ESSAY CONCERNING A METAPHYSICAL DEMONSTRATION OF THE GENERAL PRINCIPLE OF EQUILIBRIUM

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I. As equilibrium is produced by the action of forces, we must begin by establishing the correct idea of force and the manner in which it acts. We call force, that which is capable of changing the state of a body as much in motion as in rest. For since a body itself always remains in the same state, either at rest or in motion, then if some change happens, the cause of this change necessarily stems from outside the body; and whatever this cause may be is called force. For any force there are two things to consider, quantity and direction. By quantity we mean how much a force is greater or smaller than another, whereas direction tells us the bearing towards which a force acts on a body to disturb its state.

II. To better clarify this idea (Fig. 1), let A be a body, on which a force acts along the direction EF. We see that if the body were at rest, it would be led by this force precisely in the direction EF. If it were in motion, it would be diverted from its route in this same direction. Thus, the direction is always the straight line, along which the force aims to transport the body. And this tendency suffices to determine the direction since it is not a question here of actual motion, which is imparted to the body by the force. This would no doubt entail a deeper study and would no

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longer be in the realm of metaphysics. Here I shall limit myself to establising only the idea of the direction in which a force acts on a body.

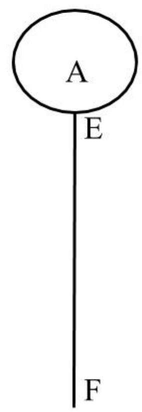

Fig. 1

III. It is also easy to formulate the correct idea of the quantity of a force in general. For since quantity can only be understood by comparison, we merely need to take a known amount of force to be unity, and this will serve as a common measure of all other forces. Thus taking as unity to mark this known force, when we know the number to attribute to the force EF, we shall then have the right idea of its quantity, since this number indicates how many times the force EF contains in itself the force that was taken to be unity.

IV. We can also consider it in this manner (Fig. 2): let us imagine body A as being attached to chord EF to which bar MM is held at right angles, which is then pulled towards another fixed bar NN by a certain number of threads, 11, 22, 33, 44, etc., each of which contracts with a force equal to that which is taken to be unity. Therefore if N indicates the number of these threads, this number indicates at the same time the force, with which body A is pulled by the chord EF, for all the forces, by which the threads tend to contract, contribute to bringing bar MM closer to the stationary bar NN. And from there, since the forces are each equal to 1 and their number $= N$, the total force that results in order to pull body A with bar MM will be $= N$, and at the same time chord EF will represent the direction of this force that acts on body A.

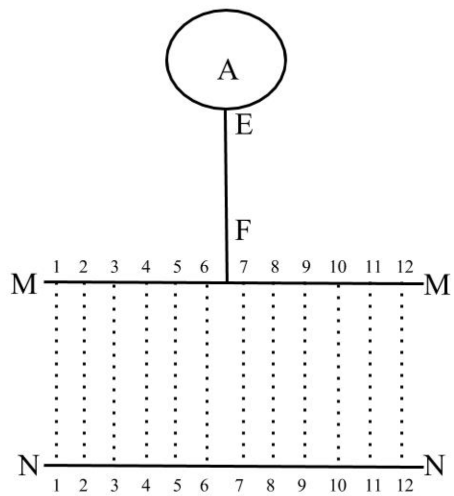

Fig. 2

V. Having established this, it is clear that the action of this force consists in the actual contraction of threads, 11, 22, 33, etc., and as body A is effectively pulled by chord EF, the action will be greater the shorter the threads become. For I suppose here that the threads always preserve the same force to contract, in such a way that the total force that results from their union remains constantly $= N$. Thus the shortening of the threads will provide the most correct measure of the action of the total force N. From there, if we suppose that they contract by the amount z, or that the length of each one of them is reduced by z, so that body A is pulled through space by an amount $= z$, this action will be expressed by the product Nz, which marks the total shortening of all the threads.

VI. Let x be the distance of body A to the fixed bar NN and b the length of cord EF, which we shall consider as a constant, so that $x - b$ is the length of each thread. Therefore the sum of the lengths of all the threads will be $= N(x-b)$, which is consequently the amount, whose reduction is the true object of force, or that the force, so much as it acts on body A tends to render this quantity $N(x - b)$ smaller and smaller. Now b being a constant, the action of the force consists in the reduction of the quantity Nx; for if the threads contract in the amount z, Nx will be reduced by the amount Nz.

VII. Therefore, this is what the goal of force is about, in a manner of speaking: it is to reduce the quantity Nx more and more, which is the product of force N with the distance of body A to the fixed bar NN. Now it is clear that this absolute distance does not strictly enter into consideration; for if we conceive of bar NN at any other distance from body A, the same contraction of threads will always produce the same reduction in the quantity Nx, provided that this bar is always perpendicular to the direction EF, along which we conceive the body is attracted by force N.

VIII. Given this idea of the action of each force, we shall easily deduce from it this general principle:

Every force acts as much as it is able.

And as soon as we understand the meaning of this proposition, we will not be able to refuse to take it as an axiom. For since the action of a force consists in the contraction of the threads, of which we conceive the force as being comprised, the threads will not cease to contract as long as they do not encounter an invincible obstacle that goes against their subsequent contraction. Therefore these threads, and consequently the force that they compose, will act as much as it can, or as much as the circumstances allow it to act.

IX. When a body or a system of bodies is in equilibrium, because the forces acting upon it are so opposed to one another that they could not possibly agitate or stir the body, it must be that the action of the forces must be greater, or that the threads, which we have conceived of as making up the force, are in their greatest contraction, in such a way that it would be impossible for them to contract even more. Thus a body or a system of bodies will be in equilibrium, when it is positioned with the forces acting upon it, in such a way that the threads are in their greatest contraction, or that the sum of the lengths of all the threads together be the smallest it can be.

X. Let us consider some force that contributes along with other forces to maintain a body in equilibrium; let this force $= N$. Now let us take an arbitrary fixed point in the direction of this force, and let x be the distance from this point to the body to which the force is applied. We have seen that through the contraction of the aforementioned threads, this quantity Nx diminishes. Therefore, if we assemble in a sum these expressions Nx, which correspond to each of the acting forces, this sum must be smallest since the contraction of all the threads taken together has to be greatest in the case of equilibrium.

XI. The force of this reasoning consists in how all forces are reduced into a certain number of identical and equal threads, which through their contracting force make up the forces themselves. Thus when the body sustaining the action of the forces is in equilibrium, it must be that according to our axiom, all these threads are in their greatest contraction. For if it were possible for them to contract even more, they would do so, and consequently the body would not be in equilibrium. Therefore, if the body is in equilibrium, it follows necessarily that the threads would not be able to contract any further, or which goes back to the same thing, the sum of all the acting forces is least.

XII. Therefore, this is a general rule for all equilibria of bodies with forces acting upon them, provided that these forces are constant, or that they pull with the same amount of effort, at some distances that the bodies find themselves in relation to these forces. According to this rule we shall consider each force separately, we shall take in its direction a fixed point and multiply the force by the distance from this point at the place of application of the force. Next, we shall assemble all the products into a sum, which will be a minimum in the case of equilibrium. Reciprocally therefore we can determine the state of equilibrium through the method of the maxima and minima when the forces are constant, or when the quantity N which has until now has stood for the force is not dependent on the quantity x, which we consider here to be a variable.

XIII. Of this kind is the force of gravity, as we make abstraction of the variation that it undergoes in greater or smaller distances from the center of the earth. Therefore, if we consider a body AB (Fig. 3), whose parts are only acted upon by gravity, we shall consider each particle M separately, as being pulled along the vertical direction MP. We shall then take an arbitrary fixed point P on the horizontal line NN and designate the distance $MP = x$. And calling the mass of the particle

$M = dM$, this dM shall mean at the same time the weight of the particle M, or the force acting upon it along MP. Therefore xdM shall be the product Nx for this particle. From there the sum of all the xdM, resulting from all the particles of the body, will be the smallest when the body is in equilibrium.

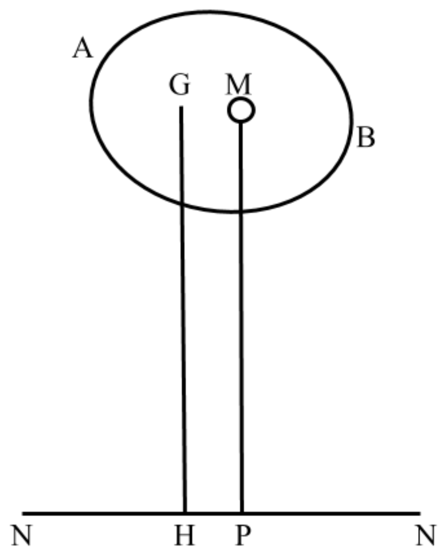

Fig. 3

XIV. But we know that the sum of all the xdM expresses the product of the entire weight of the body by the distance from its center of gravity along the same horizontal line to NN. Letting M represent the weight of the body, of which G is the center of gravity, and GH its distance to NN, the product MGH, being equal to the sum of all the xdM, will be a *minimum* in the case of equilibrium. From there we see that a weighted body would not be in equilibrium unless its center of gravity is as low as possible. Thus this grand principle of greatest descent of the center of gravity, which has been known for a long time, without ever having been proven, is a necessary consequence of what I have just established.

XV. But the rule that has been established up to now can only be applied to forces that act with the same effort at all distances or that are constant. For if the forces were not constant, their resolution into threads as used above can no longer happen, or else it must be supposed that the number of threads be variable while they contract. For this purpose what we have established above $= Nx$ must be decomposed into its elements Ndx. And since for each distance x the force or the number of threads is variable, say let it be $= P$, and having Pdx for the element of the distance dx, the integral $\int Pdx$ will be the proper value that must be taken in place of Nx, whenever the force is variable.

XVI. To further bring this to light, we only need to see how the formulas Nx taken from the constant forces become a *minimum*. Now this happens when their differentials Ndx taken together vanish. And in these differentials, it is no longer a question of whether or not the force N is constant. Therefore, if the force is variable, say $= P$, we shall have Pdx instead of Ndx to set their sum equal to zero; from there, it is clear that the formula which then becomes a *minimum*, will be composed of these $\int Pdx$, which we need to take from each of the acting forces, where it is clear that in the case when the forces are constant, that is $P = N$, we shall have the same formulae Nx that we have found above to give a *minimum*.

XVII. Thus here is the universal Principle which applies to all states of equilibrium, and whose truth has just been deduced from axioms that no one could call into question. By virtue of this principle, we shall consider each force that acts upon the body in equilibrium separately in this manner: let P, Q, R, etc. (Fig. 4), be the forces acting upon body M in the direction AF, BG, CH (respectively). Upon these we pick arbitrary fixed points F, G, and H. And having named the distances $AF = x$, $BG = y$, and $CH = z$, the state of equilibrium should always have this property, that the sum of the expressions

$$\int Pdx + \int Qdy + \int Rdz + etc.$$

is a *minimum*.

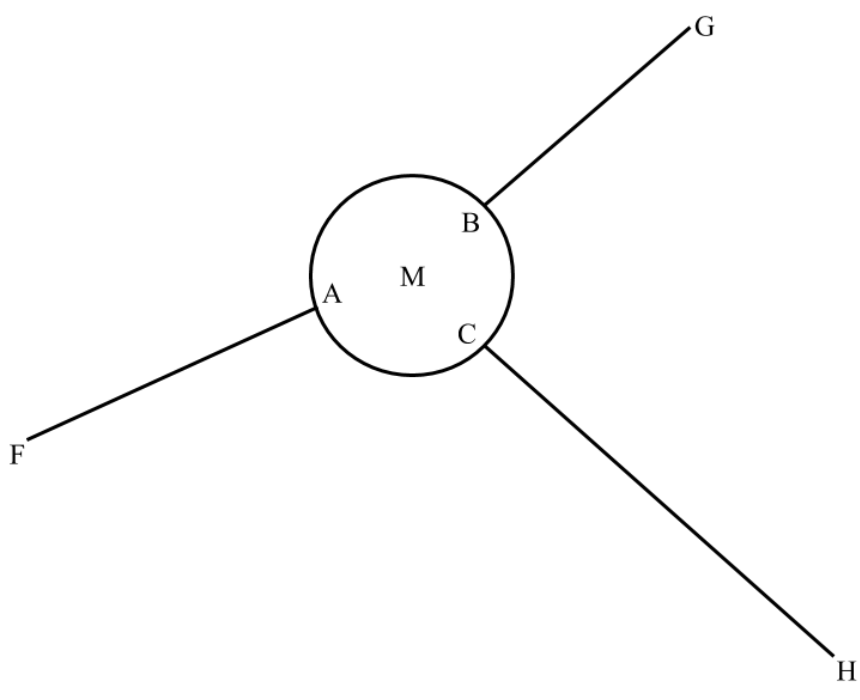

Fig. 4

XVIII. We see that it would not matter in this study what distance we take the points F, G, and H provided that they be regarded in the calculations as fixed, for the differentials dx, dy, and dz, will always stay the same. Now when the forces are variable, as we consider central forces to be, expressing them as certain functions of the distances to their centers, it shall be appropriate for convenience of calculation to take the points F, G, and H to be at the centers themselves of the forces. Thus if it were that[1] $P = \alpha x^n$, $Q = \beta y^n$, and $R = \gamma z^n$, the expression to bring to a *minimum* would be

$$\frac{\alpha}{n+1}x^{n+1} + \frac{\beta}{n+1}y^{n+1} + \frac{\gamma}{n+1}z^{n+1} + etc.$$

and this search for the *minimum* will easily come out in all cases where such forces are found.

XIX. Since each force P furnishes in the calculation such a formula $\int Pdx$, this value is indubitably something that is quite essential in the action of forces, since equilibrium depends upon it uniquely, so that, provided we have the value $\int Pdx$, without considering the force itself, we are in the position to determine the equilibrium, or that the quantity $\int Pdx$ contributes essentially to determine the equilibrium. It is therefore very reasonable to give to this quantity a particular name, which applies to its usage. And it seems to me that the term *effort* expresses the nature of this usage rather well.

XX. Therefore to assess equilibrium, we must first find the amount of effort that applies to each acting force. To do this, having taken in the direction of the force a

[1]Translators' note: In the original publication, the equation for Q was printed as $Q = \delta y^n$. This was corrected and noted in the version that appears in the *Opera Omnia*.

fixed point F and setting the distance $AF = x$, we must then simply multiply the force P itself by the differential of this distance dx, and the integral $\int P dx$ shall be the effort of the force P. Thus the universal principle of equilibrium that we have just demonstrated shall be expressed in this rather simple rule:

The sum of all the efforts, to which a body in equilibrium is subject, is a minimum.

XXI. When the body for which we seek the state of equilibrium is flexible or even fluid, all of its elements, as well as all the forces acting upon it, must be considered separately to obtain the effort to which each element is subject. Next, the sum of these efforts, or the total effort acting on the body, is found through integration, and having rendered a *minimum*, will show the conditions of equilibrium. The application of this principle that I have demonstrated on an infinity of different cases, as much in the relationship to the nature of the body as the diversity of forces, have sufficiently shown the importance and great advantages that we still hope from it.

XXII. I shall conclude with a remark that will contribute to an even clearer understanding of how this principle is linked with the state of equilibrium. This remark is the idea that we do not need to introduce into the calculation of equilibrium those forces that attach a body to fixed object or maintain it at rest. Thus, to find through this method the curve of a suspended chain, we shall not look at the forces that hold the nails from which the chain is suspended; and when it is a question of equilibrium of a fluid surrounded by a vessel, it is not necessary to consider the forces exerted by the fluid pressing on the vessel. Rather, in one and the other case, it will suffice to consider only the force of gravity to determine the state of equilibrium.

XXIII. The reason for this difference is easily understood through the way in which we have considered the action of the forces which involves the contraction of threads. Thus, if there were forces whose actions the body could not possibly respond to, such as those that support or stop the body, or those that attach an immovable object to it, such forces do not figure into our calculation, or rather their efforts must be considered as vanishing, since the parts of the body fixed by them are effectively immovable. Therefore, as these forces are excluded, there remains to consider only the forces capable of imparting some sort of movement on the body in order to find the state of equilibrium, by taking their efforts and rendering their sum a *minimum*.