DE SERIERVM DETERMINATIONE SEV NOVA METHODVS INVENIENDI TERMINOS GENERALES SERIERVM.

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§. 1.

Cum lex progressionis, quam termini cuiusque seriei tenent, in infinitum variare possit, non solum omnes diuersae serierum species, sed etiam ne genera quidem, quantumuis late extendantur, enumerari posse videntur. Hinc duae pluresque series dantur, quae etiamsi tot, quot quis voluerit, habeant terminos communes, tamen inter se discrepent, ac maxime diuersis legibus contineantur. Qui amplissimum serierum campum vel obiter inspexerit, facile intelliget, naturam seriei non determinari, quotcunque etiam eius termini exhibeantur. Sic si quaeratur, qualis sit series, quae ab his incipiat terminis: 1, 3, 5, 7, 9, 11, 13, 15; quaestio maxime est indeterminata; et praeter seriem numerorum imparium naturali ordine procedentium innumerabiles aliae assignari possunt series, quae ab iisdem terminis incipiant: neque iste determinationis defectus ad certum terminorum datorum numerum adstringitur, sed quantuscunque is fuerit, infinitis seriebus communis esse potest.

§. 2. Clarius autem hoc perspicietur, si naturam serierum ad Geometriam transferamus. Quaelibet enim series

ſeries per lineam curuam repraeſentari poteſt, cuius applicatae per ipſos ſeriei terminos exprimantur, dum abſciſſae eorum indices, ſeu numeros, qui ordinem cuiusque termini deſignant, referunt. Hoc modo quilibet ſeriei terminus punctum in linea curua delinit, quod datae abſciſſae reſpondet. Quare ſi ſeries requiratur, quae tot, quot libuerit, habeat terminos datos, quaeſtio huc redit, vt quaeratur linea curua, quae per totidem puncta data tranſeat. Perſpicuum autem eſt, quotcunque etiam data fuerint puncta, ſemper innumerabiles lineas curuas aſſignari poſſe, quae per ſingula ſimul tranſeant. Quod cum NEVTONVS *de ſolis curuis parabolicis* oſtendiſſet, ſi non ſolum omnes curuae Algebraicae, ſed etiam tranſcendentes admittantur, dubium eſt nullum, quin numerus curuarum ſatisfacientium inſuper infinities fiat maior.

§. 3. Magis mirum videbitur, ſi dixero, ſeriem nondum determinari, etiamſi innumeri eius termini dentur. Sic ſi hanc ſeriem 1 + 2 + 3 + 4 + 5 + 6 + etc. ita definiam, vt dicam, in ea omnes numeros integros naturali ordine contineri, quis non putet hanc ſeriem penitus eſſe determinatam? cum cuiuis ſeriei loco ſuus terminus ſit aſſignatus: in loco enim, qui x vnitatibus ab initio diſtat, erit terminus = ipſi numero x, ſeu terminus, cuius index $= x$, ipſe quoque erit $= x$. Quatenus autem illa ſeries ita vt factum eſt deſcribitur, plus inde non conſtat, niſi indici x, ſi fuerit x, numerus integer, reſpondere terminum $= x$: ſin autem pro indice x aſſumatur numerus fractus, nulla adhuc ratio adeſt, qua euinceretur, terminum iſti indici x reſpondentem eſſe $= x$. Oſtendam autem, ſi pro hac ſerie

 termi-

terminus indici x respondens ponatur $= y$, infinitis modis fieri posse, vt quoties x sit numerus integer, toties semper fiat $y = x$, etiamsi numeris fractis pro x sumendis, valores ipsius y ab x discrepent. Hinc etsi omnes seriei termini, qui indicibus integris respondent, sunt determinati, intermedios tamen, qui indices habent fractos infinitis variis modis definire licet, ita vt interpolatio istius seriei maneat indeterminata.

§. 4. Quod, quo clarius perspiciatur, ad arcus circulares est recurrendum: cum enim posita semicircumferentia circuli $= \pi$ cuius radius sit $= 1$; sit sinus arcus $n\pi = 0$, quoties n est numerus integer: manifestum est, si ponatur $y = x + P \sin. \pi x$, denotante P, vel quantitatem constantem, vel functionem quamcunque ipsius x; ac pro x successiue ponantur indices integri 1, 2, 3, 4, 5, etc: tum valores ipsius y futuros esse $=$ 1, 2, 3, 4, 5, etc. perinde ac si esset $P = 0$. Neque tamen termini intermedii, qui indicibus fractis respondent, his ipsis indicibus erunt aequales. Sit enim e. g. $P = xx$ et ponatur $- = \frac{1}{2}$; ob $\sin. \frac{1}{2}\pi = 1$, fiet terminus indici $\frac{1}{2}$ respondens $= \frac{1}{2} + \frac{1}{4}. 1 = \frac{3}{4}$. Infinitae autem aliae huiusmodi expressiones excogitari possunt, quae aeque satisfaciant, cuiusmodi sunt: $y = x + P \sin. \pi x + Q \sin. 2\pi x + R \sin. 3\pi x + S \sin. 4\pi x$ etc. quibus interpolatio multo magis indeterminata redditur.

§ 5. Simile exemplum seriei, quae determinata videri queat, iam ante aliquod tempus exhibui: inueneram enim expressionem, seu functionem ipsius x, quae si loco x potestas quaecunque ipsius 10. ponatur, ipsi exponenti huius potestatis aequalis fiat, siquidem hic exponens sit numerus integer affirmatiuus. Functio scilicet illa ipsius x, quam

quam littera y indicabo, ita erat comparata, vt posito $x=1$, fiat $y=0$; et, si ponatur $x=10^n$, existente n numero integro affirmatiuo, fit semper $y=n$: vnde sequi videbatur, functionem y semper fore logarithmum vulgarem ipsius x. Nihilo vero minus monstraui, si pro x non qua piam denarii potestas substituatur, valorem ipsius y saepe numero non parum a logarithmo numeri x discrepare. Facta ergo serie, cuius sint

Indices $1, 10^1, 10^2, 10^3, 10^4, 10^5, 10^6$, etc. et
termini $0, 1, 2, 3, 4, 5, 6$, etc.

ad descriptionem logarithmorum non sufficit, si quis dicat, logarithmos esse terminos medios inferioris seriei, qui indicibus in superiori serie assumtis respondeant.

§. 6. Cum igitur natura seriei non ex aliquot eius terminis, etiamsi eorum numerus sit infinitus, determinetur; propterea quod interpolatio nihilominus maneat indeterminata, infinitisque modis absolui possit; facile perspicitur, quam incertae sint omnes illae interpolandi methodi, quae negotium ex solis terminis integros indices habentibus perficere docent. Neque enim interpolatio pro certa haberi poterit, nisi ipsa seriei natura spectetur, eiusque ratio in operatione habeatur. Perfecte autem natura seriei cognoscitur, si eius terminus generalis, seu formula, quae cuiuis indici x, siue integro, siue fracto, siue etiam surdo, terminum respondentem exhibeat, fuerit cognita. Hoc enim modo non solum omnes seriei termini, qui indicibus integris respondent, determinantur, sed etiam termini, qui indicibus quibuscunque non integris conueniunt, sine ambiguitate definiuntur; sicque interpolationis negotium nulla amplius incertitudine impeditur.

§. 7.

§. 7. Habentur autem praeter terminum generalem innumerabiles alii modi ſeries formandi: interim tamen omnes iſti modi commode ad tria genera reuocari poſſunt. Ad primum genus refero eos ſerierum formandarum modos, quibus terminus quisque ſeriei per ſolum indicem reſpondentem determinatur; quod cum per certas operationes in hunc finem inſtituendas efficiatur, formula iſtas operationes in genere complectens ipſe erit terminus generalis ſeriei, quo pacto ſeriem abſolute ac perfectiſſime determinari iam notaui. Ad genus ſecundum pertineant iſti ſeries formandi modi, quibus terminus quiuis ſeriei per aliquot terminos antecedentes ſecundum certam quandam regulam determinatur, qui modus in ſeriebus imprimis recurrentibus adhiberi ſolet. Quando vero ad terminum quemuis ſeriei inueniendum non ſolum terminorum antecedentium ratio eſt habenda, ſed etiam ipſe index adhiberi debet, hinc tertium genus affirmationis ſerierum conſtituo.

§. 8. Si quilibet ſeriei terminus ex ſolo indice determinatur, tum ſiue numerus integer, ſiue fractus, pro indice aſſumatur, terminus reſpondens aeque definitur, ſicque interpolatio ſeriei, neque quicquam difficultatis, neque incertitudinis habet. Sin autem, vti in ſecundo genere poſuimus, quilibet terminus ex praecedente vel aliquot antecedentibus determinatur, tum primo vel aliquot primoribus terminis pro lubitu aſſumtis, ſinguli quidem termini, qui indicibus integris reſpondent, inuenientur, terminos vero intermedios, indicibus fractis conuenientes hinc definire non licet; quod idem de tertio genere eſt tenendum. Quamquam autem hoc modo in ſecundo et tertio

tertio genere non ſolum omnes termini, qui indicibus integris reſpondent, aſſignantur, ſed etiam lex praeſcribitur inter terminum quemuis eiusque antecedentes, quae ad terminos indicum fractorum aeque patet; tamen ne hoc quidem modo ſeries penitus determinatur, ſed pro qualibet ſerie huius generis infiniti termini generales exhiberi poſſunt, qui dum eosdem terminos pro indicibus integris praebent, tamen pro fractis diſſentiunt.

§. 9. Quod cum merito maxime paradoxon videatur, operae pretium erit, hunc determinationis defectum in ſeriebus, quarum quisque terminus ex antecedentibus definitur, diligentius perpendere. Sumamus ergo caſum ſimpliciſſimum, ſeriemque ita definiri concipiamus, vt quilibet terminus aequalis ſit antecedenti ipſi. Quodſi primus ſeriei terminus ſtatuatur $=1$, ſecundus quoque erit $=1$, omnesque ſequentes, qui indicibus integris reſpondent, vnitati aequabuntur, naſceturque haec ſeries:

Indic:	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	etc.
Term:	1,	1,	1,	1,	1,	1,	1,	1,	1,	1,	etc.

atque manifeſtum eſt, indici cuicunque integro x reſpondere terminum $=1$. Quemadmodum autem termini indicibus fractis reſpondentes ſe ſint habituri, hinc non definitur: hoc tantum conſtat, ſi terminus indici $\frac{1}{2}$ reſpondens fuerit $=a$ omnes quoque terminos, qui indicibus $\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$ etc. conueniunt, fore $=a$. Omnes enim terminos, quorum indices vnitate, vel aliquot vnitatibus, differunt, per legem praeſcriptam inter ſe aequales eſſe oportet: quia antecedens quisque terminus intelligitur is, cuius index eſt vnitate minor.

§. 10. Haec igitur ſeries ita definitur, vt, ſi terminus indici x reſpondens ponatur $= y$, ſequens vero terminus indici $x + 1$ reſpondens $= y'$, habeatur $y' = y$; tum vero praeterea aſſumitur, ſi fuerit $x = 1$, fore quoque $y = 1$. Quare ſi pro hac ſerie terminus generalis deſideretur, is eiusmodi functio ipſius x eſſe debet, quae ſit $= y$, vt ſi loco x ponatur $x + 1$ functionis y valor reſultans y' aequalis ſit futurus ipſi y, atque vt facto $x = 1$ fiat $y = 1$. Manifeſtum autem eſt, ſi generatim ponatur $y = 1$, huic conditioni ſatisfieri, hocque caſu non ſolum terminos, qui indicibus integris, ſed etiam eos, qui fractis reſpondeant, vnitati aequales fore. At vero his conditionibus infinitis quoque aliis modis ſatisfieri poteſt: ſi enim ponatur $y = 1 + \alpha \text{ ſin. } 2\pi x$, denotante π ſemicircumferentiam circuli, cuius radices $= 1$, erit $y' = 1 + \alpha \text{ ſin. } 2\pi(x+1)$; at eſt ſin. $2\pi(x+1) = \text{ſin. } 2\pi x$, ideoque $y' = y$, tum vero poſito $x = 1$ erit $y = 1$. Hoc vero caſu termini intermedii, ſeu qui indicibus fractis reſpondent, non amplius vnitati aequabuntur, poſito enim $x = \frac{1}{4}$ erit $y = 1 + \alpha$.

§. 11. Quoniam hic non ſolum α pro arbitrio aſſumi poteſt, ſed etiam innumerabiles aliae eiusmodi formulae excogitari poſſunt, quae praeſcriptas conditiones adimpleant, cuiusmodi ſunt $y = 1 + \alpha \text{ ſin. } 2\pi x + \beta \text{ ſin. } 4\pi x + \gamma \text{ ſin. } 6\pi x + \text{etc.}$ perſpicuum eſt, interpolationem vel huius ſimpliciſſimae ſeriei $1 + 1 + 1 + \text{etc.}$ quatenus aliter non definitur, niſi quod quilibet terminus antecedenti aequalis eſſe, primus vero vnitate exprimi dicatur, interpolationem maxime eſſe indeterminatam: cum termini intermedii indices habentes fractos, quibuscunque numeris

meris aequales esse queant. Interim tamen etiamsi innumerabiles termini generales pro hac serie exhiberi queant, tamen ii omnes in lege quadam generali continentur, atque sine diuinatione per analysin inueniri possunt. Methodus scilicet latissime patens tradi potest, cuius ope omnium serierum, quarum termini per antecedentes, siue sine indice, siue cum indice, definiuntur, terminos generales vniuersalissime inueniri licet: quam methodum, cum non solum pleniorem serierum cognitionem suppeditet, sed etiam non contemnenda analyseos augmenta complectatur, hic diligentius euoluere constitui: quem in finem sequentia problemata pertractabo.

Problema.

§. 12. Inuenire terminum generalem seriei, cuius quilibet terminus aequalis sit antecedenti, terminus vero primus $= 1$.

Solutio.

Sit terminus generalis seu is, qui indici x respondet $= y$, ac ponatur terminus sequens, (cuius index $= x + 1$) $= y'$ debebitque esse $y' = y$; ac posito $x = 1$, fieri debet $y = 1$. Cum iam sit y quaepiam functio ipsius x, per naturam calculi differentialis, si loco x ponatur $x + 1$, fiet:

$$y' = y + \frac{dy}{1dx} + \frac{ddy}{1.2dx^2} + \frac{d^3y}{1.2.3dx^3} + \frac{d^4y}{1.2.3.4dx^4} + \text{etc.}$$

sumto differentiali dx constante. Quocirca esse debet:

$$0 = \frac{dy}{1.dx} + \frac{ddy}{1.2dx^2} + \frac{d^3y}{1.2.3dx^3} + \frac{d^4y}{1.2.3.4dx^4} + \text{etc.}$$

Haecque aequatio omnes omnino satisfacientes valores ipsius y continet, dummodo integratio ita temperetur, vt posito $x = 1$ fiat $y = 1$, seu quod eodem redit, vt

 posito

posito $x = 0$ fiat $y = 1$. Quaestio itaque perducta est ad resolutionem istius aequationis differentialis, quae non solum infinito terminorum numero constat, sed etiam omnes differentialium gradus in se complectitur. Quia vero variabilis y vbique plus vna dimensione non habet, et alterius variabilis x non nisi differentiale dx, quod constans est assumtum, occurrit, haec aequatio eo modo tractari potest, quem in *Miscell. Berol. Tomo. VII.* exposui. Formetur igitur ponendo z loco $\frac{dy}{dx}$; z^2 loco $\frac{ddy}{dx^2}$ et generatim z^n loco $\frac{d^ny}{dx^n}$ aequatio Algebraica:

$$0 = \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \text{etc.}$$

quae sumto e pro numero cuius logarithmus hyperbolicus $= 1$, transit in hanc formam finitam $0 = e^z - 1$. Huius iam aequationis omnes radices, quarum numerus est infinitus, inuestigari, seu omnes factores formulae $e^z - 1$ assignari oportet. Est vero $e^z = (1 + \frac{z}{n})^n$, posito n numero infinito, qui valor, si substituatur, habebitur haec formula resoluenda $(1 + \frac{z}{n})^n - 1$, cuius quidem vnus factor simplex est $= \frac{z}{n}$ seu z: quem aequatio infinita statim monstrat. Ad reliquos inueniendos in subsidium vocari debet Theorema, quo demonstratur formulae binomiae $a^n - b^n$ factorem esse $aa - 2ab \cos. \frac{2k\pi}{n} + bb$, denotante k numerum quemuis integrum. Praesenti ergo casu est $a = 1 + \frac{z}{n}$ et $b = 1$, vnde formulae propositae $e^z - 1$ omnes factores continentur in hac forma generali.

$$1 + \frac{2z}{n} + \frac{zz}{nn} - 2(1 + \frac{z}{n}) \cos. \frac{2k\pi}{n} + 1$$

seu $2(1 + \frac{z}{n}) \sin. \frac{2k\pi}{n} + \frac{zz}{nn}$: vnde hunc factorem per

quan-

quantitatem constantem 2 fin. $\frac{2k\pi}{n}$ diuidendo erit factor generalis $= 1 + \frac{z}{n} + \frac{zz}{2\,nn\,\text{fin.}\,v\,\frac{2k\pi}{n}}$. Cum iam n fit numerus infinitus erit cof. $\frac{2k\pi}{n} = 1 - \frac{2kk\pi\pi}{nn}$ et fin. v, $\frac{2k\pi}{n} = \frac{2kk\pi\pi}{nn}$: quo valore fubftituto, erit factor formulae $e^z - 1$ generalis $= 1 + \frac{z}{n} + \frac{zz}{4kk\pi\pi}$, et loco k fucceffiue omnes numeros integros 1, 2, 3, 4, etc. fubftituendo orientur omnes omnino factores formulae $e^z - 1$. At primus factor z dat integralis partem conftantem, quae fit $= C$: reliqui vero factores, qui ad hanc formam reducuntur

$$4kk\pi\pi + \frac{4kk\pi\pi}{n}z + zz;$$

fi cum forma factorum, quos in ante allegata differtatione euolui, $ff - 2fz\,\text{cof.}\,\Phi + zz$ comparentur, erit $f = 2k\pi$ et cof. $\Phi = -\frac{k\pi}{n}$; et fin. $\Phi = 1$ ob n numerum infinitum, quo cafu eft cof. $\Phi = 0$. Pars ergo integralis hinc oriunda erit

$\alpha e^{\frac{-2kk\pi\pi}{n}x}\,\text{fin.}\,2k\pi x + \mathfrak{A} e^{\frac{-2kk\pi\pi}{n}x}\,\text{cof.}\,2k\pi x$; feu ob $n = \infty$ $\alpha\,\text{fin.}\,2k\pi x + \mathfrak{A}\,\text{cof.}\,2k\pi x$. Subftitutis ergo pro k fucceffiue omnibus numeris integris 1, 2, 3, 4, etc. proueniet integrale aequationis inuentae.

$$0 = \frac{1dy}{dx} + \frac{ddy}{1.2dx^2} + \frac{d^3y}{1.2.3\,dx^3} + \frac{d^4y}{1.2.3.4\,dx^4} + \text{etc.}$$

fequenti forma expreffum:

$$y = C + \alpha\,\text{fin.}\,2\pi x + \mathfrak{A}\,\text{cof.}\,2\pi x + \beta\,\text{fin.}\,4\pi x + \mathfrak{B}\,\text{cof.}\,4\pi x + \gamma\,\text{fin.}\,6\pi x + \mathfrak{C}\,\text{cof.}\,6\pi x + \text{etc.}$$

Iam conftans C ita definiatur vt pofito $x = 0$ fiat $y = 1$ reperieturque terminus generalis feriei propofitae:

 $y =$

$$y = 1 + \alpha \sin. 2\pi x + \mathfrak{A}(\cos. 2\pi x - 1) + \\ + \beta \sin. 4\pi x + \mathfrak{B}(\cos. 4\pi x - 1) + \\ + \gamma \sin. 6\pi x + \mathfrak{C}(\cos. 6\pi x - 1) + \\ \text{etc.}$$

Quicunque ergo valores loco $\alpha, \beta, \gamma, \delta$, etc. $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, etc. ſubſtituantur, ſemper prodibit formula, quae terminum generalem ſeriei propoſitae exhibet. Q. E. I.

Coroll. 1.

§. 13. Si terminus primus, cui omnes reliqui, habentes exponentes integros, ſunt aequales, non debeat eſſe vnitas, ſed quantitas quaecunque, terminus generalis ſeriei y, ſeu terminus, qui indici x reſpondet, reperitur:

$$y = a + \alpha \sin. 2\pi x + \beta \sin. 4\pi x + \gamma \sin. 6\pi x + \delta \sin. 8\pi x \\ + \mathfrak{A} \cos. 2\pi x + \mathfrak{B} \cos. 4\pi x + \mathfrak{C} \cos. 6\pi x + \mathfrak{D} \cos. 8\pi x \; \text{etc.}$$

eritque terminus quilibet indicem habens numerum integrum $= a + \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} +$ etc.

Coroll. 2.

§ 14. Quia ſinus et coſinus arcuum $4\pi x$, $6\pi x$, $8\pi x$ etc. per poteſtates ipſorum ſin. $2\pi x$ coſ. $2\pi x$ exprimi poſſunt; atque viciſſim omnes functiones rationales, ſeu quae ambiguas ſignificationes non habent, per huiusmodi ſeries, qualem pro y inuenimus, exhiberi poſſunt: terminum generalem y ita definire poterimus, vt dicamus, y eſſe functionem quamcunque ipſorum ſin. $2\pi x$ et coſ. $2\pi x$: dummodo non occurrant huiusmodi formulae $\sqrt{(1 \pm \cos. 2\pi x)}$, aliaeque ſimiles, quae involuunt ſinus vel coſinus angulorum ſubmultiplorum ipſius $2\pi x$.

Coroll.

Coroll. 3.

§. 15. His igitur casibus exclusis, si ponamus sin. $2\pi x = p$ et cos. $2\pi x = q$, erit y aequalis functioni cuicunque ipsarum p et q: vnde ista aequatio differentialis infinita

$$0 = \frac{dy}{1dx} + \frac{ddy}{1.2dx^2} + \frac{d^3y}{1.2.3dx^3} + \frac{d^4y}{1.2.3.4dx^4} + \text{etc.}$$

in genere ita integrabitur, vt sit y functio quaecunque ipsarum p et q.

Coroll. 4.

§. 16. Sin autem vocemus sin. $\pi x = r$ et cos. $\pi x = s$, erit $p = 2rs$ et $q = ss - rr$; et functiones ipsarum p et q erunt functiones parium dimensionum ipsarum r et s. Quare ex illa aequatione differentiali infinita valor ipsius y in genere aequabitur functioni cuicunque parium dimensionum ipsarum r et s, vbi notandum ob sinum totum $= 1$ esse $rr + ss = 1$.

Coroll. 5.

§. 17. Ponatur $\frac{x}{a}$ loco x, vt habeatur ista aequatio

$$0 = \frac{a\,dy}{1dx} + \frac{a^2ddy}{1.2dx^2} + \frac{a^3d^3y}{1.2.3dx^3} + \frac{a^4d^4y}{1.2.3.4dx^4} + \text{etc.}$$

Si iam ponamus sin. $\frac{\pi x}{a} = r$ et cos. $\frac{\pi x}{a} = s$, integrale istius aequationis ita describetur, vt sit $y =$ functioni cuicunque parium dimensionum ipsarum r et s.

Coroll. 6.

§. 18. Gemina ergo formula pro valore huius integralis exhiberi potest, quarum altera est:

$$y = \frac{A + Br^2 + Crs + Dss + Er^4 + Fr^3s + Gr^2s^2 + Hrs^3 + Is^4 + \text{etc.}}{\alpha + \beta r^2 + \gamma rs + \delta s^2 + \varepsilon r^4 + \zeta r^3s + \eta r^2s^2 + \theta rs^3 + \iota s^4 + \text{etc.}}$$

altera vero forma erit:

$y =$

$$y = \frac{Ar + Bs + Cr^3 + Dr^2s + Ers^2 + Fs^3 + Gr^5 + \text{etc.}}{\alpha r + \beta s + \gamma r^3 + \delta r^2 s + \varepsilon rs^2 + \zeta s^3 + \eta r^5 + \text{etc.}}$$

Coroll. 7.

§. 19. Quicunque ergo huiusmodi valor pro y in aequatione:

$$0 = \frac{a\,dy}{1\,dx} + \frac{a^2\,ddy}{1\cdot 2\,dx^2} + \frac{a^3\,d^3y}{1\cdot 2\cdot 3\,dx^3} + \frac{a^4\,d^4y}{1\cdot 2\cdot 3\cdot 4\,dx^4} + \text{etc.}$$

ſubſtituatur, aequatio prodibit identica: ſeu ſeries proueniet infinita, cuius ſumma erit $= 0$. Pro differentiationibus autem continuis tenendum eſt eſſe $\frac{dr}{dx} = \frac{\pi s}{a}$ et $\frac{ds}{dx} = -\frac{\pi r}{a}$ ideoque per ſubſtitutionem differentialia dx ſe mutuo vbique tollent.

Scholion 1.

§. 20. Notari etiam merentur factores, in quos expreſſio Algebraica infinita: $\frac{z}{1} + \frac{z^2}{1\cdot 2} + \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4} + \frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} +$ etc. ſupra eſt reſoluta. Cum enim primus factor ſimplex ſit $= z$, et reliqui trinomiales in hac forma generali contineantur: $1 + \frac{z}{n} + \frac{zz}{4kk\pi\pi}$; ſi loco k ſucceſſiue ponantur numeri 1, 2, 3, 4, etc. Ponamus breuitatis gratia:

$$Z = \frac{z}{1} + \frac{z^2}{1\cdot 2} + \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

eritque per factores infinitos:

$$Z = z\left(1 + \frac{z}{n} + \frac{zz}{4\pi\pi}\right)\left(1 + \frac{z}{n} + \frac{zz}{16\pi\pi}\right)\left(1 + \frac{z}{n} + \frac{zz}{33\pi\pi}\right)\left(1 + \frac{z}{n} + \frac{zz}{69\pi\pi}\right)\text{etc.}$$

quorum factorum, excepto primo, numerus eſt infinitus atque $= \frac{1}{2}n$. Sit igitur $\frac{1}{2}n = m$ ſeu $n = 2m$, ac ponatur $z = 2v$ erit $\frac{2v}{1} + \frac{2^2v^2}{1\cdot 2} + \frac{2^3v^3}{1\cdot 2\cdot 3} + \frac{2^4v^4}{1\cdot 2\cdot 3\cdot 4} +$ etc.

$$2v\left(1 + \frac{v}{m} + \frac{vv}{\pi\pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{4\pi\pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{9\pi\pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{16\pi\pi}\right)\text{etc.}$$

ideoque

ideoque ſequens productum infinitorum factorum, quorum numerus eſt $= m$, erit:

$$\left(1+\frac{v}{m}+\frac{vv}{\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{4\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{9\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{16\pi\pi}\right) \text{ etc.}$$
$$=1+\frac{2}{1\cdot 2}v+\frac{4}{1\cdot 2\cdot 3}v^2+\frac{8}{1\cdot 2\cdot 3\cdot 4}v^3+\frac{16}{1\cdot 2\cdot 3\cdot 4\cdot 5}v^4+\text{ etc.}$$

Quodſi iam illud productum actu euoluatur, quia factorum numerus eſt $= m$, exiſtente m numero infinito, proueniet:

$$1+v+\frac{m(m-1)}{1\cdot 2}\frac{vv}{mm}+\frac{vv}{\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{ etc.}\right)$$
$$+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\frac{v^3}{m^3}+\frac{(m-1)v^3}{m\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{ etc.}\right)$$
$$\text{etc.}$$

qui termini cum ſerie iam inuenta comparati dabunt

$$1=\frac{2}{1\cdot 2};\ \frac{1}{1\cdot 2}+\frac{1}{\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{ etc.}\right)=\frac{4}{1\cdot 2\cdot 3}$$
$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{ etc.}\right)=\frac{8}{1\cdot 2\cdot 3\cdot 4}$$

Vnde vtrinque habetur:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\text{ etc.}=\frac{1}{6}\pi\pi$$

quae eſt eadem ſummatio, quam iam ante complures annos primus inueneram; pluribusque demonſtrationibus confirmaueram. Ceterum hinc perſpicuum eſt, etiamſi ſit in his factoribus m numerus infinitus; tamen alterum terminum $\frac{v}{m}$ omitti non licere: cum in euolutione ob remplicationem infinitam ex terminis infinite paruis $\frac{v}{m}$ termini finiti exſurgant. Quando autem quilibet factor ſeorſim conſideratur, vt in formatione integralis fecimus, tum ſine errore hos terminos infinite paruos praetermittere licuit.

Scholion 2.

§. 21. Altiores quoque poteſtates terminorum ſeriei

$1 + \frac{1}{4} + \frac{1}{9} +$ etc. ex hac fonte ſummare licet, eademque expreſſiones prodibunt, quas iam olim erueram. Ne autem calculus nimis fiat prolixus, ſequenti modo facile expediri poterit. Ponatur

$$V = 1 + \frac{2v}{1 \cdot 2} + \frac{2^2 v^2}{1 \cdot 2 \cdot 3} + \frac{2^3 v^3}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{2^4 v^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \text{etc.}$$

erit $V = \frac{e^{2v} - 1}{2v}$ et $\frac{dV}{V dv} = \frac{2 e^{2v}}{e^{2v} - 1} - \frac{1}{v}$; quae reducitur ad hanc formam commodiorem: $\frac{dV}{V dv} = \frac{2 e^{v}}{e^{v} - e^{-v}} - \frac{1}{v}$

$$= \frac{1 + \frac{v}{1} + \frac{v^2}{1 \cdot 2} + \frac{v^3}{1 \cdot 2 \cdot 3} + \frac{v^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}}{\frac{v}{1} + \frac{v^3}{1 \cdot 2 \cdot 3} + \frac{v^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{v^7}{1 \cdot \ldots \cdot 7} + \text{etc.}} - \frac{1}{v}:$$ ita vt

fit $$\frac{dV}{V dv} - 1 = \frac{1 + \frac{v^2}{1 \cdot 2} + \frac{v^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{v^6}{1 \cdot 2 \cdot \ldots \cdot 6} + \text{etc.}}{\frac{v}{1} + \frac{v^3}{1 \cdot 2 \cdot 3} + \frac{v^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{v^7}{1 \cdot 2 \cdot \ldots \cdot 7} + \text{etc.}} - \frac{1}{v}$$

ſeu $$\frac{dV}{V dv} - 1 = \frac{\frac{2v}{1 \cdot 2 \cdot 3} + \frac{4v^3}{1 \cdot 2 \cdot \ldots \cdot 5} + \frac{6v^5}{1 \cdot 2 \cdot \ldots \cdot 7} + \frac{8v^7}{1 \cdot 2 \cdot \ldots \cdot 9} + \text{etc.}}{1 + \frac{v^2}{1 \cdot 2 \cdot 3} + \frac{v^4}{1 \cdot 2 \cdot \ldots \cdot 5} + \frac{v^6}{1 \cdot 2 \cdot \ldots \cdot 7} + \frac{v^8}{1 \cdot 2 \cdot \ldots \cdot 9} + \text{etc.}}$$

Ponatur $\frac{dV}{V dv} = 1 + \mathfrak{A} v - \mathfrak{B} v^3 + \mathfrak{C} v^5 - \mathfrak{D} v^7 + \mathfrak{E} v^9 -$ etc.

erit $\mathfrak{A} = \frac{2}{1 \cdot 2 \cdot 3}$

$$\mathfrak{B} = \frac{\mathfrak{A}}{1 \cdot 2 \cdot 3} - \frac{4}{1 \cdot 2 \cdot \ldots \cdot 5}$$

$$\mathfrak{C} = \frac{\mathfrak{B}}{1 \cdot 2 \cdot 3} - \frac{\mathfrak{A}}{1 \cdot 2 \cdot \ldots \cdot 5} + \frac{6}{1 \cdot 2 \cdot \ldots \cdot 7}$$

$$\mathfrak{D} = \frac{\mathfrak{C}}{1 \cdot 2 \cdot 3} - \frac{\mathfrak{B}}{1 \cdot 2 \cdot \ldots \cdot 5} + \frac{\mathfrak{A}}{1 \cdot 2 \cdot \ldots \cdot 7} - \frac{8}{1 \cdot 2 \cdot \ldots \cdot 9}$$

etc.

His valoribus inuentis conſideretur altera forma quantitatis V per factores expreſſa, haec:

$$V = \left(1 + \frac{v}{m} + \frac{vv}{1 \pi \pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{4 \pi \pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{9 \pi \pi}\right) \text{ etc.}$$

ex

ex qua per differentiationem elicitur:

$$\frac{dV}{Vdv}=\frac{\frac{1}{m}+\frac{2v}{1\pi\pi}}{1+\frac{v}{m}+\frac{vv}{1\pi\pi}}+\frac{\frac{1}{m}+\frac{2v}{4\pi\pi}}{1+\frac{v}{m}+\frac{vv}{4\pi\pi}}+\frac{\frac{1}{m}+\frac{2v}{9\pi\pi}}{1+\frac{v}{m}+\frac{vv}{9\pi\pi}}+\text{etc.}$$

Generaliter vero eſt

$$\frac{\frac{1}{m}+\frac{2v}{\lambda\pi\pi}}{1+\frac{v}{m}+\frac{vv}{\lambda\pi\pi}}=\frac{1}{m}+\left.\begin{matrix}\frac{2}{\lambda\pi\pi}\\-\frac{1}{mm}\end{matrix}\right\}v-\left.\begin{matrix}\frac{3}{m\lambda\pi\pi}\\+\frac{1}{m^3}\end{matrix}\right\}v^2+\left.\begin{matrix}\frac{4}{m^2\lambda\pi\pi}\\-\frac{1}{m^4}\\-\frac{1}{\lambda\lambda\pi^4}\end{matrix}\right\}v^3\ \text{etc.}$$

Cum autem ſit m numerus infinitus, ipſique factorum numerus aequalis, excepto primo termino, reliqui per m diuiſi ſine errore praetermitti poterunt, ita vt ſit

$$\frac{\frac{1}{m}+\frac{2v}{\lambda\pi\pi}}{1+\frac{v}{m}+\frac{vv}{\lambda\pi\pi}}=\frac{1}{m}+\frac{2v}{\lambda\pi\pi}-\frac{2v^3}{\lambda^2\pi^4}+\frac{2v^5}{\lambda^3\pi^6}-\frac{2v^7}{\lambda^4\pi^8}+\text{etc.}$$

ſubſtitutis ergo pro λ ſucceſſiue numeris quadratis 1, 4, 9, 16 etc. hisque ſeriebus, quarum numerus eſt m in vnam ſummam coniectis, reperietur:

$$\begin{aligned}\frac{dV}{Vdv}=1&+\frac{2v}{\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}\right)\\&-\frac{2v^3}{\pi^4}\left(1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\frac{1}{25^2}+\text{etc.}\right)\\&+\frac{2v^5}{\pi^6}\left(1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\frac{1}{25^3}+\text{etc.}\right)\\&-\frac{2v^7}{\pi^8}\left(1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\frac{1}{25^4}+\text{etc.}\right)\\&\text{etc.}\end{aligned}$$

Quodſi iam haec ſerie cum prius inuenta comparetur habebitur:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}=\frac{1}{2}\mathfrak{A}\pi^2=\frac{1}{6}\pi^2$$

$$1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\text{etc.}=\frac{1}{2}\mathfrak{B}\pi^4=\frac{1}{90}\pi^4$$

$$1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\text{etc.}=\frac{1}{2}\mathfrak{C}\pi^6=\frac{1}{945}\pi^6$$

$$1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\text{etc.}=\frac{1}{2}\mathfrak{D}\pi^8=\frac{1}{9450}\pi^8$$

etc.

 Hocque

Hocque modo ſummationes a me iam olim exhibitae magis confirmantur, cum non nullis principium, quo tum vſus fueram, lubricum eſſet viſum.

Problema. II.

§. 22. Inuenire terminum generalem ſeriei, cuius quilibet terminus excedat praecedentem data quantitate, et cuius terminus primus ſit datus.

Solutio.

Sit terminus primus $=a$, et exceſſus cuiusque termini ſupra praecedentem $=g$, erunt vtique termini indicibus integris reſpondentes hi:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a, & a+g; & a+2g; & a+3g; & a+4g; & a+5g; & a+6g; \text{ etc.} \end{array}$$

ita vt indici integro x conueniat terminus $y=a+(x-1)g$. At exiſtente x numero quocunque infinitae aliae formulae pro y locum inueniunt. Sit enim y' terminus indici $x+1$ reſpondens, erit:

$$y'=y+\frac{dy}{dx}+\frac{ddy}{1\cdot 2dx^2}+\frac{d^3y}{1\cdot 2\cdot 3dx^3}+\frac{d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}+\text{ etc.}$$

Cum iam per hypotheſin eſſe debeat $y'=y+g$, erit

$$g=\frac{dy}{1dx}+\frac{ddy}{1\cdot 2dx^2}+\frac{d^3y}{1\cdot 2\cdot 3dx^3}+\frac{d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}+\text{ etc.}$$

Quamuis huiusmodi aequationum, vbi praeter terminos, qui differentialia ipſius y continent, adeſt terminus, vel conſtans, vel functio quaecunque ipſius x, reſolutionem ante aliquod tempus tradiderim, tamen expediet per ſubſtitutionem $y=+gx+u$ hunc terminum g tollere, erit enim $dy=gdx+du$, $ddy=ddu$, $d^3y=d^3u$, etc. ob dx conſtans. Fiet ergo

$0=$

$$o = \frac{du}{1dx} + \frac{ddu}{1\cdot 2dx^2} + \frac{d^3u}{1\cdot 2\cdot 3dx^3} + \frac{d^4u}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.}$$

Quae aequatio cum conueniat cum ea, quam in problemate praecedente inuenimus: ſi ponamus ſin. $\pi x = r$ et coſ. $\pi x = s$: erit u functio quaecunque parium dimenſionum ipſarum r et s, cuiusmodi §. XVIII. exhibuimus; hacque inuenta erit terminus generalis quaeſitus $y = A + gx + u$, dummodo conſtans A ita definiatur, vt poſito $x = a$ fiat $y = a$ Q. E. I.

Problema. III.

§. 23. Inuenire terminum generalem ſeriei, cuius quilibet terminus prodeat, ſi praecedens per datum numerum m multiplicetur, et cuius terminus primus ſit $= a$.

Solutio.

Termini ergo huius ſeriei, qui indices habent integros, ſequentem progreſſionem Geometricam conſtituent:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ a, & ma; & m^2a; & m^3a; & m^4a; & m^5a; \text{ etc.} \end{array}$$

ita vt indici integro x reſpondeat terminus am^{x-1}. Sit igitur generatim y terminus indici x, et y' terminus indici $x+1$ conueniens: eritque $y' = my$. At eſt

$$y' = y + \frac{dy}{1dx} + \frac{ddy}{1\cdot 2dx^2} + \frac{d^3y}{1\cdot 2\cdot 3dx^3} + \frac{d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.} = my.$$

Ad hanc aequationem reſoluendam, ponatur ſecundum praecepta data 1 pro y; z pro $\frac{dy}{dx}$; z^2 pro $\frac{ddy}{dx^2}$: etc. vt prodeat ſequens aequatio Algebraica:

$$m = 1 + \frac{z}{1} + \frac{zz}{1\cdot 2} + \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

cuius radices singulas inuestigari oportet. Erit autem $m = e^z$, at sit logarithmus hyperbolicus ipsius $m = \lambda$, vt sit $m = e^\lambda$, ideoque $e^\lambda - e^z = 0$. Quia vero sumto n numero infinito est $e^\lambda = (1 + \frac{\lambda}{n})^n$ et $e^z = (1 + \frac{z}{n})^n$, habebitur ista aequatio, cuius radices sunt inuestigandae:

$$(1 + \tfrac{\lambda}{n})^n - (1 + \tfrac{z}{n})^n = 0,$$

cuius quidem statim vna radix $z - \lambda = 0$ constat, vnde pars integralis obtinetur $y = \alpha e^{\lambda x} = \alpha m^x$ ob $e^\lambda = m$. Reliquae radices sunt imaginariae et continentur in hoc factore trinomio:

$$(1 + \tfrac{\lambda}{n})^2 - 2(1 + \tfrac{\lambda}{n})(1 + \tfrac{z}{n}) \operatorname{cos.} \tfrac{2k\pi}{n} + (1 + \tfrac{z}{n})^2$$

existente k numero quocunque integro: quae forma transit in hanc:

$$2 + \tfrac{2\lambda}{n} - 2(1 + \tfrac{\lambda}{n}) \operatorname{cos.} \tfrac{2k\pi}{n} + \tfrac{\lambda\lambda}{n \cdot n}$$
$$+ \tfrac{2z}{n} - \tfrac{2z}{n}(1 + \tfrac{\lambda}{n}) \operatorname{cos.} \tfrac{2k\pi}{n} + \tfrac{zz}{nn}$$

Verum ob n numerum infinitum est $\operatorname{cos.} \frac{2k\pi}{n} = 1 - \frac{2kk\pi\pi}{nn}$. Multiplicata ergo illa forma per nn, erit factor generalis $= 2n(n + \lambda)(1 - \operatorname{cos.} \frac{2k\pi}{n}) + \lambda\lambda + 2nz(1 - \operatorname{cos.} \frac{2k\pi}{n}) - 2\lambda z \operatorname{cos.} \frac{2k\pi}{n} + zz = \lambda\lambda + 4kk\pi\pi + \frac{4kk\pi\pi z}{n} - 2\lambda z + zz$; neglectis terminis euanescentibus: quo respectu etiam terminus $\frac{4kk\pi\pi z}{n}$ omitti potest, ita vt factor generalis sit:

$$\lambda\lambda + 4kk\pi\pi - 2\lambda z + zz$$

horumque factorum numerus, si pro k successiue numeri 1, 2, 3, 4, etc. substituantur, erit $= \frac{n}{2}$. Haec autem forma cum generali in *dissertatione mea Tom. VII. Miscell.* tradita, $ff - 2fz \operatorname{cos.} \Phi + zz$ dabit $f = \sqrt{(\lambda\lambda + 4kk\pi\pi)}$

et

et cos. $\Phi = \frac{\lambda}{\sqrt{(\lambda\lambda + 4kk\pi\pi)}}$, hincque sin. $\Phi = \frac{2k\pi}{\sqrt{(\lambda\lambda + 4kk\pi\pi)}}$.
Vnde nascitur haec integralis y pars,

$$y = e^{\lambda x}(\alpha \text{ sin. } 2k\pi x + \mathfrak{A} \text{ cos. } 2k\pi x).$$

Substitutis igitur pro k successiue valoribus, ob $e^{\lambda} = m$ reperietur:

$$y = m^x \left\{ \begin{array}{l} C + \alpha \text{ sin. } 2\pi x + \beta \text{ sin. } 4\pi x + \gamma \text{ sin. } 6\pi x + \text{etc.} \\ + \mathfrak{A} \text{ cos. } 2\pi x + \mathfrak{B} \text{ cos. } 4\pi x + \mathfrak{C} \text{ cos. } 6\pi x + \text{etc.} \end{array} \right\}$$

Quare cum posito $x = 1$ fieri debeat $y = a$, erit $a = m(C + \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} + \text{etc.})$ Vnde constans C definitur. Seu si positis sin. $\pi x = r$ et cos. $\pi x = s$, sit Q functio quaecunque parium dimensionum ipsarum r et s, erit terminus generalis quaesitus $y = m^x Q$. Q. E. I.

Coroll. 1.

§. 24. In progressione ergo Geometrica, quatenus ita tantum describitur, vt quisque terminus ad praecedentem rationem constantem habere dicatur, interpolatio non est determinata, cum termini intermedii infinitis diuersis modis exprimi, imo quemuis valorem recipere queant.

Coroll. 2.

§. 25. Huius ergo aequationis differentialis infinitae
$(m-1)y = \frac{dy}{1dx} + \frac{ddy}{1.2dx^2} + \frac{d^3y}{1.2.3.dx^3} + \frac{d^4y}{1.2.3.4dx^4} + \text{etc.}$
integrale completum generaliter exprimi potest. Positis enim sin. $\pi x = r$ et cos. $\pi x = s$, si Q denotet functionem quamcunque ipsarum r et s erit $y = m^x Q$: ideoque $m^{-x} y$ functioni cuicunque parium dimensionum ipsarum r et s aequatur.

Coroll.

Coroll. 3.

§. 26. Si pro x ſcribatur $\frac{x}{a}$ prodibit haec aequatio: $(m-1)y = \frac{a\,dy}{1\,dx} + \frac{aaddy}{1.2\,dx^2} + \frac{a^3 d^3 y}{1.2.3\,dx^3} + \frac{a^4 d^4 y}{1.2.3.4\,dx^4} +$ etc. Ad quam integrandam ponatur ſin. $\frac{\pi x}{a} = r$ et coſ. $\frac{\pi x}{a} = s$. ſignificetque Q functionem quamcunque parium dimenſionum ipſarum r et s, ita vt Q eundem valorem retineat, etiamſi pro r et s ſcribantur $-r$ et $-s$. Quo facto, erit $y = m^{x:a}Q$.

Coroll. 4.

§. 27. Et huius problematis ſolutio reduci poterat ad ſolutionem problematis primi. Cum enim eſſe debeat $y' = my$, erit $ly' = ly + lm$. Ponatur $ly = v$, vt ſit $ly' = v'$, ſiatque $lm = \lambda$, oportebit eſſe $v' = v + \lambda$, vnde fit ob $v' = v + \frac{dv}{1dx} + \frac{ddv}{1.2dx^2} + \frac{d^3 v}{1.2.3dx^3} +$ etc.

$$\lambda = \frac{dv}{1.dx} + \frac{ddv}{1.2dx^2} + \frac{d^3 v}{1.2.3dx^3} + \frac{d^4 v}{1.2.3.4dx^4} + \text{etc.}$$

et poſito $v = u + \lambda x$ habebitur:

$$0 = \frac{du}{1dx} + \frac{ddu}{1.2dx^2} + \frac{d^3 u}{1.2.3dx^3} + \frac{d^4 u}{1.2.3.4dx^4} + \text{etc.}$$

quae eſt aequatio, ad quam in primo problemate peruenimus. Quodſi ergo ponatur ſin. $\pi x = r$ et coſ. $\pi x = s$, atque Q denotet functionem parium dimenſionum ipſarum r et s erit $u = Q$, hincque $v = \lambda x + Q = ly = xlm + Q$. Numeris itaque ſumendis habetur $y = m^x e^Q$, vbi cum e^Q ſit quoque functio parium dimenſionum ipſarum r et s, ſi pro ea ſcribatur Q, erit vt ante inuenimus $y = m^x Q$.

Scholion.

§. 28 Quoniam aequationis Algebraicae:

$$m = 1 + \frac{z}{1} + \frac{zz}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \text{etc.}$$

omnes

omnes radices inuenimus, poterimus inde huius expressionis infinitae:

$$Z = 1 + \frac{z}{1(1-m)} + \frac{zz}{1\cdot 2(1-m)} + \frac{z^3}{1\cdot 2\cdot 3(1-m)} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4(1-m)} + \text{etc.}$$

omnes factores exhibere. Posito enim $lm = \lambda$ primus factor simplex erit $1 - \frac{z}{\lambda}$: et reliqui factores trinomiales in hac forma generali continebuntur:

$$1 + \frac{4kk\pi\pi z}{n(\lambda\lambda + 4kk\pi\pi)} - \frac{2\lambda z + zz}{\lambda\lambda + 4kk\pi\pi}$$

quae transformatur in hanc:

$$1 + \frac{z}{n} - \frac{\lambda\lambda z}{n(\lambda\lambda + 4kk\pi\pi)} - \frac{2\lambda z + zz}{\lambda\lambda + 4kk\pi\pi)}$$

si loco k successiue substituantur numeri 1, 2, 3, 4, etc. et n est numerus infinite magnus, cuius semissis $\frac{n}{2}$ exhibet ipsum factorum numerum. Sit breuitatis gratia $\lambda\lambda + 4kk\pi\pi = \Phi$; eritque

$$Z = \left(1 - \frac{z}{\lambda}\right)\left(1 + \frac{z}{n} - \frac{\lambda\lambda z}{n\Phi} - \frac{2\lambda z}{\Phi} + \frac{zz}{\Phi}\right)$$

vbi posterior factor locum teneat omnium infinitorum, qui ex variatione quantitatis Φ ex eo nascuntur. Sumtis ergo logarithmis, iisque differentiatis, obtinebitur.

$$\frac{dZ}{Zdz} = \frac{-1}{\lambda - z} + \frac{\frac{1}{n} - \frac{\lambda\lambda}{n\Phi} - \frac{2\lambda}{\Phi} + \frac{2z}{\Phi}}{1 + \frac{z}{n} - \frac{\lambda\lambda z}{n\Phi} - \frac{2\lambda z}{\Phi} + \frac{zz}{\Phi}}.$$

Hisque terminis infinitas resolutis:

$$\begin{array}{llllllll}
\frac{dZ}{Zdz} = & -\frac{1}{\lambda} & -\frac{z}{\lambda^2} & -\frac{zz}{\lambda^3} & -\frac{z^3}{\lambda^4} & -\frac{z^4}{\lambda^5} & -\frac{z^5}{\lambda^6} & -\text{etc.} \\
 & +\frac{1}{n} & -\frac{4\lambda^2 z}{\Phi^2} & -\frac{8\lambda^3 zz}{\Phi^3} & -\frac{16\lambda^4 z^3}{\Phi^4} & -\frac{32\lambda^5 z^4}{\Phi^5} & -\frac{64\lambda^6 z^5}{\Phi^6} & \\
 & -\frac{\lambda\lambda}{n\Phi} & +\frac{2z}{\Phi} & +\frac{6\lambda zz}{\Phi\Phi} & +\frac{16\lambda^2 z^3}{\Phi^3} & +\frac{40\lambda^3 z^4}{\Phi^4} & +\frac{96\lambda^4 z^5}{\Phi^5} & \text{etc.} \\
 & -\frac{2\lambda}{\Phi} & & & -\frac{2z^3}{\Phi^2} & -\frac{10\lambda z^4}{\Phi^3} & -\frac{36\lambda^2 z^5}{\Phi^4} & \\
 & & & & & & +\frac{2z^5}{\Phi^3} &
\end{array}$$

 pona-

ponatur $\frac{dZ}{Zdz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$ et cum sit $\Phi = \lambda\lambda + 4kk\pi\pi$, vbi successiue pro k omnes numeri 1, 2, 3, 4, etc. vsque ad $\frac{1}{2}n$ substituti concipiendi sunt: eritque

$$A = \tfrac{1}{2} - \tfrac{1}{\lambda} - 2\lambda\left(\frac{1}{\lambda\lambda+4\pi\pi} + \frac{1}{\lambda\lambda+16\pi\pi} + \frac{1}{\lambda\lambda+36\pi\pi} + \text{etc.}\right)$$

Vel si breuitatis gratia statuatur:

$$\frac{1}{\lambda\lambda+4\pi\pi} + \frac{1}{\lambda\lambda+16\pi\pi} + \frac{1}{\lambda\lambda+36\pi\pi} + \text{etc.} = \mathfrak{A}$$
$$\frac{1}{(\lambda\lambda+4\pi\pi)^2} + \frac{1}{(\lambda\lambda+16\pi\pi)^2} + \frac{1}{(\lambda\lambda+36\pi\pi)^2} + \text{etc.} = \mathfrak{B}$$
$$\frac{1}{(\lambda\lambda+4\pi\pi)^3} + \frac{1}{(\lambda\lambda+16\pi\pi)^3} + \frac{1}{(\lambda\lambda+36\pi\pi)^3} + \text{etc.} = \mathfrak{C}$$
$$\frac{1}{(\lambda\lambda+4\pi\pi)^4} + \frac{1}{(\lambda\lambda+16\pi\pi)^4} + \frac{1}{(\lambda\lambda+36\pi\pi)^4} + \text{etc.} = \mathfrak{D}$$
etc.

erit: $A = \frac{1}{2} - \frac{1}{\lambda} - 2\lambda\mathfrak{A}$

$B = -\frac{1}{\lambda\lambda} + 2\mathfrak{A} - 4\lambda^2\mathfrak{B}$

$C = -\frac{1}{\lambda^3} + 6\lambda\mathfrak{B} - 8\lambda^3\mathfrak{C}$

$D = -\frac{1}{\lambda^4} - 2\mathfrak{B} + 16\lambda^2\mathfrak{C} - 16\lambda^4\mathfrak{D}$

$E = -\frac{1}{\lambda^5} - 10\lambda\mathfrak{C} + 40\lambda^3\mathfrak{D} - 32\lambda^5\mathfrak{E}$

$F = -\frac{1}{\lambda^6} + 2\mathfrak{C} - 36\lambda^2\mathfrak{D} + 96\lambda^4\mathfrak{E} - 64\lambda^6\mathfrak{F}$

etc.

Cum iam sit $Z = 1 + \frac{z}{1(1-m)} + \frac{zz}{1\cdot2(1-m)} + \frac{z^3}{1\cdot2\cdot3(1-m)} + \text{etc.}$

erit $Z = \frac{e^z - m}{1-m} = \frac{e^z - e^\lambda}{1-e^\lambda}$; et $\frac{dZ}{dz} = \frac{e^z}{1-e^\lambda}$; vnde

$$\frac{dz}{Zdz} = \frac{e^z}{e^z - e^\lambda} = \frac{1}{1-e^\lambda e^{-z}} = \frac{1}{1-me^{-z}}$$ hincque

$$\frac{dZ}{Zdz} = \frac{1}{1 - m + mz - \frac{mzz}{1\cdot2} + \frac{mz^3}{1\cdot2\cdot3} - \frac{mz^4}{1\cdot2\cdot3\cdot4} + \text{etc.}}$$

Iam

Iam ob $\frac{dZ}{Zdz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$ fiet:

$$
\begin{aligned}
1 = (1-m)A &+ (1-m)Bz + (1-m)Cz^2 + (1-m)Dz^3 + (1-m)Ez^4 + \text{etc.} \\
&+ \quad mA \quad + \quad mB \quad + \quad mC \quad + \quad mD \\
&\qquad\quad - \tfrac{1}{2}mA \quad - \tfrac{1}{2}mB \quad + \tfrac{1}{2}mC \\
&\qquad\qquad\qquad + \tfrac{1}{6}mA \quad + \tfrac{1}{6}mB \\
&\qquad\qquad\qquad\qquad\quad - \tfrac{1}{24}mA
\end{aligned}
$$

vnde sequentes prodibunt determinationes:

$$A = \frac{1}{1-m}$$

$$B = \frac{-mA}{1-m} = \frac{-m}{(1-m)^2}$$

$$C = \frac{-mB + \frac{1}{2}mA}{1-m} = \frac{mm}{(1-m)^3} + \frac{m}{2(1-m)^2}$$

$$D = \frac{-mC + \frac{1}{2}mB - \frac{1}{6}mA}{1-m} = \frac{-m^3}{(1-m)^4} - \frac{mm}{(1-m)^3} - \frac{m}{6(1-m)^2}$$

$$E = \frac{-mD + \frac{1}{2}mC - \frac{1}{6}mB + \frac{1}{24}mA}{1-m} = \frac{m^4}{(1-m)^5} + \frac{3m^3}{2(1-m)^4} + \frac{7mm}{12(1-m)^3} + \frac{m}{24(1-m)^2}$$

Sequentes ergo orientur serierum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. summationes:

I. $\frac{1}{1-m} = \frac{1}{2} - \frac{1}{\lambda} - 2\lambda\mathfrak{A}$; seu $\mathfrak{A} = \frac{1}{4\lambda} - \frac{1}{2\lambda\lambda} - \frac{1}{2\lambda(1-m)}$

II. $\frac{-m}{(1-m)^2} = -\frac{1}{\lambda\lambda} + 2\mathfrak{A} - 4\lambda\lambda\mathfrak{B} = \frac{1}{2\lambda} - \frac{2}{\lambda\lambda} - \frac{1}{\lambda(1-m)} - 4\lambda\lambda\mathfrak{B}$

vnde $\mathfrak{B} = \frac{1}{8\lambda^3} - \frac{1}{2\lambda^4} - \frac{1}{4\lambda^3(1-m)} + \frac{m}{4\lambda\lambda(1-m)^2}$

III. $\frac{mm}{(1-m)^3} + \frac{m}{2(1-m)^2} = -\frac{1}{\lambda^3} + 6\lambda\mathfrak{B} - 8\lambda^3\mathfrak{C} = \frac{3}{4\lambda\lambda} - \frac{4}{\lambda^3} - \frac{3}{2\lambda^2(1-m)}$
$+ \frac{3m}{2\lambda(1-m)^2} - 8\lambda^3\mathfrak{C}$.

Ergo $\mathfrak{E} = \frac{3}{32\lambda^5} - \frac{1}{2\lambda^6} - \frac{1}{16\lambda^5(1-m)} + \frac{3m}{16\lambda^4(1-m)^2} - \frac{m}{16\lambda^3(1-m)^2} - \frac{mm}{8\lambda^3(1-m)^3}$
Sicque ſequentes ſerierum propoſitarum ſummae $\mathfrak{D}$, $\mathfrak{E}$, etc. inuenientur.

Coroll. 1.

§. 29. Cum igitur ſit $m = e^{\lambda}$ erit:

$$\frac{1}{\lambda\lambda+4\pi\pi} + \frac{1}{\lambda\lambda+16\pi\pi} + \frac{1}{\lambda\lambda+36\pi\pi} + \text{etc.} = \frac{1}{4\lambda} - \frac{1}{2\lambda\lambda} - \frac{1}{2\lambda(1-e^{\lambda})}$$

ſit $\lambda = \frac{2\pi a}{b}$ erit:

$$\frac{bb}{4(aa+bb)\pi^2} + \frac{bb}{4(aa+4bb)\pi^2} + \frac{bb}{4(aa+9bb)\pi^2} + \text{etc.} = \frac{b}{8\pi a} - \frac{bb}{8\pi\pi aa} - \frac{b}{4\pi a(1-e^{2\pi a:b})}$$

ideoque per $\frac{4\pi\pi}{bb}$ multiplicando habebitur:

$$\frac{1}{aa+bb} + \frac{1}{aa+4bb} + \frac{1}{aa+9bb} + \text{etc.} = \frac{\pi}{2ab} - \frac{1}{2aa} + \frac{\pi}{ab(e^{2\pi a:b}-1)}$$

quam ſummam iam alibi ex alio fonte deductam exhibui.

Coroll. 2.

§. 30. Si ergo ſtatuatur $b = 1$, habetur haec ſumma

$$\frac{1}{aa+1} + \frac{1}{aa+4} + \frac{1}{aa+9} + \text{etc.} = \frac{\pi}{2a} - \frac{1}{2aa} + \frac{\pi}{a(e^{2\pi a}-1)}$$

ac ſi ponatur praeterea $a = 0$, vt prodeat haec ſeries: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ etc. huius ſumma, ob terminos in infinitum excreſcentes, ita ex formula deriuabitur: Concipiatur a infinite paruum, erit $e^{2\pi a} = 1 + 2\pi a + 2\pi\pi aa + \frac{4}{3}\pi^3 a^3$ ideoque ſumma erit $= \frac{\pi}{2a} - \frac{1}{2aa} + \frac{1}{2aa + 2\pi a^3 + \frac{4}{3}\pi^2 a^4}$ =

$$=\frac{\pi a+\pi\pi aa+\frac{2}{3}\pi^3a^3-1-\pi a-\frac{2}{3}\pi^2aa+1}{2aa(1+\pi a+\frac{1}{3}\pi\pi a^2)}=\tfrac{1}{6}\pi^2,$$ quae vt constat est summa seriei $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+$ etc.

Coroll. 3.

§. 31. Si in serie ante inuenta:

$$\frac{1}{aa+bb}+\frac{1}{aa+4bb}+\frac{1}{aa+9bb}+\text{etc.}=\frac{\pi}{2ab}-\frac{1}{2aa}+\frac{\pi}{ab(e^{2\pi a:b}-1)}$$

quantitas a vt variabilis tractetur, et differentiatio instituatur prodibit summa seriei $\mathfrak{B}$, sicque porro per continuam differentiationem ex serie $\mathfrak{A}$ reperientur summae serierum sequentium $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, etc.

Coroll. 4.

§. 32. Summa huius seriei commodius ita exprimi potest $\frac{1}{aa+bb}+\frac{1}{aa+4bb}+\frac{1}{aa+9bb}+$ etc. $=\frac{-1}{2aa}+\frac{\pi(e^{2\pi a:b}+1)}{2ab(e^{2\pi a:b}-1)}=\frac{-1}{2aa}+\frac{\pi(e^{\pi a:b}+e^{-\pi a:b})}{2ab(e^{\pi a:b}-e^{-\pi a:b})}$. Ex qua forma facile colligitur valor seriei; si b sit numerus imaginarius, sit enim $b=\frac{c}{\sqrt{-1}}$ erit:

$$\frac{1}{aa-cc}+\frac{1}{aa-4cc}+\frac{1}{aa-9cc}+\text{etc.}=\frac{-1}{2aa}+\frac{\pi(e^{\frac{\pi a\sqrt{-1}}{c}}+e^{\frac{-\pi a\sqrt{-1}}{c}})\sqrt{-1}}{2ac(e^{\frac{\pi a\sqrt{-1}}{c}}-e^{\frac{-\pi a\sqrt{-1}}{c}})}$$

At est $e^{\frac{\pi a\sqrt{-1}}{c}}+e^{\frac{-\pi a\sqrt{-1}}{c}}=2\,\text{cof.}\frac{\pi a}{c}$ atque

$$e^{\frac{\pi a\sqrt{-1}}{c}}-e^{\frac{-\pi a\sqrt{-1}}{c}}=2\sqrt{-1}.\,\text{fin.}\frac{\pi a}{c}$$

vnde fit:

$$\frac{1}{aa-cc}+\frac{1}{aa-4cc}+\frac{1}{aa-9cc}+\text{etc.}=\frac{-1}{2aa}+\frac{\pi\,\text{cof.}\,\pi a:c}{2ac\,\text{fin.}\,\pi a:c}$$