

Theorems on residues obtained by the division of powers*

Leonhard Euler[†]

Theorem 1.

1. If p is a prime number and a is prime to p , then no term of the geometric progression $1, a, a^2, a^3, a^4, a^5, a^6$, etc. is divisible by the number p .

Demonstration.

This follows from Book VII, Prop. 26 of Euclid, where it is demonstrated that if two numbers a and b are prime to p then too that their product ab is prime to p . Thus with a prime to p , by putting $b = a$, the square a^2 will be prime to p ; and then in turn by putting $b = a^2$; likewise $b = a^3$, etc. Therefore no power of a will be divisible by the prime number p .

Corollary 1.

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[†]Date of translation: August 18, 2006. Translated from the Latin by Jordan Bell, 4th year undergraduate in Honours Mathematics, School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada. Email: jbelle3@connect.carleton.ca. This translation was written during an NSERC USRA.

2. Therefore if each of the terms of the geometric progression

$$1; a^2; a^3; a^4; a^5; a^6; a^7; a^8; \text{ etc.}$$

were divided by a prime number p , division never happens without a residue, but rather a residue arises from each term.

Scholion.

3. I have resolved to carefully study the residues which emerge from the division of all the terms of the given geometric progression by the prime number p . First indeed, it is apparent from the nature of division that each of these residues will be less than the number p ; also, no residue will be $= 0$, because no term is divisible by p . For if residues are produced which are greater than the number p , the way in which they may be reduced to less than it is clear from arithmetic. Thus the residue of $p+r$ equals the residue r , and in general the residue $np+r$ returns the residue r ; and if r is greater than p , this residue is reprised as $r-p$, or $r-2p$, or $r-3p$, etc., until a number less than p is reached. Therefore all the residues $r \pm np$ reflect the same residue as r . In particular, all the residues may be referred to as positive numbers less than p . In fact however, on many occasions it is convenient for negative residues to be considered: as if r were a residue remaining from division by a certain number p such that $r < p$, the residue will be $r-p$, namely a negative number; thus the residue of the positive r equals the residue of the negative $r-p$. In this way the residues can be exhibited such that none exceed half of p , and on the other hand if a residue r is larger than $\frac{1}{2}p$, its place is taken by the residue of the negative $r-p$, since this will be less than half of p .

Corollary 2.

4. Because all the residues are integral numbers which are less than p , it follows that no more than $p-1$ different residues can arise. Since the terms geometric series $1; a; a^2; a^3; a^4; a^5; \text{ etc.}$ consist of infinitely many different numbers, it is necessary that multiple terms exhibit the same residues.

Corollary 3.

5. If a^μ and a^ν are two terms which produce the same residue r , so that $a^\mu = mp + r$ and $a^\nu = np + r$, it will be that $a^\mu - a^\nu = (m - n)p$, and thus that the difference $a^\mu - a^\nu$ of these terms will be divisible by p . Therefore the difference between two terms of the given geometric progression will be divisible by the number p in innumerably many ways.

Corollary 4.

6. If the power a^μ were to give the residue r and the power a^ν the residue s , and it were $r + s = p$, we say in this case of the residues r and s for each to be the complement of the other, in which case the sum of the powers $a^\mu + a^\nu$ will be divisible by p . For if it were $a^\mu = mp + r$ and $a^\nu = np + s$, it will then be $a^\mu + a^\nu = (m + n)p + r + s = (m + n + 1)p$, and thus it has p as a factor.

Theorem 2.

7. If the power a^μ divided by p has the residue r , and the power a^ν the residue s , the power $a^{\mu+\nu}$ will have the residue rs .

Demonstration.

Were it $a^\mu = mp + r$ and $a^\nu = np + s$, it will be $a^{\mu+\nu} = mnpp + mps + npr + rs$; and thus if $a^{\mu+\nu}$ is divided by p , the residue will be rs ; and if it were larger than p , by subtracting p as many times as possible, it can be reduced to a residue less than p . Q.E.D.

Corollary 1.

8. Hence the residue of the base a divided by p can be expressed by a (for if $a < p$, a will be a residue properly called, and if however $a > p$, nevertheless the residue can be expressed by a , because $a - p$ or $a - np$ are likewise treated), if the residue of the power a^μ divided by p were r , the residue of the power $a^{\mu+1}$ will be ar , and in the same way the residue of the power $a^{\mu+2}$ will be a^2r , $a^{\mu+3}$ will be a^3r , etc.

Corollary 2.

9. From this it also follows that if the residue of the power a^μ divided by p is $= r$, for the residue of the power $a^{2\mu}$ to be $= rr$, the residue of the power $a^{3\mu}$ to be $= r^3$, etc. Thus if the residue of the power a^μ is $= 1$, 1 will be the identical residue of all the powers $a^{2\mu}, a^{3\mu}, a^{4\mu}, a^{5\mu}$, etc.

Corollary 3.

10. But if the residue of the power a^μ divided by p is $= p - 1$, which we have seen is able to be expressed by -1 , then the residue of the power $a^{2\mu}$ will be $= +1$, the residue of the power $a^{3\mu}$ will be $= -1$, and in turn the residue of the power $a^{4\mu} = +1$. And thus in general the residue of the power $a^{n\mu}$ will be either $+1$, if n is an even number, or -1 , if n is an odd number.

Scholion.

11. Hence the way is inferred for the residues to be easily found which remain from the division of any power by a given number. For say we want to investigate the residue which arises from the division of the power 7^{160} by 641.

powers	residues	since of course the first power of 7 leaves 7,
7^1	7	the powers $7^2, 7^3, 7^4$ leave 49,343 and 478, or -163 ;
7^2	49	the square of this, 7^8 , leaves 163^2 or 288, and the square
7^3	343	of this, 7^{16} , leaves 288^2 or 255. In the same way
7^4	478	the power 7^{32} leaves 255^2 or 284, and
7^8	288	the residue of the power 7^{64} will be -110 , and from
7^{16}	255	7^{128} arises 110^2 or -79 , which residue is
7^{32}	284	multiplied by 288, which gives the residue
7^{64}	-110	of the power $7^{128+32} = 7^{160}$,
7^{128}	-79	which will be 640 or -1
7^{160}	-1	

We know therefore that if the power 7^{160} is divided by 641 then the residue will be 640, or -1 , from which we conclude for the residue of the power 7^{320} to be $+1$. Therefore in general the residue of the power 7^{160n} divided by 641 will be either $+1$, if n is an even number, or -1 , if n is an odd number.

Theorem 3.

12. If the number a is prime to p , and the geometric progression $1; a; a^2; a^3; a^4; a^5; a^6; a^7; \text{ etc}$ were formed, innumerably many terms occur in it which leave the residue 1 when divided by p , and the exponents of these terms constitute an arithmetic progression.

Demonstration.

Since the number of terms is infinite, but on the other hand no more than $p-1$ different residues can arise, it is necessary that several, and indeed infinitely many, terms produce the same residue r . Were a^μ and a^ν two such terms leaving the same residue r , $a^\mu - a^\nu$ will be divisible by p . As $a^\mu - a^\nu = a^\nu(a^{\mu-\nu} - 1)$, and since the product is divisible by p yet the one factor a^ν is prime to p , it is necessary that the other factor $a^{\mu-\nu} - 1$ is divisible by p ; from this, the power $a^{\mu-\nu}$ when divided by p will have a residue $= 1$. If it were $\mu - \nu = \lambda$, so that the residue of the power a^λ were $= 1$, it will then likewise occur that all the powers $a^{2\lambda}, a^{3\lambda}, a^{4\lambda}, a^{5\lambda}, \text{ etc.}$ will have the same residue $= 1$. And thus unity will be the residue of all of these powers:

$$1; a^\lambda; a^{2\lambda}; a^{3\lambda}; a^{4\lambda}; a^{5\lambda}; a^{6\lambda}; \text{ etc.}$$

whose exponents advance as an arithmetic progression.

Corollary 1.

13. Therefore if a single power a^λ is found which when divided by p produces a residue $= 1$, from it infinitely many other powers can be exhibited which leave unity when divided by p . And indeed also, the least power of this type is $a^0 = 1$.

Corollary 2.

14. But even if no power of a aside from unity appears, that is which when divided by p leaves unity as its residue, we still find for infinitely many powers of this type to be given.

Corollary 3.

15. From the demonstration it is thus apparent how to give a power a^λ which presents a residue = 1, whose exponent λ is less than p . For if the geometric progression is continued up to the term a^{p-1} , since the number of terms is p , it is necessary that at least two terms, say a^μ and a^ν , have the same residue; from this the power $a^{\mu-\nu}$ will have a residue = 1, and because $\mu < p$ and $\nu < p$, it is certain to be $\mu - \nu < p$.

Theorem 4.

16. If the power a^μ divided by p leaves a residue = r , and the residue of a higher power $a^{\mu+\nu}$ is = rs , the residue of the power a^ν , with which this was increased, will be = s .

Demonstration.

As the power a^ν gives a residue, which may be set = t , and since the residue of the power a^μ is = r , the residue of the power $a^{\mu+\nu}$ will be = rt , which should be equal to rs itself. Therefore it is $rt = rs + np$, supposing that we set the residues r, t to be less than the divisor p . Therefore it shall be $t = s + \frac{np}{r}$: but since a and p are mutually prime, all the residues which arise from powers of a divided by p will likewise be prime to p , except in the case they are = 1. Therefore for $\frac{np}{r}$ to be an integral number, it is necessary that $\frac{n}{r}$ be an integral number; putting this = m , it will therefore be $t = s + mp$ and thus $t = s$. Whence if the residue of the power a^μ is = r and the residue of the power $a^{\mu+\nu}$ is = rs , then it follows that the residue of the power a^ν will be = s .

Corollary 1.

17. Therefore if $s = 1$, that is if the two powers a^μ and $a^{\mu+\nu}$ have the same residue r , it will follow that if the larger is divided by the smaller, the quotient a^ν will possess a residue = 1, the demonstration of which rests on the preceding theorem.

Corollary 2.

18. If $r = 1$ and $s = 1$, that is if the two powers a^μ and $a^{\mu+\nu}$ have the same residue $= 1$, then also the power a^ν , whose exponent is the difference of the previous exponents, will likewise have a residue $= 1$.

Scholion.

19. The theorem can also be disclosed in this way. As a^μ leaves the residue r when divided by p , and in the same way $a^{\mu+\nu} = np + rs$; then it will be $a^{\mu+\nu} - a^\mu s = np - mps = (n - ms)p$; and thus the number $a^{\mu+\nu} - a^\mu s = a^\mu(a^\nu - s)$ will be divisible by p : but since the first factor a^μ is not divisible by p , therefore the other factor $a^\nu - s$ will be divisible by p , and it follows that the power a^ν gives a residue $= s$ when divided by p .

Theorem 5.

20. If after unity a^λ is the smallest power which leaves unity when divided by p , then no other powers may leave the same residue $= 1$, unless they occur in this geometric progression.

$$1; a^\lambda; a^{2\lambda}; a^{3\lambda}; a^{4\lambda}; a^{5\lambda}; \text{etc.}$$

Demonstration.

For we may put some other power a^μ to give a residue $= 1$ when divided by p , and with $\mu > \lambda$, neither however is this equal to any multiple of λ . Thus this exponent μ can be expressed as $\mu = n\lambda + \delta$, where $\delta < \lambda$, and neither will it be $\delta = 0$. Therefore since the power $a^{n\lambda}$, and to the same extent $a^\mu = a^{n\lambda+\delta}$, will leave unity when divided by p , by §18 the power a^δ will have unity as its residue, and it would therefore be that a^λ was not the smallest power of this type, contrary to the hypothesis. Whence if a^λ is the smallest power giving a residue $= 1$, then no other powers will have this same property except those whose exponents are multiples of λ .

Corollary 1.

21. Therefore if the second term of the geometric progression $1, a, a^2, a^3, a^4,$ etc. leaves 1 when divided by p , then all the terms will yield the same residue $= 1$: therefore no residue aside from 1 will occur in any of these numbers.

Corollary 2.

22. If the residue of the third term a^2 is $= 1$, that is if $a^2 = np + 1$, then the alternate terms $1, a^2, a^4, a^4,$ etc., whose exponents are even, will all have the same residue $= 1$, and indeed all the other terms, unless a^1 also has a residue $= 1$, will all produce different residues.

Corollary 3.

23. It can happen, therefore, that many fewer numbers may occur among the residues than the number $p - 1$ contains unities: on the other hand, more than $p - 1$ distinct numbers cannot occur.

Theorem 6.

24. If the power a^{2n} , whose exponent is an even number, divided by a prime number p leaves a residue $= 1$, then the power a^n divided by the same number p will give a residue either $= +1$ or $= -1$.

Demonstration.

Indeed, we put r to be a residue which remains from the division of the power a^n by the prime number p , and the residue of the power a^{2n} will be $= rr$, which by the hypothesis is $= 1$. From this it will be that $rr = 1 + mp$, and so $rr - 1 = mp$; from which since $rr - 1 = (r + 1)(r - 1)$ must be divisible by p , either the factor $r + 1$ or the factor $r - 1$ shall be divisible by p . In the first case it will be $r + 1 = \alpha p$, and $r = \alpha p - 1$, and hence $r = -1$. In the latter case it will be $r - 1 = \alpha p$ and $r = \alpha p + 1$ and hence $r = +1$. Therefore if the power a^{2n} yields a residue $= +1$, the power a^n will have a residue $= +1$ or $= -1$, if indeed p were a prime number.

Corollary 1.

25. Consequently if a^{2n} were the minimum power which leaves a residue $= +1$ when divided by the prime number p , then the power a^n will give a residue $= -1$. Therefore if the exponent λ of the minimum power a^λ yielding a residue $= 1$ is even, then among the residues of the terms of the geometric progression $1, a, a^2, a^3, a^4$, etc. the number -1 will also appear.

Corollary 2.

26. But if on the other hand the exponent λ of the minimum power a^λ yielding the residue 1 were an odd number, then no power whatsoever will leave a residue $= -1$. For if a^μ were such a power, which would give a residue $= -1$, then the power $a^{2\mu}$ would give a residue $= +1$, whence $2\mu = n\lambda$, and since λ is an odd number it would be $2\mu = 2m\lambda$, and so $\mu = m\lambda$. But then the power $a^{m\lambda}$ leaves a residue $= +1$, and therefore the residue -1 does not occur anywhere.

Theorem 7.

27. If a^λ is the minimum power of a which yields a residue $= 1$ when divided by the number p , then all the residues which result from the terms of the geometric progression $1, a, a^2, a^3, \dots, a^{\lambda-1}$, continued onto the power a^λ , will be mutually distinct.

Demonstration.

For if, say, a^μ and a^ν are two powers whose exponents μ and ν are less than λ and which give the same residue, then the difference $a^\mu - a^\nu$ of them would be divisible by p , and thus the power $a^{\mu-\nu}$ divided by p would leave a residue $= +1$, for which it would be $\mu - \nu < \lambda$, contrary to the hypothesis; from this it is clear for all the powers whose exponents are less than λ to yield different residues.

Theorem 8.

28. If a^λ were a certain power of a , which divided by the number p produces a residue $= 1$, and also the geometric progression were divided into sections according to the powers $a^\lambda, a^{2\lambda}, a^{3\lambda}, a^{4\lambda}$, etc. in this way:

$$1, a, a^2, \dots, a^{\lambda-1} | a^\lambda, \dots, a^{2\lambda-1} | a^{2\lambda}, \dots, a^{3\lambda-1} | a^{3\lambda}, \dots, a^{4\lambda-1} | \text{etc.}$$

such that each section contains λ terms, then in each section the same residues will appear, and also the same order will be repeated.

Demonstration.

Indeed, the first terms $1, a^\lambda, a^{2\lambda}, a^{3\lambda}$, etc. of all the sections produce the same residue $= 1$. Then the second terms $a, a^{\lambda+1}, a^{2\lambda+1}, a^{3\lambda+1}$, etc. of all the sections will likewise give the same residue; for if r were a residue arising from the term a^1 , since $a^{\lambda+1} = a^\lambda \cdot a^1$, the residue arising from this term will be equal to $1 \cdot r = r$; and in a similar way it is apparent for the residues of all the terms $a^{2\lambda+1}, a^{3\lambda+1}$, etc. to be $= r$. And if in general a^μ were a certain term of the first section, the residue arising from which were $= r$, then too the residue of the term $a^{n\lambda+\mu}$ will be $= r$, since the residue of the term $a^{n\lambda}$ is $= 1$: hence the analogous terms $a^{\lambda+\mu}, a^{2\lambda+\mu}, a^{3\lambda+\mu}$, etc. of all the sections will have the same residue.

Corollary 1.

29. And if so many many of the terms contained in the first section were known, then too the residues of all the terms which constitute the remaining sections will be known.

Corollary 2.

30. For if a term a^x is given, whose exponent x is an arbitrarily large number, the residue of it can be easily found. For this exponent x can be reduced to the form $n\lambda + \mu$, such that $\mu < \lambda$, and thus the residue of the term a^x will be the same as the that of the term a^μ .

Corollary 3.

31. Moreover for this number μ which is chosen as less than λ , if the number x is divided by p , then for the residue which remains from this division which is sought will be this very number μ .

Corollary 4.

32. Moreover powers a^λ may always be given which when divided by p leave unity, whose exponent λ is less than the given number p , and thus for finding the residues of all the terms of the geometric progression, it is not necessary to continue beyond the term a^p .

Corollary 5.

33. If moreover the power a^μ is the minimum of those which leave unity when divided by the number p , then as all the terms less than a^λ yield different residues, in all the residues, no more and no less different numbers occur than λ . Therefore if λ were less than $p - 1$, not all numbers occur in the residues: rather certain numbers will clearly never be able to remain in the division of the terms of the geometric progression $1, a, a^2, a^3$, etc.

Corollary 6.

34. Therefore if the diversity of the residues is considered, it can happen that only a single residue occurs among all the powers of a , or exactly two different residues, or three etc. are produced, though no more than $p - 1$ can occur. But however many residues should be produced, unity will always appear among them.

Theorem 9.

35. If p were a prime number and a were prime to p , and also all the numbers less than p itself appear among the residues which arise from the division of all the powers of a by the number p , then a^{p-1} will be the minimum power which leaves unity when divided by p .

Demonstration.

Were a^λ the minimum power which leaves unity when divided by p , it is clear from the preceding that $\lambda < p$ (15). Now since the number of all the residues of divisors is $= \lambda$, and all the numbers less than p is $= p - 1$, it is clear that if it were $\lambda < p - 1$, then not all the numbers less than p could occur in the residues; therefore it will not be $\lambda < p - 1$, and neither indeed is it $\lambda > p - 1$, since in general it is $\lambda < p$. From this it follows that $\lambda = p - 1$. Whenceforth if all the numbers less than p occur in the residues, the power a^{p-1} will be the minimum which leaves unity when divided by p .

Scholion.

36. The nature of this theorem requires that p be a prime number; for unless this is the case, it is not possible to make it that all the numbers less than p occur in the residues. For it may be clearly seen, which is to be carefully considered, that if p is a composite number, to which however a is prime, for no aliquot part of p to occur in the residues: for if some power a^μ should give a residue r which was an aliquot part of p , as $a^\mu = mp + r$, this power a^μ would have the divisor r , and thus neither it nor its root a would be prime to p , which is against the hypothesis.

Theorem 10.

37. If the number of different residues which arise from the division of the powers $1, a, a^2, a^3, a^4, a^5$, etc. by a prime number p is less than $p - 1$, then there are at least as many numbers which are not residues as which are residues.

Demonstration.

Were a^λ the minimum power which leaves unity when divided by p , and also $\lambda < p - 1$, the number of all the residues of the divisor will be $= \lambda$, and thus less than $p - 1$. Therefore as the number of all the numbers less than p itself are $= p - 1$, it is clear in the given case for numbers to be given which do not occur as residues. I say moreover for the number of numbers of this type at a minimum to be $= \lambda$. So that this may be exhibited, we shall relate the residues by the terms from which they arise, and they will be

$$\text{these residues } 1, a, a^2, a^3, a^4, \dots, a^{\lambda-1}$$

the number of which is $= \lambda$, and these residues, if reduced to the usual form, will all be less than p and mutually distinct. Therefore as it was $\lambda < p - 1$ by hypothesis, a number is certainly given which does not appear in these residues. Let k be such a number; I say now that if k is not a residue, then neither ak , nor a^2k , nor a^3k , etc., nor $a^{\lambda-1}k$ will occur in the residues. For let $a^\mu k$ be a residue arising from the power a^α , and it shall be $a^\alpha = np + a^\mu k$, that is $a^\alpha - a^\mu k = np$, and thus $a^\alpha - a^\mu k = a^\mu(a^{\alpha-\mu} - k)$ is divisible by p . But since a^μ is not divisible by p , $a^{\alpha-\mu}$ will therefore be divisible by p , that is the power $a^{\alpha-\mu}$ when divided by p will leave the residue k , which opposes the hypothesis. Whence it is clear that all these numbers: $k, ak, a^2k, a^3k, \dots, a^{\lambda-1}k$, that is all the numbers derived from it, to not be residues. And these numbers, whose multitude is $= \lambda$, are all mutually distinct; for if two, say $a^\mu k$ and $a^\nu k$, convene, and reduce to the same residue r , it would be $a^\mu k = mp + r$ and $a^\nu k = np + r$, and thus $a^\mu k - a^\nu k = (m - n)p$, that is $(a^\mu - a^\nu)k = (m - n)p$ would be divisible by p . Neither indeed is k divisible by p , as we took p as a prime number and $k < p$; were $a^\mu - a^\nu$ divisible by p , or should $a^{\mu-\nu}$ leave unity when divided by p , since however $\mu < \lambda - 1$ and $\nu < \lambda - 1$, it would be $\mu - \nu < \lambda$, which is absurd. Therefore all these numbers $k, ak, a^2k, a^3k, \dots, a^{\lambda-1}k$, if reduced, will be mutually distinct, and the multitude of them is $= \lambda$. Therefore at a minimum λ numbers are given which will not occur in the residues, supposing that it were $\lambda < p - 1$.

Corollary 1.

38. Consequently were there λ different numbers which are residues, and just as many different numbers which are not residues, and all were smaller than p , the number of them joined together cannot be larger than 2λ , or $p - 1$: for there are no more numbers less than p given than $p - 1$.

Corollary 2.

39. Therefore if a^λ were the minimum power which leaves unity when divided by the prime number p , it would be $\lambda < p - 1$, and then it will certainly not be that $\lambda > \frac{p-1}{2}$: it will therefore be either $\lambda = \frac{p-1}{2}$ or $\lambda < \frac{p-1}{2}$.

Corollary 3.

40. From before we see for the exponent λ of the minimum power to necessarily be less than p ; it will therefore be either $\lambda = p - 1$ or $\lambda < p - 1$; in the case that $\lambda < p - 1$, we know at once for it now to be either $\lambda = \frac{p-1}{2}$ or $\lambda < \frac{p-1}{2}$. And thus no number is contained beyond the limits $p - 1$ and $\frac{p-1}{2}$ which could ever be a value of λ .

Theorem 11.

41. If p were a prime number, and also a^λ were the minimum power of a which leaves unity when divided by p , and it were $\lambda < \frac{p-1}{2}$; then it cannot happen that the exponent λ is larger than $\frac{p-1}{3}$: and it will therefore be either $\lambda = \frac{p-1}{3}$ or $\lambda < \frac{p-1}{3}$.

Demonstration.

Were a^λ the minimum power which leaves unity when divided by a prime number p , then no more than λ different numbers occur in the residues, which are left from this terms

$$1; a; a^2; a^3; a^4; \dots; a^{\lambda-1}$$

if each were divided by p ; whence since $\lambda < p - 1$, $p - 1 - \lambda$ numbers will occur which are not residues, and if one of these is $= r$, we have seen that all these numbers

$$r; ar; a^2r; a^3r; a^4r; \dots; a^{\lambda-1}r,$$

according as they are reduced to numbers less than p itself when divided by p , to not be contained in the residues. Then moreover at least λ numbers are excluded from the residues; whence as it was $\lambda < \frac{p-1}{2}$, it will be $\lambda < p - 1 - \lambda$, and thus aside from these numbers others are also given which are not contained in the residues. Let s be a number of this type, which was neither a residue or contained in the preceding series of non-residues; then too all these numbers

$$s; as; a^2s; a^3s; a^4s; \dots; a^{\lambda-1}s$$

will not be residues: and these numbers, which we have displayed in the preceding demonstration, will all be mutually distinct. Neither indeed shall any of these numbers, say $a^\mu s$, be contained now in the preceding series of non-residues, that is it is not $a^\mu s = a^\nu r$. For indeed if it were $a^\nu r = a^\mu s$, it would become $s = a^{\nu-\mu}r$, or $s = a^{\lambda+\nu-\mu}r$, supposing that it were $\mu > \nu$, from which now s shall be contained in the prior series contrary to the hypothesis. Wherefore if $\lambda < \frac{p-1}{2}$, at least λ numbers are given which are not residues, and since we have λ residues and 2λ non-residues, and all these numbers were less than p itself, it cannot happen that the sum 3λ of them would be greater than $p - 1$, that is it will not be that $\lambda > \frac{p-1}{3}$. Therefore it will be either $\lambda = \frac{p-1}{3}$ or $\lambda < \frac{p-1}{3}$, if indeed $\lambda < \frac{p-1}{2}$ and p were a prime number.

Corollary 1.

42. Therefore if it is not $\lambda < \frac{p-1}{3}$, then it will certainly be $\lambda = \frac{p-1}{3}$, just as when $\lambda < \frac{p-1}{2}$. And by removing this condition, if we know that it is not $\lambda < \frac{p-1}{3}$, then it necessarily follows that either $\lambda = \frac{p-1}{3}$, $\lambda = \frac{p-1}{2}$, or $\lambda = p - 1$.

Corollary 2.

43. Moreover if it were either $\lambda = \frac{p-1}{3}$ or $\lambda = \frac{p-1}{2}$, the power a^{p-1} divided by p will leave unity. For if a^λ leaves unity, then also $a^{2\lambda}$ and $a^{3\lambda}$ will give unity as residues.

Theorem 12.

44. If a^λ were the minimum power of a which leaves unity when divided by a prime number p , and it were $\lambda < \frac{p-1}{3}$, then it will certainly not be $a > \frac{p-1}{4}$, and therefore it will be either $\lambda = \frac{p-1}{4}$ or $\lambda < \frac{p-1}{4}$.

Demonstration.

Because the number of all the different residues which come forth from the division of all the powers of a by the prime number p is $= \lambda$, and in fact arise from these terms: $1; a; a^2; a^3; a^4; \dots; a^{\lambda-1}$: on account of $\lambda < \frac{p-1}{3}$, immediately twice as many numbers will occur which are not residues, which arise from these two progressions

$$r; ar; a^2r; a^3r; a^4r; \dots; a^{\lambda-1}r$$

and

$$s; as; a^2s; a^3s; a^4s; \dots; a^{\lambda-1}s$$

the number of these numbers, both residues and non-residues, is $= 3\lambda$, and thus less than $p - 1$, therefore numbers will still be left over which will not be residues. Were t such a number, we display as before that too all these numbers

$$t; at; a^2t; a^3t; a^4t; \dots; a^{\lambda-1}t$$

will not be residues, the number of which is $= \lambda$. And these numbers are not only all mutually distinct, since p was a prime number, but moreover they are different from the preceding, and thus the multitude of all these numbers, either residues or non-residues, is $= 4\lambda$, and also all these numbers shall be less than p ; it is impossible that it were $4\lambda > p - 1$; therefore it will be either $\lambda = \frac{p-1}{4}$ or $\lambda < \frac{p-1}{4}$, if indeed, as we have assumed, $\lambda < \frac{p-1}{3}$ and p were a prime number.

Corollary 1.

45. In a similar way it can be demonstrated, that if $\lambda < \frac{p-1}{4}$ then it is impossible that $\lambda > \frac{p-1}{5}$, and to thus be either $\lambda = \frac{p-1}{5}$ or $\lambda < \frac{p-1}{5}$.

Corollary 2.

46. In general if it is known that $\lambda < \frac{p-1}{n}$, it may be demonstrated in the same way that it cannot be $\lambda > \frac{p-1}{n+1}$, and therefore will be either $\lambda = \frac{p-1}{n+1}$ or $\lambda < \frac{p-1}{n+1}$.

Corollary 3.

47. From this it is clear for the number of all those numbers which cannot be residues to be either $= 0$, or $= \lambda$, or $= 2\lambda$, or another such multiple of λ : for if they were more numbers of this type than $n\lambda$, then because from each λ others follows, the number of all of these would be $= (n+1)\lambda$; and if these are not the only numbers contained in the non-residues, then again λ others follow simultaneously.

Theorem 13.

48. If p were a prime number, and a^λ the minimum power of a which leaves unity when divided by p , the exponent λ will be a divisor of the number $p-1$.

Demonstration.

Therefore the number of all the residues of divisors is $= \lambda$, from which the number of the remaining numbers less than p which are not residues will be $= p-1-\lambda$; but this number (47) is a multiple of λ , say put as $n\lambda$, thus it would be that $p-1-\lambda = n\lambda$, from which it would be $\lambda = \frac{p-1}{n+1}$. It is therefore evident for the exponent λ to be a divisor of the number $p-1$, from which if it were not $\lambda = p-1$, it will be certain for the exponent λ to be equal to some aliquot part of the number $p-1$.

Theorem 14.

49. If p were a prime number, and a prime to p , then the power a^{p-1} will leave unity when divided by p .

Demonstration.

Were a^λ the minimum power of a which leaves unity when divided by p , as we have seen $\lambda < p$, and indeed we have demonstrated above for it to be either that $\lambda = p - 1$ or for λ to be an aliquot part of the number $p - 1$. In the first case the proposition is obvious, as the power a^{p-1} leaves unity when divided by p . In the latter case, in which λ is an aliquot part of the number $p - 1$, it will be $p - 1 = n\lambda$, but since the power a^λ leaves unity when divided by p , so too all the powers $a^{2\lambda}, a^{3\lambda}, a^{4\lambda}$, etc. and therefore $a^{n\lambda}$, that is a^{p-1} , shall leave unity when divided by p . Therefore the power a^{p-1} will always leave unity when divided by p .

Corollary 1.

50. Since the power a^{p-1} leaves unity when it is divided by the prime number p , the formula $a^{p-1} - 1$ will be divisible by the prime number p , supposing that a were prime to p , that is that a were not divisible by p .

Corollary 2.

51. Therefore if p were a prime number, all the powers of the exponent $p - 1$, such as n^{p-1} , when divided by p will leave either unity or nothing. The first happens if n is a number prime to p , the latter indeed if n is a number divisible by p .

Corollary 3.

52. If p were a prime number, and a and b were numbers prime to p , the difference of the powers $a^{p-1} - b^{p-1}$ will be divisible by the number p . For since both $a^{p-1} - 1$ and $b^{p-1} - 1$ are divisible by p , then too the differences of these formulas, that is $a^{p-1} - b^{p-1}$, will be divisible by p .

Scholion.

53. Behold therefore a new demonstration of the extraordinary theorem, found before by Fermat, which differs greatly from that which I gave in the Comment. Acad. Petropol. Volume VIII. For there I called upon the expansion of the binomial $(a + b)^n$ into a series by means of the method of Newton, which reasoning seems quite remote from the proposition; here indeed I have demonstrated the same theorem from the properties of powers alone, by which this demonstration seems much more natural. In addition, other important properties about the residues of powers when divided by prime numbers may appear to us. For it is clear that if p were a prime number, then not only will the formula $a^{p-1} - 1$ be divisible by p , but it will also sometimes happen that the simpler formula $a^\lambda - 1$ will be divisible by p , and then for the exponent λ to be an aliquot part of the exponent $p - 1$.

Theorem 15.

54. If q were a prime number, and also the power a^q left unity when divided by a prime number p , then a^q will be the minimum power of a which leaves unity when divided by p , except for the singular case in which the number a leaves unity when divided by p .

Demonstration.

For were a^λ the minimum power of a which left unity when divided by the prime number p , then no other powers will be gifted with this property, except for $a^{2\lambda}, a^{3\lambda}, a^{4\lambda}$, etc. Truly none of these will be able to be equal to the power a^q , unless it were $\lambda = 1$, since q was a prime number, and thus it is necessary that it be $q = \lambda$, and thus that a^q is the minimum power which leaves unity when divided by p . However the case in which $\lambda = 1$ is excepted, that is in which the number a leaves unity when divided by p : for in this case each power a^n , either when the exponent n was a prime number or a composite number, shall leave unity when division by p is carried out.

Corollary 1.

55. Therefore if the power a^q , whose exponent is a prime number, leaves unity when divided by a prime number p , then q will be an aliquot part of the number $p - 1$, in which case the formula $a^q - 1$ will be divisible by the prime number p .

Corollary 2.

56. Were q an aliquot part of the number $p - 1$, it will be $p - 1 = nq$, and $p = nq + 1$. And if therefore the formula $a^q - 1$, in which q is a prime number, were divisible by this prime number p , it will always have to have a divisor of such form $p = nq + 1$, unless $p = a - 1$: for $a - 1$ is always a divisor of the formula $a^q - 1$.

Corollary 3.

57. Therefore the formula $a^q - 1$, with q a prime number, will not admit prime divisors other than $p - 1$ unless they are contained in the form $nq + 1$; and as q is a prime number, and therefore odd, unless $q = 2$, except for which even numbers can be taken for n , and therefore all the divisors, if it has any, will be contained in the form $2nq + 1$.

Corollary 4.

58. Therefore since a divisor of the formula $a^q - 1$ is

$$a^{q-1} + a^{q-2} + a^{q-3} + a^{q-4} + \dots + a^2 + a + 1$$

this form will be contained in $2nq + 1$, and therefore this expression: $a^{q-1} + a^{q-2} + a^{q-3} + \dots + a^2 + a$ will be divisible by the prime number p , whatever number a is, though if $a = q$ or $a = mq$ this is manifest by itself.

Scholion 1.

59. This is moreover clear, if a were neither q nor mq ; then for the formula which has been found to turn into

$$a(a^{q-2} + a^{q-3} + a^{q-4} + \cdots + a + 1)$$

of which the second factor, which transforms into $\frac{a^{q-1}-1}{a-1}$, is divisible by q : because indeed by this it is evident; for since q was a prime number, the formula $a^{q-1} - 1$ is divisible by it; and when this is further divided by $a - 1$, it will remain divisible by p , unless $a - 1$ has q as a divisor, which case is excepted now as before. For it is to be noted for the form $a^{q-1} + a^{q-2} + a^{q-3} + \cdots + a^2 + a + 1$ to be contained in the form $2nq + 1$, insofar as this is either a prime number, or composed from prime numbers of the form $2nq + 1$. But if these formula should have now a factor $a - 1$, by which the form $a^q - 1$ is divisible, then the form $2nq + 1$ shall not agree with it. But if $a - 1 = mq$, that is $a = mq + 1$, then this form will be divisible by q , because the number of terms is $= q$, and therefore this will not be contained in the form $2nq + 1$.

Scholion 2.

60. It is also most interesting to know the divisors of the formula $a^q - 1$, when q is a prime number, because usually these, except the divisor $a - 1$ which reveals itself at once, are most difficult to investigate, and indeed it is possible to happen that often such a formula, after it is divided by $a - 1$, shall be a prime number. And if q is not a prime number, but should have divisors of itself m, n , then clearly the formulas $a^m - 1$ and $a^n - 1$ will be divisors of the formula $a^q - 1$. Therefore in these cases the investigation of the further divisors is reduced to the formulas $a^m - 1$ and $a^n - 1$, in which the exponents m and n are prime numbers. We know therefore that if it is wanted to be attempted to investigate the divisors of the formula $a^q - 1$, an attempt should be made with no other prime numbers, unless they were contained in the form $2nq + 1$, which operation elsewhere is most difficult, and is not easily lightened.

Theorem 16.

61. If the power a^m , divided by the number p , leaves a residue $= r$, then moreover the power $(a \pm \alpha p)^m$, divided by p , will leave the same residue r .

Demonstration.

If the power $(a \pm \alpha p)^m$ is expanded, it will produce

$$a^m \pm m\alpha a^{m-1}p \pm \frac{m(m-1)}{1 \cdot 2} \alpha^2 a^{m-2} p^2 \pm \text{etc.}$$

all of whose terms, besides the first, are divisible by p : whence this quantity will leave the same residue when divided by p as if just the first term a^m were divided by p . Therefore as the power a^m leaves a residue $= r$, so too the power $(a \pm \alpha p)^m$ will leave a residue $= r$.

Corollary 1.

62. If m were an even number, the demonstration would also be valid for the formula $(-a + \alpha p)^m$, therefore in this case also the formula $(\alpha p - a)^m$, when divided by p will leave the same residue r that the formula a^m leaves.

Corollary 2.

63. But if m were an odd number, because the formula $-a^m$ leaves a residue $= -r$ when divided by p , also the formula $(\alpha p - a)^m$ shall leave a residue $= -r$.

Theorem 17.

64. If it were $a = c^n \pm \alpha p$, then the formula $a^{\frac{p-1}{n}}$ divided by the prime number p shall leave unity, supposing that n were a divisor of the number $p - 1$.

Demonstration.

Since it is $a = c^n \pm \alpha p$, the power $a^{\frac{p-1}{n}}$, or $(c^n \pm \alpha p)^{\frac{p-1}{n}}$, when divided by p will leave the same residue as the power $c^{n \cdot \frac{p-1}{n}}$ or c^{p-1} , and as p is a prime number, the power c^{p-1} will leave unity when divided by p , therefore also the power $a^{\frac{p-1}{n}}$ shall leave unity, supposing indeed that $a = c^n \pm \alpha p$, and neither a or c were divisible by p .

Corollary 1.

65. Therefore from this theorem the case may be understood in which the powers of the numbers, whose exponents are less than $p - 1$, leave unity when divided by the prime number p .

Corollary 2.

66. Therefore if it were $a = cc + \alpha p$, with p taken as a prime number, then the power $a^{\frac{p-1}{2}}$ will leave unity when divided by p , that is the formula $a^{\frac{p-1}{2}} - 1$ will be divisible by p . For were p a prime number, unless it were $= 2$, the exponent $\frac{p-1}{2}$ will always be an integral number.

Corollary 3.

67. If it were $a = c^3 \pm \alpha p$, then the power $a^{\frac{p-1}{3}}$ will leave unity when divided by p , that is the formula $a^{\frac{p-1}{3}} - 1$ will be divisible by p . This case will occur if the prime number p is disposed such that $p - 1$ were divisible by 3.

Theorem 18.

68. If it were $ab^n = c^n \pm \alpha p$, and p were a prime number, then the power $a^{\frac{p-1}{n}}$ will leave unity when divided by p , providing that $\frac{p-1}{n}$ is an integral number.

Demonstration.

The power $(c^n \pm \alpha p)^{\frac{p-1}{n}}$, or $a^{\frac{p-1}{n}} b^{p-1}$, will leave the same residue when divided by p as the power $c^{n \cdot \frac{p-1}{n}} = c^{p-1}$, and since this power leave unity, therefore so will the power $a^{\frac{p-1}{n}} b^{p-1}$. Moreover the factor b^{p-1} of this likewise leaves leave unity; therefore it is necessary for the other factor $a^{\frac{p-1}{n}}$, if divided by p , to leave unity, unless either b or c were divisible by p .

Corollary 1.

69. Therefore if $ab^n = c^n \pm \alpha p$, that is $ab^n - c^n$, or $c^n - ab^n$, were divisible by the prime number p , then too this formula $a^{\frac{p-1}{n}} - 1$ will be divisible by p .

Corollary 2.

70. Were p a prime number, where it may be put $p = mn + 1$, if the formula $ab^n - c^n$ or $c^n - ab^n$ divisible by p , then also the formula $a^m - 1$ will be divisible by this prime number p .

Corollary 3.

71. Therefore provided that numbers b and c are given of this type, that $ab^n - c^n$ or $c^n - ab^n$ admit division by a prime number $p = mn + 1$, then it is certain for the formula $a^m - 1$ to be divisible by the same prime number $p = mn + 1$.

Theorem 19.

72. If the formula $a^m - 1$ were to be divisible by a prime number $p = mn + 1$, then there will always be given numbers x and y of such a type that $ax^n - y^n$ will be divisible by this prime number p .

Demonstration.

For since x^{mn} and y^{mn} leave unity when divided by p , the formula $a^m x^{mn} - y^{mn}$ will always be divisible by p , provided that neither x or y are divisible by p . Since also by factors it would be $a^m x^{mn} - y^{mn} = (ax^n - y^n)(a^{m-1}x^{mn-n} + a^{m-2}x^{mn-2n}y^n + a^{m-3}x^{mn-3n}y^{2n} + \dots + y^{mn-1})$, if this ever denies the prime factor $ax^n - y^n$ to be divisible by p , it is compelled to affirm for the other factor to always be divisible by p , providing that the numbers taken for x and y are not divisible by p . Any value whatsoever may be retained for x , and for y we should take the successive numbers 1, 2, 3, 4, onto $p - 1 = mn$, not ever shall a value divisible by p be obtained, and were it for the sake of brevity

$$\begin{aligned}
A &= a^{m-1}x^{mn-n} + a^{m-2}x^{mn-2n} + \dots + 1 \\
B &= a^{m-1}x^{mn-n} + a^{m-2}x^{mn-2n}2^n + \dots + 2^{mn-n} \\
C &= a^{m-1}x^{mn-n} + a^{m-2}x^{mn-2n}3^n + \dots + 3^{mn-n} \\
&\vdots \\
N &= a^{m-1}x^{mn-n} + a^{m-2}x^{mn-2n}(mn)^n + \dots + (mn)^{mn-n}
\end{aligned}$$

and also all these quantities, which constitute an algebraic progression of order $mn - n$, would be divisible by p , and then moreover the first differences of these, the second, third and each in this order. But the differences of this series of order $mn - n$, which are defined only by the $mn - n + 1$ terms of this series, do not involve the term $(mn + 1)^{mn-n}$, or p^{mn-n} , because p is not able to be a value of y , it follows:

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots (mn - n)$$

which plainly is not divisible by the prime number $p = mn + 1$, because its only prime divisors are less than $mn - n$. Therefore since the difference of this order $mn - n$ is not divisible by p , it follows that not all the terms of the series A, B, C, D, \dots, D are divisible by p . For this case therefore, indeed for the cases of this y , in which the terms of this series are not divisible by p , by necessity the other factor $ax^n - y^n$ will not be divisible by p .

Corollary 1.

73. Therefore for whatever number is taken for x , if only not divisible by p , for y is always given a value $< p$ which renders the formula $ax^n - y^n$

divisible by p . And in a similar way, if any number we please is taken for y , it can be demonstrated that for x a number $< p$ can always be found by which that formula avoids divisibility by p .

Corollary 2.

74. Therefore if $a^m - 1$ would be divisible by a prime number $mn + 1 = p$, and for x were taken any number b not divisible by p , a number y can always be found so that the form $ab^n - y^n$, or $y^n - ab^n$, shall be divisible by $p = mn + 1$.

Corollary 3.

75. In a similar way if the form $a^m - 1$ were to be divisible by a prime number $p = mn + 1$, and for y were taken any number c not divisible by p , a number x will always be able to be found so that the form $ax^n - c^n$, or $c^n - ax^n$, shall be divisible by $p = mn + 1$.

Theorem 20.

76. If the form $ab^n - c^n$, or $c^n - ab^n$, were to be divisible by a prime number $p = mn + 1$, then by taking d to be any number we please, providing that it were not divisible by p , a number x will always be able to be found so that either the form $ax^n - d^n$, or $ad^n - x^n$, or $d^n - ax^n$, or $x^n - ad^n$ shall be divisible by this prime number $p = mn + 1$.

Demonstration.

Since this form $ab^n - c^n$, or $c^n - ab^n$, was divisible by a prime number $p = mn + 1$, then further the number $a^m - 1$ will be divisible by the same prime number $p = mn + 1$. (71) Truly if $a^m - 1$ is divisible by p , the number x will be given by taking any number d not divisible by p , so that either the form $ax^n - d^n$, or on the other hand the form $ad^n - x^n$, or $d^n - ax^n$, or $x^n - ad^n$, shall be thus divisible by the prime number $p = mn + 1$.

Corollary.

77. Therefore by putting $d = 1$, if the formula $ab^n - c^n$ were divisible by a prime number $p = mn + 1$, then a number x will be given so that either the form $ax^n - 1$, $a - x^n$, or $x^n - a$ shall be divisible by this prime number p .

Scholion.

78. The nineteenth theorem, which is the inverse of the eighteenth theorem, I shall now propose in another way, but without demonstration, and even though I have attempted the demonstration of it in many ways, I had not however been able to uncover it, until I took up the use of this method: which therefore is seen as most noteworthy, and there should be no doubt that this shall be applied to many other arcane journeys of numbers. For this method, which is encountered in the consideration of differences, was recently brought to life by me, which I demonstrated in the service of the most beautiful theorem of Fermat, by which every prime number of the form $4n + 1$ is affirmed to be the sum of two squares; before which I had been unable to persevere in any other way.