since this now is the outstanding criterion of the truth of this form, it will be advantageous also to have noted, since the form of the root contains $n-1$ arbitrary quantities, that just as many arbitrary quantities also enter into the rational equation, whence it is clear that these quantities can be determined in such a way that a rational equation may have the coefficients Δ, A, B, C etc., which are thereby given, that is, that a general equation of this degree is obtained. If this determination can in fact be established, we shall then find a general solution of equations of any degree, from which at least the possibility of reaching a solution in this way shines forth. Significant difficulties indeed will occur in this undertaking, which we shall recognize the more clearly if, beginning from the most simple, we should adjust our form to any degree. Moreover taking into consideration the simplicity and elegance of the calculation, let us omit the rational part ω of the root, so that in any degree we may reach rational equations of this sort, in which the second term is absent, by which very fact the scope of the solution must not be considered to be restricted.

I. SOLUTION OF EQUATIONS OF THE SECOND DEGREE

20. Thus, in order that we may begin from equations of the second degree, if we let $n = 2$ and set $\omega = 0$, our form of the root will be

$$x = \mathfrak{A}\sqrt{v},$$

which made rational gives

$$xx = \mathfrak{A}\mathfrak{A}v.$$

Let this equation be compared with the general form of the second degree

$$xx = A$$

with the second term absent, with the result that

$$\mathfrak{A}\mathfrak{A}v = A;$$

in order that this may be satisfied let it be stipulated that $\mathfrak{A} = 1$, and the result will be

$$v = A.$$

Whence, given the proposed equation $xx = A$, if it be assumed that $\mathfrak{A} = 1$ and $v = A$, one root of it will be

$$x = \mathfrak{A}\sqrt{v} = \sqrt{A},$$

and because $\sqrt{1}$ has two values, 1 and -1, a second root will be

$$x = -\mathfrak{A}\sqrt{v} = -\sqrt{A},$$

which indeed is self evident.

II. SOLUTION OF EQUATIONS OF THE THIRD DEGREE

21. Now having set $n = 3$, the form of the root for this case will be

$$x = \mathfrak{A}\sqrt[3]{v} + \mathfrak{B}\sqrt[3]{v^2};$$

whence, so that a rational equation may be derived, first let the cube be taken

$$x^3 = \mathfrak{A}^3 v + 3\mathfrak{A}\mathfrak{A}\mathfrak{B}v\sqrt[3]{v} + 3\mathfrak{A}\mathfrak{B}\mathfrak{B}v\sqrt[3]{v^2} + \mathfrak{B}^3 v^2.$$

Let now this cubic equation be considered

$$x^3 = Ax + B,$$

whence by substituting the assumed value for x, also will arise

$$x^3 = A\mathfrak{A}\sqrt[3]{v} + A\mathfrak{B}\sqrt[3]{v^2} + B,$$

which form is to be rendered equal to the former by setting equal to each other the rational, as well as the irrational parts of each type $\sqrt[3]{v}$ and $\sqrt[3]{v^2}$.

22. Moreover, a comparison of rational terms produces

$$B = \mathfrak{A}^3 v + \mathfrak{B}^3 v^2$$

and from the comparison of irrationals the result is

$$A\mathfrak{A} = 3\mathfrak{A}\mathfrak{A}\mathfrak{B}v \quad \text{and} \quad A\mathfrak{B} = 3\mathfrak{A}\mathfrak{B}\mathfrak{B}v,$$

each of which equations yields

$$A = 3\mathfrak{A}\mathfrak{B}v.$$

Hence, if this cubic equation is proposed

$$x^3 = 3\mathfrak{A}\mathfrak{B}vx + \mathfrak{A}^3 v + \mathfrak{B}^3 v^2,$$

one root of it will be

$$x = \mathfrak{A}\sqrt[3]{v} + \mathfrak{B}\sqrt[3]{v^2},$$

and if 1, $\mathfrak{a}$, $\mathfrak{b}$, are the three cube roots of unity, the two remaining roots will be

$$x = \mathfrak{A}\mathfrak{a}\sqrt[3]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[3]{v^2}, \quad x = \mathfrak{A}\mathfrak{b}\sqrt[3]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[3]{v^2}.$$

It is moreover the case that

$$\mathfrak{a} = \mathfrak{b}^2 = \frac{-1+\sqrt{-3}}{2} \quad \text{and} \quad \mathfrak{b} = \mathfrak{a}^2 = \frac{-1-\sqrt{-3}}{2}.$$

23. Moreover, if the cubic equation

$$x^3 = Ax + B,$$

be proposed, from the coefficients A and B, the quantities $\mathfrak{A}$, $\mathfrak{B}$ and v can in turn be determined, so that thereby all three roots of this equation may be obtained. To this end, moreover, because only two equations must be satisfied, one of the letters $\mathfrak{A}$ and $\mathfrak{B}$ can be taken at will. Therefore let $\mathfrak{A} = 1$ and the equation

$$A = 3\mathfrak{A}\mathfrak{B}v = 3\mathfrak{B}v$$

produces

$$\mathfrak{B} = \frac{A}{3v},$$

whence occurs

$$\mathfrak{B}^3 = \frac{A^3}{27v^3},$$

which value substituted in the first equation

$$B = v + \mathfrak{B}^3v^2$$

gives

$$B = v + \frac{A^3}{27v} \quad \text{or} \quad vv = Bv - \frac{1}{27}A^3,$$

whence it follows that

$$v = \frac{1}{2}B \pm \sqrt{\frac{1}{4}BB - \frac{1}{27}A^3};$$

moreover, this is the case, whichever of these two values is employed.

24. Moreover, once the value of $v = \frac{1}{2}B \pm \sqrt{\frac{1}{4}BB - \frac{1}{27}A^3}$ has been discovered, it will be the case that $\mathfrak{B} = \frac{A}{3v}$ and $\mathfrak{B}\sqrt[3]{v^2} = \frac{A}{3\sqrt[3]{v}}$ and hence the three roots of the proposed equation

$$x^3 = Ax + B$$

will be

$$\text{I.} \quad x = \sqrt[3]{v} + \frac{A}{3\sqrt[3]{v}}, \qquad \text{II.} \quad x = \mathfrak{a}\sqrt[3]{v} + \frac{\mathfrak{b}A}{3\sqrt[3]{v}}, \qquad \text{III.} \quad x = \mathfrak{b}\sqrt[3]{v} + \frac{\mathfrak{a}A}{3\sqrt[3]{v}}.$$

However, since it is the case that

$$\frac{1}{v} = \frac{\frac{1}{2}B \mp \sqrt{\frac{1}{4}BB - \frac{1}{27}A^3}}{\frac{1}{27}A^3},$$

it will follow that

$$\sqrt[3]{v} = \sqrt[3]{\frac{1}{2}B \pm \sqrt{\frac{1}{4}BB - \frac{1}{27}A^3}}$$

and

$$\frac{A}{3\sqrt[3]{v}} = \sqrt[3]{\frac{1}{2}B \mp \sqrt{\frac{1}{4}BB - \frac{1}{27}A^3}}$$

and hence will arise the familiar formulas for the solution of cubic equations.

III. SOLUTION OF EQUATIONS OF THE FOURTH DEGREE

25. Having set $n = 4$, let us consider this form of the root

$$x = \mathfrak{A}\sqrt[4]{v} + \mathfrak{B}\sqrt[4]{v^2} + \mathfrak{C}\sqrt[4]{v^3}$$

and let us seek an equation of the fourth degree of which this form is a root. Indeed in this case a calculation is easily devised whereby irrationalities are removed; for because $\sqrt[4]{v^2} = \sqrt{v}$, let this equation be assumed

$$x - \mathfrak{B}\sqrt{v} = \mathfrak{A}\sqrt[4]{v} + \mathfrak{C}\sqrt[4]{v^3},$$

which, when squared, gives

$$xx - 2\mathfrak{B}x\sqrt{v} + \mathfrak{B}\mathfrak{B}v = \mathfrak{A}\mathfrak{A}\sqrt{v} + 2\mathfrak{A}\mathfrak{C}v + \mathfrak{C}\mathfrak{C}v\sqrt{v},$$

which, when the irrational parts are transferred to the same side, becomes

$$xx + (\mathfrak{B}\mathfrak{B} - 2\mathfrak{A}\mathfrak{C})v = 2\mathfrak{B}x\sqrt{v} + (\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v)\sqrt{v},$$

and if they are assumed to be squared again, this rational equation will emerge

$$\begin{aligned} &x^4 + 2(\mathfrak{B}\mathfrak{B} - 2\mathfrak{A}\mathfrak{C})vxx + (\mathfrak{B}\mathfrak{B} - 2\mathfrak{A}\mathfrak{C})^2vv \\ &= 4\mathfrak{B}\mathfrak{B}vxx + 4(\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v)\mathfrak{B}vx + (\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v)^2v, \end{aligned}$$

which, arranged in order, will result in this form

$$\begin{aligned} x^4 = {}& 2(\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})vxx + 4(\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v)\mathfrak{B}vx \\ &+ \mathfrak{A}^4v - \mathfrak{B}^4vv + \mathfrak{C}^4v^3 + 4\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv - 2\mathfrak{A}\mathfrak{A}\mathfrak{C}\mathfrak{C}vv. \end{aligned}$$

26. Therefore one root of this biquadratic equation is

$$x = \mathfrak{A}\sqrt[4]{v} + \mathfrak{B}\sqrt[4]{v^2} + \mathfrak{C}\sqrt[4]{v^3},$$

and if the biquadratic roots of unity should be denoted 1, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, such that

$$\mathfrak{a} = +\sqrt{-1}, \quad \mathfrak{b} = -1 \quad \text{and} \quad \mathfrak{c} = -\sqrt{-1},$$

it will be the case that

$$\begin{aligned} \mathfrak{a}^2 &= -1 = \mathfrak{b}, & \mathfrak{a}^3 &= -\sqrt{-1} = \mathfrak{c}, \\ \mathfrak{b}^2 &= +1, & \mathfrak{b}^3 &= -1 = \mathfrak{b}, \\ \mathfrak{c}^2 &= -1 = \mathfrak{b}, & \mathfrak{c}^3 &= +\sqrt{-1} = \mathfrak{a}, \end{aligned}$$

whence the three remaining roots of the same equation will be

$$x = \mathfrak{A}\mathfrak{a}\sqrt[4]{v} + \mathfrak{B}\mathfrak{b}\sqrt[4]{v^2} + \mathfrak{C}\mathfrak{c}\sqrt[4]{v^3},$$

$$x = \mathfrak{A}\mathfrak{b}\sqrt[4]{v} + \mathfrak{B}\sqrt[4]{v^2} + \mathfrak{C}\mathfrak{b}\sqrt[4]{v^3},$$

$$x = \mathfrak{A}\mathfrak{c}\sqrt[4]{v} + \mathfrak{B}\mathfrak{b}\sqrt[4]{v^2} + \mathfrak{C}\mathfrak{a}\sqrt[4]{v^3}.$$

27. Thus, moreover, in turn any biquadratic equation can be reduced to that form and its roots can be written down. For, let this equation be proposed

$$x^4 = Axx + Bx + C$$

and the values of the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ along with the quantity v are to be sought, and once they have been found the roots of this equation will become known. Moreover, it will be the case that

$$\begin{aligned} A &= 2(\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})v, \\ B &= 4(\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v)\mathfrak{B}v, \\ C &= \mathfrak{A}^4 v - \mathfrak{B}^4 vv + \mathfrak{C}^4 v^3 + 4\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv - 2\mathfrak{A}\mathfrak{A}\mathfrak{C}\mathfrak{C}vv \end{aligned}$$

or

$$C = (\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v)^2 v - (\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})^2 vv + 8\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv.$$

Thus moreover it follows that

$$(\mathfrak{B}\mathfrak{B} + 2\mathfrak{A}\mathfrak{C})v = \frac{1}{2}A$$

and

$$\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v = \frac{B}{4\mathfrak{B}v},$$

which values here substituted give

$$C = \frac{BB}{16\mathfrak{B}\mathfrak{B}v} - \frac{1}{4}AA + 8\mathfrak{A}\mathfrak{B}\mathfrak{B}\mathfrak{C}vv.$$

Moreover, the first formula produces $4\mathfrak{A}\mathfrak{C}v = A - 2\mathfrak{B}\mathfrak{B}v$, which value again substituted yields

$$C = \frac{BB}{16\mathfrak{B}\mathfrak{B}v} - \frac{1}{4}AA + 2A\mathfrak{B}\mathfrak{B}v - 4\mathfrak{B}^4 vv,$$

so that now two letters $\mathfrak{A}$ and $\mathfrak{C}$ have been eliminated.

28. Since up to this point two unknowns $\mathfrak{B}$ and v remain, the value of $\mathfrak{B}$ is left to our discretion. Therefore let $\mathfrak{B} = 1$ and the quantity v ought to be determined from the following cubic equation

$$v^3 - \frac{1}{2}Av^2 + \frac{1}{4}(C + \frac{1}{4}AA)v - \frac{1}{64}BB = 0.$$

Moreover once the root v has been hereby discovered, the letters $\mathfrak{A}$ and $\mathfrak{C}$ should be sought from the prior equations. Since therefore it is the case that

$$\mathfrak{A}\mathfrak{A} + \mathfrak{C}\mathfrak{C}v = \frac{B}{4v} \quad \text{and} \quad 2\mathfrak{A}\mathfrak{C}\sqrt{v} = \frac{A - 2v}{2\sqrt{v}},$$

by applying addition as well as subtraction and extracting the square root, it will be the case that

$$\mathfrak{A} + \mathfrak{C}\sqrt{v} = \sqrt{\frac{B}{4v} + \frac{A}{2\sqrt{v}} - \sqrt{v}}$$

and

$$\mathfrak{A} - \mathfrak{C}\sqrt{v} = \sqrt{\frac{B}{4v} - \frac{A}{2\sqrt{v}} - \sqrt{v}},$$

whence it will be found that

$$\mathfrak{A} = \frac{1}{4\sqrt{v}}\sqrt{B + 2A\sqrt{v} - 4v\sqrt{v}} + \frac{1}{4\sqrt{v}}\sqrt{B - 2A\sqrt{v} + 4v\sqrt{v}}$$

and

$$\mathfrak{C} = \frac{1}{4\sqrt{v}}\sqrt{B + 2A\sqrt{v} - 4v\sqrt{v}} - \frac{1}{4\sqrt{v}}\sqrt{B - 2A\sqrt{v} + 4v\sqrt{v}}.$$

29. Since it is the case that

$$\mathfrak{A}\sqrt[4]{v} \pm \mathfrak{C}\sqrt[4]{v^3} = \left(\mathfrak{A} \pm \mathfrak{C}\sqrt{v}\right)\sqrt[4]{v},$$

there will be, of the proposed equation

$$x^4 = Axx + Bx + C,$$

after the value v has been found from the equation

$$v^3 - \frac{1}{2}Av^2 + \frac{1}{4}(C + \frac{1}{4}AA)v - \frac{1}{64}BB = 0,$$

four roots

$$\text{I.}\quad x = \sqrt{v} + \frac{1}{2\sqrt{v}}\sqrt{B\sqrt{v} + 2Av - 4vv},$$

$$\text{II.}\quad x = \sqrt{v} - \frac{1}{2\sqrt{v}}\sqrt{B\sqrt{v} + 2Av - 4vv},$$

$$\text{III.}\quad x = -\sqrt{v} + \frac{1}{2\sqrt{v}}\sqrt{-B\sqrt{v} + 2Av - 4vv},$$

$$\text{IV.}\quad x = -\sqrt{v} - \frac{1}{2\sqrt{v}}\sqrt{-B\sqrt{v} + 2Av - 4vv}.$$

In this manner, it is agreed, the solution of a biquadratic equation is reduced to the solution of a cubic equation.

IV. SOLUTION OF EQUATIONS OF THE FIFTH DEGREE

30. Having set $n = 5$, our form of the root will be

$$x = \mathfrak{A}\sqrt[5]{v} + \mathfrak{B}\sqrt[5]{vv} + \mathfrak{C}\sqrt[5]{v^3} + \mathfrak{D}\sqrt[5]{v^4}$$

and first should be sought the equation of the fifth degree of which this will be the root, or, which amounts to the same thing, the radical signs should be eliminated from this form. In this, however, the highest difficulty presents itself, since this operation of elimination can in no way be carried out by the method which I have employed in equations of the fourth degree. This is indeed clear, because all powers of x involve the same radical signs, if the sought after equation be expressed

$$x^5 = Ax^3 + Bx^2 + Cx + D,$$

then, by substituting an assumed value for x, four equations may be obtained, by the aid of which, the four radical signs may be eliminated; but then these assumed letters A , B, C, and D will be determined individually with the greatest difficulty.

31. Having considered these difficulties, I have come upon another way of carrying out this operation, which is constructed in such a way that it extends equally to all forms of roots, of whatever degree they may be, and from which at the same time it will be clearly evident that the rational equation never will go beyond the degree which is indicated by the exponent n. Moreover, this method rests upon the very nature of equations, whereby the coefficients of the individual terms are determined from all the roots. Since, therefore, all five roots of the equation that we seek are agreed upon, from them also the coefficients of its individual terms can be formed by known rules. Let then 1, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ be the five fifth roots of unity or the roots of this equation $z^5 - 1 = 0$ and by letting α, β, γ, δ, ϵ denote the roots of the equation that we seek, it will be the case that

$$\begin{aligned}
\alpha &= \mathfrak{A}\sqrt[5]{v} + \mathfrak{B}\sqrt[5]{v^2} + \mathfrak{C}\sqrt[5]{v^3} + \mathfrak{D}\sqrt[5]{v^4},\\
\beta &= \mathfrak{A}\mathfrak{a}\sqrt[5]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{a}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{a}^4\sqrt[5]{v^4},\\
\gamma &= \mathfrak{A}\mathfrak{b}\sqrt[5]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{b}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{b}^4\sqrt[5]{v^4},\\
\delta &= \mathfrak{A}\mathfrak{c}\sqrt[5]{v} + \mathfrak{B}\mathfrak{c}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{c}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{c}^4\sqrt[5]{v^4},\\
\epsilon &= \mathfrak{A}\mathfrak{d}\sqrt[5]{v} + \mathfrak{B}\mathfrak{d}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{d}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{d}^4\sqrt[5]{v^4}.
\end{aligned}$$

32. Once these five roots have been set forth, if an equation of the fifth degree having these roots is set down

$$x^5 - \Delta x^4 + Ax^3 - Bx^2 + Cx - D = 0,$$

these coefficients [13] are determined from the roots α, β, γ, δ, ϵ, such that

$\Delta =$ the sum of the roots,
$A =$ the sum of the products, taken in pairs,
$B =$ the sum of the products, taken in threes,
$C =$ the sum of the products, taken in fours,
$D =$ the product of all five [roots].

However, in order that we may more easily find these values, let us deduce them from the sums of the the powers of the roots. Therefore let

$$\begin{aligned}
P &= \alpha + \beta + \gamma + \delta + \epsilon,\\
Q &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2,\\
R &= \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3,\\
S &= \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \epsilon^4,\\
T &= \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \epsilon^5.
\end{aligned}$$

[13] Translators: We have corrected a misprint in the version of this paper that appears in *Opera Omnia*, which gives the fourth term of this equation as Bx^3, whereas Euler wrote Bx^2.

For once these values have been determined, it will be the case, as we know, [14] that

$$\begin{aligned} \Delta &= P, \\ A &= \frac{\Delta P - Q}{2}, \\ B &= \frac{AP - \Delta Q + R}{3}, \\ C &= \frac{BP - AQ + \Delta R - S}{4}, \\ D &= \frac{CP - BQ + AR - \Delta S + T}{5}. \end{aligned}$$

33. Now to investigate the values P, Q, R, S, T we ought first to collect into one sum all the powers of the roots of unity 1, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$; since these are the roots of the equation $z^5 - 1 = 0$, it will be the case that

$$\begin{aligned} 1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} &= 0, \\ 1 + \mathfrak{a}^2 + \mathfrak{b}^2 + \mathfrak{c}^2 + \mathfrak{d}^2 &= 0, \\ 1 + \mathfrak{a}^3 + \mathfrak{b}^3 + \mathfrak{c}^3 + \mathfrak{d}^3 &= 0, \\ 1 + \mathfrak{a}^4 + \mathfrak{b}^4 + \mathfrak{c}^4 + \mathfrak{d}^4 &= 0, \\ 1 + \mathfrak{a}^5 + \mathfrak{b}^5 + \mathfrak{c}^5 + \mathfrak{d}^5 &= 5. \end{aligned}$$

The sums of the sixth, seventh, etc. powers, up to the tenth again disappear, but the sum of the tenth again becomes $= 5$, since $\mathfrak{a}^5 = 1$, $\mathfrak{b}^5 = 1$, $\mathfrak{c}^5 = 1$ and $\mathfrak{d}^5 = 1$. For the sake of brevity in this calculation we shall be able clearly to omit the radical signs, provided that in turn we remember $\sqrt[5]{v}$, $\sqrt[5]{v^2}$, $\sqrt[5]{v^3}$, and $\sqrt[5]{v^4}$ must be joined with the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$. [15]

34. Now then by adding the roots α, β, γ, δ, ϵ we shall have

$$P = \mathfrak{A}(1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d}) + \mathfrak{B}(1 + \mathfrak{a}^2 + \mathfrak{b}^2 + \mathfrak{c}^2 + \mathfrak{d}^2) + \text{etc.} = 0.$$

Moreover, by taking the remaining powers, we shall further deduce

$$\begin{aligned} P &= 0, \\ Q &= 10(\mathfrak{A}\mathfrak{D} + \mathfrak{B}\mathfrak{C}), \\ R &= 15(\mathfrak{A}\mathfrak{A}\mathfrak{C} + \mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}\mathfrak{D}^2 + \mathfrak{C}^2\mathfrak{D}), \\ S &= 20(\mathfrak{A}^3\mathfrak{B} + \mathfrak{A}\mathfrak{C}^3 + \mathfrak{B}^3\mathfrak{D} + \mathfrak{C}\mathfrak{D}^3) + 30(\mathfrak{A}^2\mathfrak{D}^2 + \mathfrak{B}^2\mathfrak{C}^2) + 120\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}, \\ T &= 5(\mathfrak{A}^5 + \mathfrak{B}^5 + \mathfrak{C}^5 + \mathfrak{D}^5) + 100(\mathfrak{A}^3\mathfrak{C}\mathfrak{D} + \mathfrak{A}\mathfrak{B}^3\mathfrak{C} + \mathfrak{B}\mathfrak{C}^3\mathfrak{D} + \mathfrak{A}\mathfrak{B}\mathfrak{D}^3) \\ &\quad + 150(\mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + \mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}^2\mathfrak{D}). \end{aligned}$$

Here other products do not occur, except those which, by adding in the radical signs, produce a rational power of v; or if we should assign one dimension to letter $\mathfrak{A}$, two dimensions to letter $\mathfrak{B}$, three to letter $\mathfrak{C}$ and four to letter $\mathfrak{D}$, in all these products the number of

[14] Rudio: See E153.

[15] Translators: That is, in the computations that follow, Euler will abbreviate $\mathfrak{A}\sqrt[5]{v}$ as $\mathfrak{A}$, $\mathfrak{B}\sqrt[5]{v^2}$ as $\mathfrak{B}$, etc.

dimensions is divisible by five, and moreover the coefficient of any product is the quintuple of its coefficient, which by the law of combinations agrees with the same product. [16]

35. Since, then, it is the case that $P = 0$, it will also be true that $\Delta = 0$ and for the remaining coefficients we shall have

$$A = -\frac{1}{2}Q,\ \ B = \frac{1}{3}R,\ \ C = -\frac{1}{4}AQ - \frac{1}{4}S \text{ and } D = -\frac{1}{5}BQ + \frac{1}{5}AR + \frac{1}{5}T.$$

Hence, therefore, it will be the case that

$$\begin{aligned}
&A = -5(\mathfrak{A}\mathfrak{D} + \mathfrak{B}\mathfrak{C}),\\
&B = 5(\mathfrak{A}^2\mathfrak{C} + \mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}\mathfrak{D}^2 + \mathfrak{C}^2\mathfrak{D}),\\
&C = -5(\mathfrak{A}^3\mathfrak{B} + \mathfrak{B}^3\mathfrak{D} + \mathfrak{A}\mathfrak{C}^3 + \mathfrak{C}\mathfrak{D}^3) + 5(\mathfrak{A}^2\mathfrak{D}^2 + \mathfrak{B}^2\mathfrak{C}^2) - 5\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D},\\
&D = \mathfrak{A}^5 + \mathfrak{B}^5 + \mathfrak{C}^5 + \mathfrak{D}^5 - 5(\mathfrak{A}^3\mathfrak{C}\mathfrak{D} + \mathfrak{A}\mathfrak{B}^3\mathfrak{C} + \mathfrak{B}\mathfrak{C}^3\mathfrak{D} + \mathfrak{A}\mathfrak{B}\mathfrak{D}^3)\\
&\qquad + 5(\mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + \mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2 + \mathfrak{A}^2\mathfrak{B}^2\mathfrak{D}),
\end{aligned}$$

with which terms now the required powers of v ought to be joined, so that their proper values may be obtained. [17]

36. But if, then, after the signs of the coefficients A and C have been changed, this equation be proposed

$$x^5 = Ax^3 + Bx^2 + Cx + D,$$

whose coefficients have these values

$$\begin{aligned}
&A = 5(\mathfrak{A}\mathfrak{D} + \mathfrak{B}\mathfrak{C})v,\\
&B = 5(\mathfrak{A}^2\mathfrak{C} + \mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}\mathfrak{D}^2 v + \mathfrak{C}^2\mathfrak{D}v)v,\\
&C = 5(\mathfrak{A}^3\mathfrak{B} + \mathfrak{B}^3\mathfrak{D}v + \mathfrak{A}\mathfrak{C}^3 v + \mathfrak{C}\mathfrak{D}^3 vv)v - 5(\mathfrak{A}^2\mathfrak{D}^2 + \mathfrak{B}^2\mathfrak{C}^2)v^2 + 5\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}v^2,\\
&D = \mathfrak{A}^5 v + \mathfrak{B}^5 v^2 + \mathfrak{C}^5 v^3 + \mathfrak{D}^5 v^4 - 5(\mathfrak{A}^3\mathfrak{C}\mathfrak{D} + \mathfrak{A}\mathfrak{B}^3\mathfrak{C} + \mathfrak{B}\mathfrak{C}^3\mathfrak{D}v + \mathfrak{A}\mathfrak{B}\mathfrak{D}^3 v)v^2\\
&\qquad + 5(\mathfrak{A}^2\mathfrak{B}^2\mathfrak{D} + \mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + \mathfrak{A}\mathfrak{C}^2\mathfrak{D}^2 v + \mathfrak{B}^2\mathfrak{C}\mathfrak{D}^2 v)v^2,
\end{aligned}$$

its five roots will be

$$\begin{aligned}
\text{I.}\quad & x = \mathfrak{A}\sqrt[5]{v} + \mathfrak{B}\sqrt[5]{v^2} + \mathfrak{C}\sqrt[5]{v^3} + \mathfrak{D}\sqrt[5]{v^4},\\
\text{II.}\quad & x = \mathfrak{A}\mathfrak{a}\sqrt[5]{v} + \mathfrak{B}\mathfrak{a}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{a}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{a}^4\sqrt[5]{v^4},\\
\text{III.}\quad & x = \mathfrak{A}\mathfrak{b}\sqrt[5]{v} + \mathfrak{B}\mathfrak{b}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{b}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{b}^4\sqrt[5]{v^4},\\
\text{IV.}\quad & x = \mathfrak{A}\mathfrak{c}\sqrt[5]{v} + \mathfrak{B}\mathfrak{c}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{c}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{c}^4\sqrt[5]{v^4},\\
\text{V.}\quad & x = \mathfrak{A}\mathfrak{d}\sqrt[5]{v} + \mathfrak{B}\mathfrak{d}^2\sqrt[5]{v^2} + \mathfrak{C}\mathfrak{d}^3\sqrt[5]{v^3} + \mathfrak{D}\mathfrak{d}^4\sqrt[5]{v^4}
\end{aligned}$$

with $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ being, besides unity, the four remaining fifth roots of unity, the values of which are understood to be imaginary.

37. If now, in turn, from the given coefficients, A, B, C, D the quantities $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ could be determined along with the letter v, a general solution of all equations of the fifth

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[16] Translators: For example, in the computation of Q, all terms whose dimension is not a multiple of 5 will drop out, leaving $Q = 2\mathfrak{A}\mathfrak{D}(1+\mathfrak{a}^5+\mathfrak{b}^5+\mathfrak{c}^5+\mathfrak{d}^5) + 2\mathfrak{B}\mathfrak{C}(1+\mathfrak{a}^5+\mathfrak{b}^5+\mathfrak{c}^5+\mathfrak{d}^5)$. Since $1+\mathfrak{a}^5+\mathfrak{b}^5+\mathfrak{c}^5+\mathfrak{d}^5 = 5$, the effect is to multiply the coefficient 2 by 5.

[17] Translators: That is, the radical signs that were omitted, for the sake of brevity, in Sec. 33 should now be restored.

degree would be achieved. Truly in this very matter the highest difficulty exists, since no way lies open of successively eliminating the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, of which one may be taken at will, so that an equation may result involving only the unknown v with the given A, B, C, D, which indeed would include no superfluous roots. It may be safely supposed, however, that if this elimination be carried out correctly, one could finally arrive at an equation of the fourth degree, whereby the value of v may be determined. For if an equation of a higher degree were to result, then the value of v would also involve radical signs of the same degree, which seems absurd. Since, however, the multitude of terms renders this task so difficult that it may not even be attempted with any success, it will scarcely be off the point to develop certain less general cases, which do not lead to such complicated formulas.

38. With the intention of getting down to particular cases, let us assign to the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ values of such a sort that the calculation may be reduced to a shortened version; and first indeed let $\mathfrak{B}=0$, $\mathfrak{C}=0$ and $\mathfrak{D}=0$, whence we shall find that

$$A=0,\ \ B=0,\ \ C=0,\ \ \text{and}\ \ D=\mathfrak{A}^5v.$$

Hence, then it becomes the case that

$$\mathfrak{A}\sqrt[5]{v}=\sqrt[5]{D}.$$

Therefore, if this equation is proposed

$$x^5=D,$$

the five roots of this equation will be

$$\text{I.}\quad x=\sqrt[5]{D},\quad \text{II.}\quad x=\mathfrak{a}\sqrt[5]{D},\quad \text{III.}\quad x=\mathfrak{b}\sqrt[5]{D},\quad \text{IV.}\quad x=\mathfrak{c}\sqrt[5]{D},\quad \text{V.}\quad x=\mathfrak{d}\sqrt[5]{D};$$

since this case is self-evident, it has seemed appropriate to start from it, so that it may be clear how our method includes known cases.

39. Let now two of the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ vanish; for if three [of them], whichever ones are taken, are set equal to zero, we are always brought back to the preceding case. Therefore let $\mathfrak{C}$ and $\mathfrak{D}$ be equal to nothing; or let an equation be sought, the root of which will be

$$x=\mathfrak{A}\sqrt[5]{v}+\mathfrak{B}\sqrt[5]{v^2},$$

and we shall obtain

$$A=0,\ \ B=5\mathfrak{A}\mathfrak{B}^2v,\ \ C=5\mathfrak{A}^3\mathfrak{B}v,\ \ D=\mathfrak{A}^5v+\mathfrak{B}^5v^2,$$

whence the proposed root will satisfy this equation

$$x^5=5\mathfrak{A}\mathfrak{B}^2vx^2+5\mathfrak{A}^3\mathfrak{B}vx+\mathfrak{A}^5v+\mathfrak{B}^5v^2.$$

If this equation is compared with this form

$$x^5=5Pxx+5Qx+R,$$

it will be the case that

$$\mathfrak{A}\mathfrak{B}^2v=P,\quad \mathfrak{A}^3\mathfrak{B}v=Q,$$

whence is deduced

$$\mathfrak{A}^5v=\frac{QQ}{P}\quad\text{and}\quad \mathfrak{B}^5v^2=\frac{P^3}{Q},$$

so that

$$R=\frac{QQ}{P}+\frac{P^3}{Q}.$$

40. Hereby we are then brought to the solution of this special equation of the fifth degree

$$x^5 = 5Pxx + 5Qx + \frac{QQ}{P} + \frac{P^3}{Q},$$

the five roots of which, because $\mathfrak{A}\sqrt[5]{v} = \sqrt[5]{\frac{QQ}{P}}$ and $\mathfrak{B}\sqrt[5]{vv} = \sqrt[5]{\frac{P^3}{Q}}$, will be

$$\text{I.}\quad x = \sqrt[5]{\frac{QQ}{P}} + \sqrt[5]{\frac{P^3}{Q}},$$

$$\text{II.}\quad x = \mathfrak{a}\sqrt[5]{\frac{QQ}{P}} + \mathfrak{a}^2\sqrt[5]{\frac{P^3}{Q}},$$

$$\text{III.}\quad x = \mathfrak{b}\sqrt[5]{\frac{QQ}{P}} + \mathfrak{b}^2\sqrt[5]{\frac{P^3}{Q}},$$

$$\text{IV.}\quad x = \mathfrak{c}\sqrt[5]{\frac{QQ}{P}} + \mathfrak{c}^2\sqrt[5]{\frac{P^3}{Q}},$$

$$\text{V.}\quad x = \mathfrak{d}\sqrt[5]{\frac{QQ}{P}} + \mathfrak{d}^2\sqrt[5]{\frac{P^3}{Q}}.$$

Moreover this equation is not much unlike the formula of de Moivre, and because it does not allow itself to be resolved into factors, its solution presented here all the more deserves notice.

41. We shall be able to free this equation from fractions if we should set

$$P = MN \quad \text{and} \quad Q = M^2N;$$

for then it will be the case that

$$x^5 = 5MNxx + 5M^2Nx + M^3N + MN^2,$$

whose root will be

$$x = \sqrt[5]{M^3N} + \sqrt[5]{MN^2},$$

and if $\mathfrak{a}$ should denote any other fifth root of unity, any other root of this equation will be

$$x = \mathfrak{a}\sqrt[5]{M^3N} + \mathfrak{a}^2\sqrt[5]{MN^2}.$$

Thus if, for example, we let $M = 1$ and $N = 2$, then any root of the following equation

$$x^5 = 10xx + 10x + 6$$

is

$$x = \mathfrak{a}\sqrt[5]{2} + \mathfrak{a}^2\sqrt[5]{4};$$

and this equation is constructed in such a way that it seems capable of being solved through no known method.

42. If $\mathfrak{B}$ and $\mathfrak{D}$ should be equal to nothing, we return to the same situation. For it will be the case that

$$A = 0, \quad B = 5\mathfrak{A}^2\mathfrak{C}v, \quad C = 5\mathfrak{A}\mathfrak{C}^3vv \quad \text{and} \quad D = \mathfrak{A}^5v + \mathfrak{C}^5v^3;$$

whence, if this equation should be proposed

$$x^5 = 5Pxx + 5Qx + R,$$

so that

$$P = \mathfrak{A}^2\mathfrak{C}v \quad \text{and} \quad Q = \mathfrak{A}\mathfrak{C}^3vv,$$

it will be the case that

$$\frac{QQ}{P} = \mathfrak{C}^5v^3 \quad \text{and} \quad \frac{P^3}{Q} = \mathfrak{A}^5v$$

and hence it happens, as before, that

$$R = \frac{QQ}{P} + \frac{P^3}{Q}$$

and again the same roots are found. Furthermore the same equation is found whether one should set $\mathfrak{A} = 0$ and $\mathfrak{B} = 0$, or $\mathfrak{A} = 0$ and $\mathfrak{C} = 0$. But if, on the other hand, either $\mathfrak{A}$ and $\mathfrak{D}$ or $\mathfrak{B}$ and $\mathfrak{C}$ should be assumed to vanish, in each case in fact the same equation results, but different from the previous cases, which for that reason it will be fitting to develop. [18]

43. Let therefore both $\mathfrak{B} = 0$ and $\mathfrak{C} = 0$ and thus we shall obtain the following values

$$A = 5\mathfrak{A}\mathfrak{D}v, \quad B = 0, \quad C = -5\mathfrak{A}^2\mathfrak{D}^2v^2 \quad \text{and} \quad D = \mathfrak{A}^5v + \mathfrak{D}^5v^4.$$

Whence, if we should stipulate that $\mathfrak{A}\mathfrak{D}v = P$, it will be the case that

$$A = 5P \quad \text{and} \quad C = -5PP;$$

then indeed it will follow that

$$DD - 4P^5 = (\mathfrak{A}^5v - \mathfrak{D}^5v^4)^2 \quad \text{and} \quad \mathfrak{A}^5v - \mathfrak{D}^5v^4 = \sqrt{DD - 4P^5}$$

and therefore

$$\mathfrak{A}^5v = \frac{1}{2}D + \frac{1}{2}\sqrt{DD - 4P^5} \quad \text{and} \quad \mathfrak{D}^5v^4 = \frac{1}{2}D - \frac{1}{2}\sqrt{DD - 4P^5}.$$

Hence, if this equation should be proposed

$$x^5 = 5Px^3 - 5PPx + D,$$

any one of its roots is

$$x = \mathfrak{a}\sqrt[5]{\frac{1}{2}D + \frac{1}{2}\sqrt{DD - 4P^5}} + \mathfrak{a}^4\sqrt[5]{\frac{1}{2}D - \frac{1}{2}\sqrt{DD - 4P^5}}$$

and this is the very equation, whose solution the celebrated de Moivre proved. [19]

44. From the general form, moreover, innumerable equations of the fifth degree can be deduced, the roots of which may be assigned, even if those equations themselves cannot be resolved into factors. For, given an equation of the fifth degree

$$x^5 = Ax^3 + Bx^2 + Cx + D,$$

[18] Translators: That is, the two cases $\mathfrak{A} = \mathfrak{B} = 0$ and $\mathfrak{A} = \mathfrak{C} = 0$, like the case where $\mathfrak{B} = \mathfrak{D} = 0$, produce the same equation as the case where $\mathfrak{C} = \mathfrak{D} = 0$; the two cases $\mathfrak{A} = \mathfrak{D} = 0$ and $\mathfrak{B} = \mathfrak{C} = 0$ produce the same equation as each other, but that equation is different from the one found in the other four cases.

[19] Rudio: Letting $P = -\frac{1}{4}$ and $D = \frac{1}{4}$ yields an equation equivalent de Moivre's equation $5y + 20y^3 + 16y^5 = 4$.

whose coefficients have the following values [20]

$$A = \frac{5}{gk}(g^3 + k^3),$$

$$B = \frac{5}{mnr}\left((m+n)(m^2g^3 - n^2k^3) - (m-n)rr\right),$$

$$C = \frac{5}{mnggkkrr}\left(g^3(m^2g^3 - n^2k^3)^2 - \left(m(m+n)g^6 - (m^2+mn-n^2)g^3k^3 + n(m-n)k^6\right)rr - k^3r^4\right),$$

$$D = \frac{gg}{mmnk^4r^3}\left((m^2g^3 - n^2k^3)^3 - (m^2g^3 - n^2k^3)(m^2g^3 + 2n^2k^3)rr - n^2k^3r^4\right) + \frac{kk}{mnng^4r}\left(m^2g^3(m^2g^3 - n^2k^3) - (2m^2g^3 + n^2k^3)r^2 + r^4\right) + \frac{5(m-n)(g^3-k^3)(m^2g^3 - n^2k^3)}{mngkr} - \frac{5(m+n)(g^3-k^3)r}{mngk},$$

its roots can always be determined.

45. For the sake of brevity, let

$$T = (m^2g^3 - n^2k^3)^2 - 2(m^2g^3 + n^2k^3)rr + r^4$$

and let it be that [21]

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \frac{(m^2g^3 - n^2k^3)^3 - (m^2g^3 - n^2k^3)(m^2g^3 + 2n^2k^3)rr - n^2k^3r^4 \pm \left((m^2g^3 - n^2k^3)^2 - n^2k^3rr\right)\sqrt{T}}{2mmnr^3},$$

$$\left.\begin{matrix} R \\ S \end{matrix}\right\} = \frac{(m^2g^3 - n^2k^3)m^2g^3 - (2m^2g^3 + n^2k^3)rr + r^4 \pm (m^2g^3 - rr)\sqrt{T}}{2mnnr},$$

where the upper signs are valid for the values P and R, the lower for Q and S, and any root of the equation will be

$$x = \mathfrak{a}\sqrt[5]{\frac{gg}{k^4}P} + \mathfrak{a}^2\sqrt[5]{\frac{kk}{g^4}R} + \mathfrak{a}^3\sqrt[5]{\frac{kk}{g^4}S} + \mathfrak{a}^4\sqrt[5]{\frac{gg}{k^4}Q}.$$

46. So that we may illustrate the matter by examples, from these forms the following can be formulated:

I. A root of the equation

$$x^5 = 40x^3 + 70xx - 50x - 98$$

is

$$x = \sqrt[5]{-31 + 3\sqrt{-7}} + \sqrt[5]{-18 + 10\sqrt{-7}} + \sqrt[5]{-18 - 10\sqrt{-7}} + \sqrt[5]{-31 - 3\sqrt{-7}}.$$

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[20] The formulas for B and D reflect Rudio's corrections.

[21] The formula for P and Q reflects a correction by Rudio.

II. A root of the equation [22]

$$x^5 = 2625x + 61500$$

is

$$x = \sqrt[5]{75(5+4\sqrt{10})} + \sqrt[5]{225(35+11\sqrt{10})} + \sqrt[5]{225(35-11\sqrt{10})} + \sqrt[5]{75(5-4\sqrt{10})},$$

which examples are all the more noteworthy, because these equations can be solved in no other way.

Moreover, in similar fashion, investigations of this sort can be extended to equations of higher degrees and it will be easy to exhibit countless equations of any degree, insolvable by other methods, of which not only a single root, but all the roots, can be clearly displayed with the aid of this method.

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[22] This equation reflects a correction by Rudio.