DISCOVERY OF A NEW PRINCIPLE OF MECHANICS

by Leonhard Euler

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1. A body is called solid whose interior is not subject to any change, or which is such that all its parts constantly maintain the same relative position, whatever motion the whole body may have. Despite that firm and invariable connection of its parts, a solid body can receive an infinity of different motions. To determine those motions, and the laws which they follow, is the object of Mechanics or Dynamics, and that is what distinguishes this science from Hydrodynamics or Hydraulics, which deal with the investigation of the motion of fluid bodies, all the parts of which are so disengaged from one another that each may have its own separate motion. Between these two types of bodies one can set up an intermediate type, which is composed of flexible bodies, whose shape is susceptible to an infinity of changes. Nevertheless, the study of the motion of these bodies reduces easily to Mechanics and can be developed by the same principles, so that in this Science all that matters is the laws of motion which concern solid bodies.

2. Among the infinity of motions to which a solid body is susceptible, the first which we must consider is that in which all the parts remain constantly directed toward the same points of absolute space. That is to say, if we conceive a straight line determined by two arbitrary points of the body, that line will always maintain the same direction, or what amounts to the same thing, it will remain perpetually parallel to itself. Such a motion is called purely progressive; it has the property that, at each instant, all the parts of the body move with an equal speed in the same direction. Thus, when a body moves with a purely progressive motion, it suffices to know the motion of one of its elements, that is, the path which it traverses together with the speed at each point, in order to know the motion of the whole body.

3. Now a body, even though solid, can receive an infinity of other motions, as it can occur that, one point of the body remaining motionless, the whole body turns about that point, and it is clear that, in that case, the speed of the different parts of the body will no longer be the same, and that the direction of motion will be different in the different parts of the body. Nonetheless, as soon as we know the motion of a single point of the body, while another point remains at rest, we will be able to determine the motion of all the other points of the body at that same instant;[1] for, since the body is solid, it is necessary that all its points preserve the same position in relation to those two, of which one is at rest and the motion of the other is known. This rotational motion can also be coupled with a purely progressive motion, whence results a mixed motion, such as we observe in the Earth, all the parts of which move in such a way that the motion of each is different from the motion of all the others, both in respect to speed and to direction.

4. But since rotational motion can be infinitely varied, only a single type has been considered in Mechanics up to this time, for lack of principles sufficient to reduce the others to calculation. This type comprises the case in which a body turns about an axis, which is either immobile or which remains always parallel to itself while the body moves with a progressive motion. For in the latter case we can decompose the motion of the body into two, one of which is purely progressive, while the other takes place about a fixed axis, always keeping the same direction. It is in this way that we represent the motion of the Earth, decomposing it into the annual motion, which we regard as purely progressive, and the diurnal motion, which takes place about the Earth's axis, so that we regard that axis as directed constantly toward the same points of the Heavens, abstracting from the precession of the equinoxes, as well as from the nutation of the Earth's axis.

5. However complex the motion of a solid body may be, one can always decompose it into a progressive motion and a rotational motion. The former is characterized by the motion of the center of gravity of the

[1] This is clearly incorrect; the first point has two degrees of freedom, while the group $\mathbf{SO}(3)$ of rotations has dimension 3. *Tr.*

body, and it is always permissible to consider that motion separately and independently from the other rotational motion; and this circumstance furnishes us the advantage that we can always, conversely, consider the rotational motion independently of the progressive motion, if there is one; in other words, we can undertake the study of the rotational motion just as though the body had no progressive motion at all. In order to do this, we need only, in our imagination, impart to the space in which the body is found, a progressive motion equal and contrary to the motion of the center of gravity of the body, and by that means we will obtain the case in which the center of gravity of the body remains at rest, whatever rotational motion the body might have.

6. Thus, whatever motion has initially been imparted to a solid body, and by whatever forces it will be acted on thereafter, in order to determine its motion at each instant, we will begin by considering the body as though all its mass were concentrated at its center of gravity, and then we will determine by the known principles of Mechanics the motion of that point which is produced by the acting forces; that will be the progressive motion of the body. After that, we will set aside the progressive motion, and we will consider the same body as though its center of gravity were motionless, in order to determine the rotational motion, taking account of the motion imparted at the beginning, as well as of the forces by which it should be altered afterwards. And when we have arrived at the result of this investigation, by combining together the two motions which we have found separately, we will be able for each instant to specify the true motion which the body must undergo.

7. Supposing then the center of gravity of a given solid body to be at rest, that body will nevertheless be susceptible to an infinity of different motions. Now I will demonstrate in what follows that, whatever may be the motion of such a body, not only will the center of gravity be at rest at each instant, but there will always be in addition an infinity of points lying on a straight line passing through the center of gravity, which are all without motion as well. That is to say, whatever may be the motion of the body, there will be at each instant a rotational motion about an axis which passes through the center of gravity, and all the diversity which can occur in that motion will depend, apart from the diversity of the speed, on the variability of that axis about which the body turns at each instant; so that the question reduces to determining whether the body will rotate constantly about the same axis, which will thus be immobile, or whether the axis of rotation itself will change position, so that the body rotates successively about different lines passing through its center of gravity.

8. In order to make clear now how much progress has been made, up to this time, in the determination of the rotational motions to which a solid body is susceptible, I remark that the principles of Mechanics, which have been established up to the present, are sufficient only for the case in which the rotational motion takes place continually about the same axis. Thus, since we may designate the extremities of that axis as the poles of the body, this is the case in which the poles of the body also remain constantly at rest, along with the center of gravity. But as soon as the axis of rotation no longer remains the same, and the poles about which the body rotates themselves vary, then the principles of Mechanics which are known up to this time no longer suffice to determine the motion. It is necessary therefore to find and to establish new principles appropriate to this purpose; and this investigation will be the subject of this Memoir, which I have finally worked out after several futile attempts, which I have made over a long period of time.

9. But, before proceeding to that study, it will be appropriate to determine more precisely the case in which a solid body can rotate about a fixed axis passing through its center of gravity, in order to be better able to judge on which occasions the known principles of Mechanics can be employed with success, and hence we will be able to see at the same time that in all other cases those principles will no longer suffice, but that it will be necessary to have recourse to new principles, which I have here undertaken to determine. Now, in order to decide whether a body can rotate about a fixed axis or not, it is necessary to take into account the constitution of the body itself, as well as the forces which are acting on it. For, even if there are no forces acting on the body, as soon as it begins to rotate about a given axis, each particle will be impelled by its centrifugal force, and it is only in the case in which all those centrifugal forces cancel each other out, that the motion about an immobile axis can subsist.

10. Consider therefore an arbitrary solid body, which rotates freely about an immobile axis Aa, which passes through its center of gravity O (Fig. 1). I will take that axis Aa to be perpendicular to the plane of the diagram, in which the center of gravity O lies, and in that plane I imagine two other axes BOb and

COc, normal to each other as well as to the first axis AOa, and it is with respect to these three axes that I will determine the position of each element of the body at a given instant. It is clear, indeed, that the two other axes BOb and COc, inasmuch as they pass through the body, will also rotate with the body about the first axis AOa, which is the axis of rotation, and which remains motionless by hypothesis. Thus the point C of the axis OC will move, by virtue of the rotational motion, along a circular arc $C\gamma$, whose center is at O in the plane BOC, and if we let the distance $OC = f$ and the speed of the point C be that due to the height v, so that the speed itself will be expressed by $\surd v$, then the speed of every other point of the body will be to its distance from the axis AOa as $\surd v$ to f; in other words, $\frac{\surd v}{f}$ will always represent what is called the angular speed, or the speed of rotation.

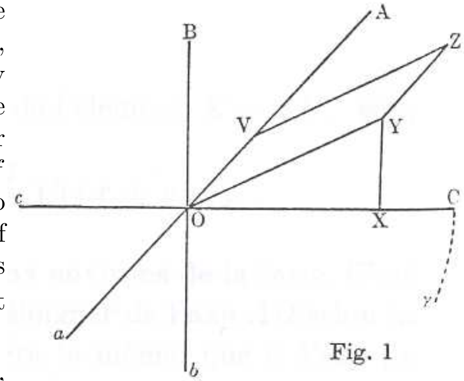

Fig. 1

11. This being so, let Z be an arbitrary element of the body, and let its mass be denoted by dM, the mass of the entire body being $= M$; from the point Z to the plane BOC drop the perpendicular ZY, which will be parallel to the axis of rotation AOa, and from the point Y to OC drop the perpendicular YX; and in addition let $OX = x$, $XY = y$ and $YZ = z$, which will be the three orthogonal coordinates by which the location of the point Z is determined. Let the line ZY be perpendicular to the axis of rotation AO,[1] and it will be parallel and equal to the line VO, and consequently $OV = YZ = z$. Thus the distance from the point Z to the axis of rotation will be $ZV = OY = \surd(xx + yy)$, whence the speed with which the point Z will rotate about the axis AO will be to its distance $\surd(xx + yy)$ as $\surd v$ is to f. Consequently the speed of the point Z will be

$$= \frac{\surd v(xx + yy)}{f},$$

and the height corresponding to that speed

$$= \frac{v(xx + yy)}{ff}.$$

12. From this it follows that the centrifugal force of the element $Z = dM$ will be

$$= \frac{2v(xx + yy)}{ff} \cdot \frac{dM}{\surd(xx + yy)} = \frac{2vdM}{ff}\surd(xx + yy),$$

where dM denotes the weight that that element would have near the earth. Consequently, that is the force with which the element Z will endeavor to recede from the axis AO in the direction VZ, and hence the effect of that force will be the same as if the axis of rotation were acted on at the point V in the direction VZ by a force

$$= \frac{2vdM}{ff}\surd(xx + yy).$$

Thus, since each element of the body must give rise to a similar force acting on the axis of revolution AOa, it is necessary, in order that that axis remain immobile, that all these forces must mutually cancel out. For if that did not occur, it is clear that the axis of rotation could not remain immobile, but since it is supposed to be free, it would have to yield to the resultant force, and so would fall into the case which has not yet been worked out by the already established principles of Mechanics.

13. Let us decompose each of the forces VZ into two others, whose directions are parallel to the axes OC and OB, and since VZ is parallel and equal to $OY = \surd(xx + yy)$, the force which will act on the axis of rotation OA at V in the direction parallel to OC will be

$$= \frac{x}{\surd(xx + yy)} \cdot \frac{2vdM}{ff}\surd(xx + yy) = \frac{2vxdM}{ff}$$

[1] The text seems to be confused here. Evidently, it is ZV, not ZY, which is perpendicular to AO. *Tr.*

and the force in the direction parallel to OB will be

$$= \frac{2vydM}{ff}.$$

Having thus reduced all the centrifugal forces into two species, one acting on the axis of rotation in the directions parallel to OC and the other in the directions parallel to OB, it is necessary, in order that the axis of rotation not be changed, that all the forces of each species must mutually cancel each other. In the first place, then, the sum of all the forces of one or the other species must vanish, so that

$$\frac{\int 2vxdM}{ff} = 0,$$

or

$$\frac{2v}{ff}\int xdM = 0 \quad \text{and} \quad \frac{2v}{ff}\int ydM = 0,$$

whence it is necessary that $\int xdM = 0$ and $\int ydM = 0$. Now this condition will be satisfied as long as we suppose that the axis of rotation AOa passes through the center of gravity O of the body.

14. But this condition alone does not suffice to maintain the axis of rotation AOa at rest; in addition, all the moments of all the forces of each species must mutually cancel out. For the first condition only frees the axis of rotation of any progressive motion, and this second condition is needed in order to prevent it from inclining to one side or the other toward the plane BCO. Now the moment of the forces $\frac{2vxdM}{ff}$, which act on the point V of the axis, being reduced to the center of gravity O becomes $= \frac{2vxzdM}{ff}$, and the moment of each force of the other species will be $= \frac{2vyzdM}{ff}$. Thus this second condition, which has to prevent the inclination of the axis of rotation, requires that both

$$\frac{\int 2vxzdM}{ff} = 0 \quad \text{and} \quad \frac{\int 2vyzdM}{ff} = 0,$$

or since for the present instant $\frac{2v}{ff}$ is a constant quantity, it is necessary that $\int xzdM = 0$ and $\int yzdM = 0$.

15. Thus, in order that a solid body be able to rotate freely about an immobile axis AOa, it is necessary first that that axis pass through the center of gravity O of the body, and in addition it is necessary that the material of which the body is composed be so disposed about the axis that both[1)]

$$\int xydM = 0 \quad \text{and} \quad \int yzdM = 0.$$

It is clear that this latter condition can fail to hold, even if the axis of rotation passes through the center of gravity; and consequently in this case, it will be impossible to determine by the known principles of Mechanics the continuation of the motion, after the body has received an arbitrary motion about such an axis. For in that case the axis will be inclining from the start, and the body at each instant will rotate about another axis, which will make impossible the application of the principles which one ordinarily uses in the determination of the rotational motion.

16. But if the axis of rotation not only passes through the center of gravity of the body, but in addition both $\int xzdM = 0$ and $\int yzdM = 0$, then whatever rotational motion the body might have received about that axis, that motion will continue uniformly, and the axis will not suffer the least change; thus the body will rotate about that immobile axis with a uniform motion, provided that the body is not acted on by any external force. Now it can happen that external forces act on the body without disturbing the position of the axis; that is when the mean direction of these forces falls in the plane BOC, perpendicular to the axis of rotation at the very center of gravity O of the body. For then the forces will not have a moment with respect either to the axis BO or to CO, and thus all the force will be employed either to accelerate or to retard the

1) The first equation should read $\int xz\,dM = 0$. *Tr.*

rotational motion about the axis AO, without changing the axis itself. And this is the case in which we can determine by the use of the known principles of Mechanics those changes caused by the external forces.

17. But if the mean direction of the forces acting on the body does not lie in the plane BOC, the axis of rotation AOa will not be able to remain immobile, but will incline toward the side to which it is forced by the moment of those forces. Thus, even though the axis may have those properties which have just been described, the forces will render it mobile, and without the discovery of new principles of Mechanics, we will not be in a position to develop the case in which the axis of rotation no longer remains immobile. Thus, each time the axis does not satisfy the stated conditions, or the acting forces produce a moment to incline the axis of rotation, or both, it will be necessary to have recourse to new principles in order to determine the changes which will be produced both in the rotational motion and in the position of the axis about which the body will rotate at each instant. But during all these changes we may always suppose that the center of gravity of the body remains motionless.

18. Although the principles involved here are new, in that they are not yet known or expounded by the Authors who have treated Mechanics, we can see nevertheless that the foundation of these principles could not be new, but that it is absolutely necessary that these principles be deduced from the first principles, or rather the axioms, upon which the whole doctrine of motion has been established. These axioms refer to infinitely small bodies, or those which are not susceptible of any motion other than progressive motion, and it is from these that all the other principles of motion must be deduced, those which serve to determine the motions of solid bodies as well as of fluids: all these other derived principles being no more than applications of the axioms, according to the various ways in which the bodies are composed of their elements, and according to the diversity of motion to which all the parts of the bodies are susceptible.

19. We find ordinarily several such principles, which it appears must be admitted to the rank of axioms of Mechanics, because they refer to the motions of infinitely small bodies; but I remark that all these principles reduce to a single one, which can be regarded as the unique foundation of all of Mechanics and of the other Sciences which treat the motion of arbitrary bodies. And it is on that sole principle that all the other principles must be established, both those which have already been received in Mechanics and Hydraulics, and which one currently uses to determine the motion of solid bodies and fluids; as well as those which are not yet known, and which we will need in order to develop the above-mentioned case of solid bodies, as well as several others which occur with fluid bodies. For in each case, it is only a matter of adroitly applying the fundamental principle which I have just mentioned, and which I will go on to explain more carefully.

EXPLICATION OF THE GENERAL AND FUNDAMENTAL PRINCIPLE OF ALL OF MECHANICS

20. Consider an infinitely small body, or one whose mass is contained in a single point, that mass being $= M$; suppose that that body has received an arbitrary motion, and that it is acted on by arbitrary forces. In order to determine the motion of the body, we need only take account of the distance of that body from a fixed and motionless plane; let the distance of the body from this plane at the present instant $= x$; let all the forces which act on the body be decomposed along the directions which are either parallel to the plane, or perpendicular to it, and let P be the force which results from that decomposition in the direction perpendicular to the plane, and which consequently endeavors to make the body recede from or approach the plane. After the element of time dt, let $x + dx$ be the distance from the body to the plane, and taking this element dt to be constant, we will have

$$2M\, ddx = \pm P\, dt^2,$$

according to whether the force P tends to make the body recede from or approach the plane. And it is this formula alone, which contains all the principles of Mechanics.

21. In order to be able to understand better the force of this formula, we should discuss the units in which the various quantities M, P, x, and t, which occur in it, are expressed. To begin with, we must remark that M, denoting the mass of the body, expresses at the same time the weight that the body would have near the surface of the earth; so that, the force P being also reduced to the force of a weight, the letters M and P contain homogeneous quantities. Further, the speed with which the body recedes from the plane will

be as $\frac{dx}{dt}$; if we suppose this speed to be equal to that which a heavy body acquires falling from a height v, we should take $\frac{dx^2}{dt^2} = v$, so that the element of time will be $dt = \frac{dx}{\sqrt{v}}$; whence we know the relation between the time t and the distance x.

22. Since this formula determines only how the body recedes from or approaches one arbitrary fixed plane, in order to find the actual place of the body at each instant, we will have only to refer it simultaneously to three fixed planes, which are mutually perpendicular. Thus, as x denotes the distance of the body from one of these planes, let y and z be its distances from the two other planes, and, after having decomposed all the forces acting on the body along the directions perpendicular to these three planes, let P be the resulting force perpendicular to the first, Q to the second, and R to the third. We may suppose that all these forces tend to make the body recede from the three planes, for in case they tend to make it approach them, we need only take the forces to be negative. This being so, the motion of the body will be contained in the following three formulas:

$$\text{I. } 2M\,ddx = P\,dt^2, \qquad \text{II. } 2M\,ddy = Q\,dt^2, \qquad \text{III. } 2M\,ddz = R\,dt^2.$$

23. If the body is not impelled by any force, so that $P = 0$, $Q = 0$, $R = 0$, the three formulas which we have found, since dt is constant, will reduce by integration to these

$$M\,dx = A\,dt, \quad M\,dy = B\,dt \quad \text{and} \quad M\,dz = C\,dt,$$

whence we see immediately that in this case the body will move in a straight line with a uniform motion; and thus these formulas contain in themselves the first law of motion, according to which each body which is at rest will remain so; or if in motion, the body will continue uniformly in the same direction, so long as it is not acted on by any external force. But it is clear that our formulas are not restricted to that great law, but contain in addition the laws according to which arbitrary forces act on the body. Consequently, the principle which I have just established contains all by itself all the principles which make possible the knowledge of the motion of all bodies, of whatever nature they may be.

24. It is thus from this same great principle that it will be necessary to derive all the rules which we need in order to determine the motion of a solid body whose axis of rotation does not remain immobile. For this purpose, it will be necessary to consider not only all the elements of the body, but also their mutual connection, in virtue of which all the elements maintain among themselves the same order and the same distances. For the motion of the whole body is composed of the motions of all its elements, and the motion of each of these must follow the principle which I have just explained, given that each element participates in the forces which act on the body, and, in addition, is affected by certain forces which prevent it from losing its connection with the others. But before we can determine the effect of the forces to which the elements are subjected, we must consider in general the motion to which such a body is susceptible.

GENERAL DETERMINATION OF THE MOTION TO WHICH A SOLID BODY IS SUSCEPTIBLE, WHILE ITS CENTER OF GRAVITY REMAINS AT REST

25. Thus let O be the center of gravity of the body, which we suppose to be at rest, and consider three fixed planes which intersect orthogonally in the point O. In other words, let us consider three axes AO, BO, CO which intersect at O at right angles, of which the two BO, CO lie in the plane of the table, while the third AO is perpendicular to that plane. These three axes will determine the three planes (Fig. 2) normal to one another, which I am referring to, namely AOB, AOC and BOC, of which the last is the plane of the table. Now I will suppose that these planes, and consequently also these three axes, remain immobile, while the body moves with an arbitrary motion; thus its center of gravity always remains motionless in O. In this way we have a fixed framework, with respect to which we can determine at each instant the configuration of the body.

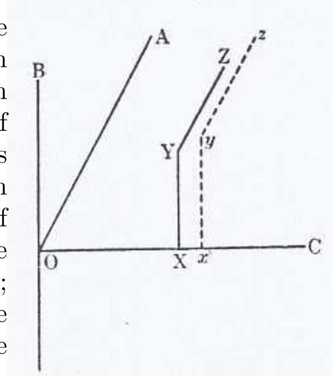

Fig. 2

26. Take now an arbitrary element of the body at Z, from which we drop onto the plane BO the perpendicular ZY, and from the point Y to OC the normal YX, and let $OX = x$, $XY = y$ and $YZ = z$, and $OX = x$ will represent the distance from the point Z to the plane AOB, $XY = y$ its distance to the plane AOC and $YZ = z$ its distance to the plane BOC. At present, whatever may be the motion of the point Z, we can resolve it along the directions of these three axes; suppose thus that after an infinitely small time $= dt$, its distance to the plane AOB becomes $= x + P\,dt$, to the plane $AOC = y + Q\,dt$ and to the plane $BOC = z + R\,dt$, or that the point Z recedes in the time dt from the plane AOB by the element $= P\,dt$, from the plane AOC by the element $= Q\,dt$ and from the plane BOC by the element $= R\,dt$, or what amounts to the same thing, let P, Q, R be the speeds which which the point Z recedes from each of the fixed planes AOB, AOC and BOC.

27. Now since the body is supposed solid, the point Z must remain always at the same distance from the center of gravity O. But at the beginning of the time dt the distance OZ was $= \sqrt{(xx + yy + zz)}$ and at the end of the time it will be $= \sqrt{\left((x + P\,dt)^2 + (y + Q\,dt)^2 + (z + R\,dt)^2\right)}$. Consequently, these two distances must be equal to one another, whence, expunging the terms which vanish with respect to the others, the equation

$$2xPdt + 2yQdt + 2zRdt = 0$$

will result, which reduces to

$$Px + Qy + Rz = 0.$$

These letters P, Q, R denote functions of our three variables x, y, z and so, once we have determined the nature of these functions, we will be in a position to find the motion of each point of the body at the proposed instant, that is, at the beginning of the time dt.

28. Now consider in addition, at the same instant, another point of the body at z, which is infinitely close to the preceding Z. For this point z, take the three variables $Ox = x + dx$, $xy = y + dy$, and $yz = z + dz$, where we should remark that the differentials dx, dy, and dz are mutually independent, since the new point z has been taken at will. The distance from this point to the first point Z will therefore be $= \sqrt{(dx^2 + dy^2 + dz^2)}$; now differentiate the functions P, Q, R, giving dx, dy, dz the values determined by the position of the point z with respect to Z; during the time dt, this point z will advance in the direction OC by the element $(P + dP)dt$, in the direction OB by the element $(Q + dQ)dt$, in the direction OA by the element $= (R + dR)dt$. So after the time dt, the distances will be

	from the point z	or from the point Z
to the plane	$AOB = x + dx + (P + dP)dt$	$= x + Pdt,$
to the plane	$AOC = y + dy + (Q + dQ)dt$	$= y + Qdt,$
to the plane	$BOC = z + dz + (R + dR)dt$	$= z + Rdt.$

29. Thus at the end of the time dt the distance between the points Z and z will be

$$= \sqrt{\left((dx + dPdt)^2 + (dy + dQdt)^2 + (dz + dRdt)^2\right)},$$

which because of the solidity of the body must be the same as at the beginning. Therefore setting that expression equal to $\sqrt{(dx^2 + dy^2 + dz^2)}$ we will obtain the equation

$$2dxdPdt + 2dydQdt + 2dzdRdt = 0,$$

neglecting the terms which are incomparably smaller than these. This equation therefore reduces to the simpler form

$$dPdx + dQdy + dRdz = 0,$$

which, taken together with the one we have already found, namely

$$Px + Qy + Rz = 0,$$

will enable us to determine the nature of the functions P, Q, R, whence we will be able to know all the motions to which all the parts of a solid body are susceptible at once, while its center of gravity remains immobile at O. From this it is clear first of all that, setting $x = 0$, $y = 0$, $z = 0$, all these three functions P, Q, R, must also vanish.

30. In order to understand these functions better, suppose that we have $Ox = OX$ and $XY = xy$, or $dx = 0$ and $dy = 0$, and the equation derived in the last paragraph will give $dRdz = 0$ and hence $dR = 0$, whence we see that the function R cannot contain the variable z. To prove this, suppose that

$$dR = Ldx + Mdy + Ndz,$$

and it will follow in the present case that $dR = Ndz$, since $dx = 0$ and $dy = 0$, and therefore $N = 0$, so that in general $dR = Ldx + Mdy$ and consequently the function R will not contain the variable z. Similarly, setting $dx = 0$ and $dz = 0$, it must be the case that $dQdy = 0$ or $dQ = 0$, whence we see that the function Q cannot contain the variable y. And finally if we consider the case where $dy = 0$ and $dz = 0$, we will find in a similar way that since $dPdx = 0$, the function P cannot contain the variable x.

31. Having recognized these properties of the functions P, Q, R, let us now set

$$dP = Ady + Bdz, \quad dQ = Cdz + Ddx, \quad dR = Edx + Fdy,$$

and if these values are substituted into the equation

$$dPdx + dQdy + dRdz = 0,$$

they will produce the following equation

$$\left.\begin{array}{l} +\,Adxdy + Bdxdz + Cdydz \\ +\,Ddxdy + Edxdz + Fdydz \end{array}\right\} = 0,$$

whence it is clear that we must have $D = -A$, $E = -B$, $F = -C$, since that equation must hold, whether we set $dx = 0$ or $dy = 0$ or $dz = 0$. Thus the differentials of our functions will be

$$dP = Ady + Bdz, \quad dQ = Cdz - Adx, \quad dR = -Bdx - Cdy.$$

Now since these formulas must be integrable, it is clear from the first that neither A nor B can contain x; from the second we see that neither C nor A can contain y, and finally the third shows that neither B nor C can contain z. Thus A, which contains neither x nor y, will be a function of z, B a function of y, and C a function of x.

32. In view of what we have just found, let us therefore let

$$dA = Ldz, \quad dB = Mdy \quad \text{and} \quad dC = Ndx,$$

and because $Ady + Bdz$ must be an integrable differential, we must have

$$\frac{dA}{dz} = \frac{dB}{dy},$$

or $L = M$. Similarly, the integrability of the second formula $Cdz - Adx$ gives

$$\frac{dC}{dx} = -\frac{dA}{dz},$$

or $N = -L$; finally, the integrability of the third formula $-Bdx - Cdy$ gives

$$-\frac{dB}{dy} = -\frac{dC}{dx}$$

or $-M = -N$. Having thus $M = L$, $N = -L$ and $M = N$ or $L = -L$, it is clear that $L = 0$ and consequently also $M = 0$ and $N = 0$. Thus the letters A, B, C will represent constant quantities, whence we get as a result, by integrating

$$P = Ay + Bz, \quad Q = Cz - Ax, \quad R = -Bx - Cy,$$

expressions which already satisfy in themselves the first condition

$$Px + Qy + Rz = 0.$$

33. Thus, every motion which a solid body can receive, while its center of gravity remains immobile, must always have this property. If, given an arbitrary point Z of the body, we take the three orthogonal coordinates $OX = x$, $XY = y$ and $YZ = z$, along the three mutually perpendicular axes OA, OB, OC and decompose the motion of the point Z along the same three directions, letting the speed of the motion along $OC = P$, that of the motion along $OB = Q$ and that of the motion along $OA = R$; these letters P, Q, R will never be able to have any other values than those which are contained in the following formulas

$$P = Ay + Bz, \quad Q = Cz - Ax, \quad R = -Bx - Cy.$$

Thus all the diversity which can take place in the motion of the body comes only from the diverse values which the constant quantities A, B, C can receive.

34. Since these formulas will serve to determine the motion of each point of the body during the element of time dt, let us see whether there are, aside from the center of gravity O, any other points devoid of all motion; or for which we have $P = 0$, $Q = 0$ and $R = 0$. But letting $P = 0$, we will have $Ay + Bz = 0$ and consequently $z = Au$ and $y = -Bu$, where u represents an arbitrary new variable; thus taking these values for y and z whatever value be given to x, the corresponding points of the body will undergo no change of distance from the plane AOB. In addition let $Q = 0$ and it will result that $Cz = Ax$, or $x = Cu$; and the same value results for x, on setting $R = 0$. Thus it follows that all the points of the body, which are contained in these formulas $x = Cu$, $y = -Bu$, $z = Au$ will remain at rest during the time dt. Now all these points lie on a straight line, which passes through the center of gravity O; thus this immobile straight line will be the axis of rotation, about which the body rotates at the present instant.

35. In order to find the rotational motion of the body about this axis which we have just found, let $YZ = z = 0$ and suppose that the point Y is so situated that the line OY becomes perpendicular to the axis of rotation. This requires that $y : x = Cu : Bu$. Therefore we take $x = Bu$ and $y = Cu$ and the distance from the point Y to the axis of rotation will be $= u\sqrt{(BB + CC)}$. Now since $x = Bu$, $y = Cu$ and $z = 0$, the three speeds of the point Y along the three directions OC, OB, OA will be

$$P = ACu, \quad Q = -ABu, \quad R = -BBu - CCu.$$

Thus the actual speed of the point Y being $= \sqrt{(PP + QQ + RR)}$ will be

$$= u\sqrt{(BB + CC)(AA + BB + CC)},$$

which being divided by the distance $OY = u\sqrt{(BB + CC)}$ will give the angular or rotational speed of the body about the axis of rotation, which consequently will be $= \sqrt{(AA + BB + CC)}$.

36. It follows that whatever motion may be imparted to a solid body, its center of gravity remaining at rest, that motion will take place at each instant about an axis which will remain immobile during that instant, and which will pass through the center of gravity of the body; and the rotational speed about that axis will be $= \sqrt{(AA + BB + CC)}$. From this it will be easy to determine the actual speed of each of the points of the body; one need only find the distant from a given point to the axis of rotation, and that distance being set $= s$, the actual speed of the point will be $= s\sqrt{(AA + BB + CC)}$, while its direction will be known from the nature of the rotational motion. It is thus impossible that all the points of a body which turns about itself, or about its center of gravity, can all be in motion at once; because there is always a straight line, all of whose points will be at rest, at least for an instant, and the motion of the other points of the body will be the more rapid, the further they are from the axis of rotation.

37. Without entering into the details of the calculation which I have just explained, we can also prove the same truth by Geometry alone. For this, we consider in the body a spherical shell (Fig. 3), centered at the center of gravity of the body, for it is clear that if we know the motion of that spherical surface, the motion of the body as a whole will be determined. Let AB be any arc of a great circle on that spherical surface, which by the motion of the body arrives at ab after the time dt, so that $ab = AB$ and from that it will be easy to determine the locations to which all the other points of the spherical surface will be transported. Extending the two arcs AB and ab to C, where they intersect, and taking $ac = AC$, the point C will be transported to c and the arc CAB to cab during the time dt. Furthermore, if we consider off the circle CAB another point M taken arbitrarily, and if we draw from that point to C the great circle arc MC, in order to find where the point M will be transported during the same time dt, we need only construct at c an arc $cm = CM$, so that the angle acm is equal to the angle ACM and it is clear that m will be the position of the point M after the time dt.

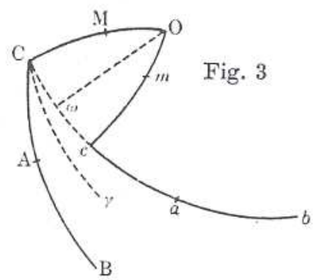

Fig. 3

38. It is also clear that the two arcs CM and cm being extended, will meet at some point O, and consequently in the time dt the entire arc CMO will be transported to cmO, and supposing that $CMO = cmO$, the point O will remain immobile. Now certainly we can always set up the arc CMO in such a way that, having described its corresponding arc cmO, we will have $cmO = CMO$. For in order for this to occur, we need only set up the arc CMO in such a way that the angle cCO becomes equal to the angle CcO; so that the spherical triangle COc becomes isosceles, and consequently the sides CO and cO will be equal. In order to find this position, we should remark that the angle

$$cCO = ACO - ACc \quad \text{and} \quad CcO = 180^\circ - acO,$$

and from that, since the angle $ACO = acO$ and $cCO = CcO$, we deduce that

$$2cCO = 180^\circ - ACc$$

and consequently $cCO = 90^\circ - \frac{1}{2}ACc$. Thus, if we divide the angle ACc into two equal parts by the arc $C\gamma$, the angle γCO will become right, and thus the arc CMO must be perpendicular to the arc $C\gamma$.

39. Having thus determined the position of the arc CMO, in order to find the point O itself, which remains immobile during the time dt, the shortest method will be to draw from the point O to the midpoint ω of the base Cc the perpendicular arc $O\omega$. For then knowing the side $C\omega = \frac{1}{2}Cc$ and the angle $\omega CO = 90^\circ - \frac{1}{2}ACc$ in the spherical triangle $C\omega O$ with right angle at ω, we will conclude that

$$\operatorname{tang} C\omega = \operatorname{tang} CO \cos \omega CO$$

or

$$\operatorname{tang} CO = \frac{\operatorname{tang} C\omega}{\cos \omega CO} = \frac{\operatorname{tang} \frac{1}{2}Cc}{\sin \frac{1}{2}ACc}.$$

But since the time dt is supposed infinitely small, both the interval Cc and the angle ACc will also be infinitely small, and hence equal to their sines or tangents. Thus the tangent of the arc CMO will be $= \frac{Cc}{ACc}$, in other words

$$\operatorname{tang} CO = \frac{Cc}{ACc}.$$

In this way we will easily determine the size of the arc CMO. As that arc is perpendicular to the arc $C\gamma$, or to the arc BA itself, the angle ACc being infinitely small, we will know the point O, through which passes the axis of rotation, about which the body turns during the element of time dt.

INVESTIGATION OF THE FORCES REQUIRED TO MAINTAIN THE BODY IN A GIVEN MOTION

40. Having referred (Fig. 4) the body to three fixed mutually perpendicular axes AO, BO, CO, which cross at the center of gravity O of the body, which remains always immobile; let $OX = x$, $XY = y$ and

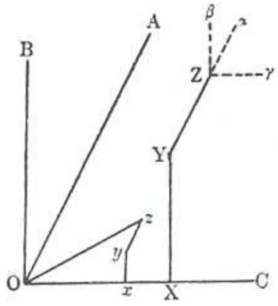

Fig. 4

$YZ = z$ be the three orthogonal coordinates which determine the position of the arbitrary element of the body at Z, at the present instant, and let dM be the mass of that element. Assume that the motion of that element is such that, decomposing it along the directions of our three fixed axes, the speed along OC is $= \lambda y - \mu z$; the speed along $OB = \nu z - \lambda x$, and the speed along $OA = \mu x - \nu y$, taking for the letters A, B, C of paragraph 33 the letters λ, $-\mu$, ν. Thus the axis of rotation at the present instant will be determined by taking $x = \nu u$, $y = \mu u$ and $z = \lambda u$ and the rotational speed about that axis will be

$$= \sqrt{(\lambda\lambda + \mu\mu + \nu\nu)}.$$

Now I suppose that these speeds being positive tend to make the point Z recede from the point O along the directions of the three axes.

41. Thus during the element of time $= dt$, the element dM at Z recedes from the plane AOB by the element of space $= (\lambda y - \mu z)dt$, from the plane AOC by the element $= (\nu z - \lambda x)dt$ and from the plane BOC by the element $= (\mu x - \nu y)dt$. Taking thus $x + dx$, $y + dy$ and $z + dz$ for the coordinates which determine the position of our element dM after the time dt, we will have for the differentials dx, dy and dz the following values

$$dx = (\lambda y - \mu z)dt, \quad dy = (\nu z - \lambda x)dt, \quad dz = (\mu x - \nu y)dt,$$

which have the property, as we have seen, that all the elements of the body remain at the same distance, both among themselves, and from the center of gravity O. From this it is clear that it is impossible for all three expressions to be positive simultaneously, even though we may regard them as such; for it is required that always

$$(x + dx)^2 + (y + dy)^2 + (z + dz)^2 = xx + yy + zz$$

or in other words

$$xdx + ydy + zdz = 0.$$

42. Now for the continuation of the motion, taking the element of time dt to be constant, the general principle of motion requires that the element dM at Z be acted on by three forces along the directions of our three axes. For the mass of this element being taken $= dM$, it must be impelled in the direction OC by a force $= \frac{2dM\,ddx}{dt^2}$, in the direction OB by the force $= \frac{2dM\,ddy}{dt^2}$ and in the direction OA by the force $= \frac{2dM\,ddz}{dt^2}$. These forces contain in themselves the external forces acting on the body from without, as well as the internal forces by which the parts of the body are connected among themselves, so that they do not change their relative positions. Now it is to be remarked that the internal forces mutually cancel one another, so that the continuation of the motion requires only the external forces, to the extent that those forces do not mutually cancel.

43. In order to render our investigation general, suppose that the axis of rotation changes during the time dt in an arbitrary manner, as does the angular speed, which will happen when the letters λ, μ, ν no longer represent constant quantities. Thus let the quantities λ, μ, ν be variable, and for the differentio-differentials ddx, ddy and ddz, the values of dx, dy, dz will give us

$$ddx = (\lambda dy + yd\lambda - \mu dz - zd\mu)dt,$$
$$ddy = (\nu dz + zd\nu - \lambda dx - xd\lambda)dt,$$
$$ddz = (\mu dx + xd\mu - \nu dy - yd\nu)dt,$$

or in other words

$$ddx = (yd\lambda - zd\mu)dt + (\lambda dy - \mu dz)dt,$$
$$ddy = (zd\nu - xd\lambda)dt + (\nu dz - \lambda dx)dt,$$
$$ddz = (xd\mu - yd\nu)dt + (\mu dx - \nu dy)dt.$$

44. Substitute for dx, dy, dz in these formulas the values given above, and we will obtain

$$ddx = (yd\lambda - zd\mu)dt + \big(\lambda\nu z + \mu\nu y - (\lambda\lambda + \mu\mu)x\big)dt^2,$$
$$ddy = (zd\nu - xd\lambda)dt + \big(\mu\nu x + \lambda\mu z - (\nu\nu + \lambda\lambda)y\big)dt^2,$$
$$ddz = (xd\mu - yd\nu)dt + \big(\lambda\mu y + \lambda\nu x - (\mu\mu + \nu\nu)z\big)dt^2.$$

Thus if the lines $Z\alpha$, $Z\beta$, $Z\gamma$ are drawn to represent the three forces which are required to act on the element dM at Z along the directions of the three axes OA, OB, OC we will have for these forces the following expressions

$$\begin{aligned} \text{force } Z\gamma &= \frac{2dM}{dt}(yd\lambda - zd\mu) + 2dM\big(\lambda\nu z + \mu\nu y - (\lambda\lambda + \mu\mu)x\big), \\ \text{force } Z\beta &= \frac{2dM}{dt}(zd\nu - xd\lambda) + 2dM\big(\mu\nu x + \lambda\mu z - (\nu\nu + \lambda\lambda)y\big), \\ \text{force } Z\alpha &= \frac{2dM}{dt}(xd\mu - yd\nu) + 2dM\big(\lambda\mu y + \lambda\nu x - (\mu\mu + \nu\nu)z\big). \end{aligned}$$

45. In order to reduce these expressions to a sum, that is, to find the total forces, we must remark that the only variables occurring in these integrations are the element dM and the coordinates x, y, z, which determine the position of that element, and that dM must range successively over all the elements of the body, so that the integral $\int dM$ gives the mass of the entire body M, and in all these integrations, in which the point Z is the only variable element, the quantities λ, μ, ν together with their differentials $d\lambda$, $d\mu$, $d\nu$ and the element of time dt are to be considered as invariable. From this it is clear that the integral of each of these three forces will become $= 0$, because by the nature of the center of gravity we have

$$\int xdM = 0, \quad \int ydM = 0 \quad \text{and} \quad \int zdM = 0.$$

Thus, whatever may be the forces which are required to act on the body, it follows that if they are applied at the center of gravity O, each in its own direction, they must mutually cancel.

46. Thus, in order to know precisely the effect of these forces, we need only consider their moments with respect to our three axes OA, OB, OC. Now the force $Z\alpha$ gives for the axis OC a moment in the sense $BA = Z\alpha \cdot y$ and for the axis OB a moment in the sense $CA = Z\alpha \cdot x$. Similarly the force $Z\beta$ gives for the axis OC a moment in the sense $AB = Z\beta \cdot z$ and for the axis OA a moment in the sense $CB = Z\beta \cdot x$. Finally the force $Z\gamma$ gives for the axis OB a moment in the sense $AC = Z\gamma \cdot z$ and for the axis OA a moment in the sense $BC = Z\gamma \cdot y$. Thus, resulting from the forces $Z\alpha$, $Z\beta$, $Z\gamma$, there will be for the axis OA a moment in the sense $BC = Z\gamma \cdot y - Z\beta \cdot x$; for the axis OB a moment in the sense $CA = Z\alpha \cdot x - Z\gamma \cdot z$ and for the axis OC a moment in the sense $AB = Z\beta \cdot z - Z\alpha \cdot y$.

47. If we substitute for these forces the values we have found, the resulting moment for the axis OA in the sense BC will be

$$\begin{aligned} &= \frac{2dM}{dt}(yyd\lambda + xxd\lambda - yzd\mu - xzd\nu) \\ &+ 2dM\big(\lambda\nu yz - \lambda\mu xz + \mu\nu yy - \mu\nu xx - (\mu\mu - \nu\nu)xy\big). \end{aligned}$$

The resulting moment for the axis OB in the sense CA will be

$$\begin{aligned} &= \frac{2dM}{dt}(xxd\mu + zzd\mu - xyd\nu - yzd\lambda) \\ &+ 2dM\big(\lambda\mu xy - \mu\nu yz + \lambda\nu xx - \lambda\nu zz - (\nu\nu - \lambda\lambda)xz\big). \end{aligned}$$

Finally the resulting moment for the axis OC in the sense AB will be

$$\begin{aligned} &= \frac{2dM}{dt}(zzd\nu + yyd\nu - xzd\lambda - xyd\mu) \\ &+ 2dM\big(\mu\nu xz - \lambda\nu xy + \lambda\mu zz - \lambda\mu yy - (\lambda\lambda - \mu\mu)yz\big). \end{aligned}$$

Now when I say that there is a moment for the axis OC in the sense AB, it is to be understood that the force of that moment tends to turn the body about the axis OC, and that in the sense AB.

48. Now we have only to take the integrals of the three formulas we have found, in order to have the total moments of the forces which must act on the body, so that its motion will be that motion which we have postulated. Now these integrals reduce to the integration of formulas which depend solely on the shape

of the body and on the distribution of the matter of which it is composed, with respect to our three fixed axes OA, OB, OC. Thus suppose that we have

$$\begin{array}{ll} \int dM(xx+yy) = Mff & \int xydM = Mll \\ \int dM(xx+zz) = Mgg & \int xzdM = Mmm \\ \int dM(yy+zz) = Mhh & \int yzdM = Mnn \end{array}$$

where it should be remarked that Mff is the moment of inertia of the body with respect to the axis OA, Mgg the moment of inertia with respect to the axis OB and Mhh the moment of inertia with respect to the axis OC. The three other formulas contain the centrifugal forces which the body would have if it rotated about one of these three axes.

49. From this, the total moments acting on the body can be expressed in the following way:

I. The moment for the axis OA in the sense BC will be

$$2M\left(\frac{ffd\lambda}{dt} - \frac{nnd\mu}{dt} - \frac{mmd\nu}{dt} + \lambda\nu nn - \lambda\mu mm - (\mu\mu - \nu\nu)ll + \mu\nu(hh-gg)\right).$$

II. The moment for the axis OB in the sense CA will be

$$2M\left(\frac{ggd\mu}{dt} - \frac{lld\nu}{dt} - \frac{nnd\lambda}{dt} + \lambda\mu ll - \mu\nu nn - (\nu\nu - \lambda\lambda)mm + \lambda\nu(ff-hh)\right).$$

III. The moment for the axis OC in the sense AB will be

$$2M\left(\frac{hhd\nu}{dt} - \frac{mmd\lambda}{dt} - \frac{lld\mu}{dt} + \mu\nu mm - \lambda\nu ll - (\lambda\lambda - \mu\mu)nn + \lambda\mu(gg-ff)\right).$$

From this we see that the moments of the forces depend both on the quantities λ, μ, ν, which correspond to the axis of rotation and to the rotational motion, and on their instantaneous changes $d\lambda$, $d\mu$, $d\nu$, which take place in the element of time dt.

50. Now to relate these formulas to the axis of rotation, let it be, say, Oz, and we have seen that $Ox = \nu u$, $xy = \mu u$ and $yz = \lambda u$, and that the angular speed about that axis is $= \sqrt{(\lambda\lambda+\mu\mu+\nu\nu)}$; now to find the direction of that speed, consider the element of the body situated at X, whose speed along OB or XY, since $y=0$ and $z=0$, will be $=-\lambda x$ and along $OA = \mu x$. Thus this point will rise above the plane BOC, and from this we will easily determine in which sense the body rotates about the axis Oz. Now let the angular speed about this axis $Oz = v$, so that $\sqrt{(\lambda\lambda+\mu\mu+\nu\nu)} = v$. Further let ζ, η, ϑ be the angles AOz, BOz, COz which the axis Oz forms with the three axes OA, OB, OC, and because

$$Oz = u\sqrt{(\lambda\lambda+\mu\mu+\nu\nu)} = uv,$$

we will have

$$\cos\zeta = \frac{\lambda}{v}, \quad \cos\eta = \frac{\mu}{v} \quad \text{and} \quad \cos\vartheta = \frac{\nu}{v},$$

whence we see that it will always be the case that

$$\cos\zeta^2 + \cos\mu^2 + \cos\vartheta^2 = 1;$$

and thus we will have

$$\lambda = v\cos\zeta, \quad \mu = v\cos\eta, \quad \nu = v\cos\vartheta.$$

51. Supposing now, because of the variability of the axis of rotation Oz, that the angles ζ, η, ϑ are variable, and in addition that the angular speed v is variable, we will obtain

$$d\lambda = dv\cos\zeta - vd\zeta\sin\zeta, \quad d\mu = dv\cos\eta - vd\eta\sin\eta,$$
$$d\nu = dv\cos\vartheta - vd\vartheta\sin\vartheta.$$

Now since $\cos\zeta^2 + \cos\mu^2 + \cos\vartheta^2 = 1$, we will have

$$d\zeta \sin\zeta \cos\zeta + d\eta \sin\eta \cos\eta + d\vartheta \sin\vartheta \cos\vartheta = 0.$$

Now because our formulas would become too complicated as a result of these substitutions, and since the position of our three axes is arbitrary, let us assume that at the present instant, or at the beginning of the element of time dt, the body has rotated exactly about the axis OA, so that $\mu = 0$ and $\nu = 0$, and that the motion thus takes place in the sense BC, with angular speed v, the value of λ being positive. We will then have $\zeta = 0$, $\eta = 90°$, $\vartheta = 90°$, whence $\lambda = v$, $\mu = 0$, $\nu = 0$; now after the element of time dt, the axis of rotation has deviated by an infinitely small amount from the axis OA, so that it forms with the axis OA an angle $= d\zeta$, with the axis OB an angle $= 90° + d\eta$ and with the axis OC an angle $= 90° + d\vartheta$, and it must be the case that $d\zeta^2 = d\eta^2 + d\vartheta^2$.

52. From this assumption we will thus have

$$d\lambda = dv, \quad d\mu = -vd\eta \quad \text{and} \quad d\nu = -vd\vartheta.$$

And these values being substituted into our preceding expressions from paragraph 49 will give:

I. The moment required for the axis OA in the sense BC:

$$2M\left(\frac{ffdv}{dt} + \frac{nnvd\eta}{dt} + \frac{mmvd\vartheta}{dt}\right).$$

II. The moment required for the axis OB in the sense CA:

$$2M\left(-\frac{nndv}{dt} - \frac{ggvd\eta}{dt} + \frac{llvd\vartheta}{dt} + mmvv\right).$$

III. The moment required for the axis OC in the sense AB:

$$2M\left(-\frac{mmdv}{dt} + \frac{llvd\eta}{dt} - \frac{hhvd\vartheta}{dt} - nnvv\right).$$

Here it should be remarked that the rotational motion about the axis AO is assumed to occur in the sense BC, with angular speed $= v$.

53. Hence, in order for the body to rotate constantly about the same axis OA, or in order that $d\eta = 0$ and $d\vartheta = 0$, but with a variable motion, it is necessary that the body be subject to forces which produce[1)]

I. For the axis OA a moment in the sense $BC = \dfrac{2Mffdv}{dt}$.

II. For the axis OB a moment in the sense $CA = \left(mmvv - \dfrac{nndv}{dt}\right)$.

III. For the axis OC a moment in the sense $BA = \left(nnvv + \dfrac{mmdv}{dt}\right)$.

Thus we see that in order to accelerate the rotational motion, it requires a moment of forces for the axis of rotation OA which is proportional to Mff, that is to say, to the moment of inertia of the body with respect to the axis OA. But, in order that the body rotate about the immobile axis OA with a uniform motion, it is necessary that the body be acted on from without by forces not having any moment for the axis OA, but which give for the axis OB a moment in the sense $CA = 2Mmmvv$, and for the axis OC a moment $= 2Mnnvv$. Thus, this motion cannot subsist without the aid of these external forces, unless we have $mm = 0$ and $nn = 0$, or $\int xzdM = 0$ and $\int yzdM = 0$, which is precisely the case remarked above, where the centrifigal forces mutually cancel.

INVESTIGATION OF THE MOTION OF A SOLID BODY ABOUT ITS CENTER OF GRAVITY, THE FORCES ACTING ON IT

1) In II and III, EULER has dropped the factor $2M$. *Tr.*

BEING GIVEN

54. The body being referred (Fig. 5) to the three fixed axis OA, OB, OC, which cut one another perpendicularly at the center of gravity O, assume, as we have done above, that we designate the coordinates $OX = x$, $XY = y$, $YZ = z$ and the element of the body at $Z = dM$.

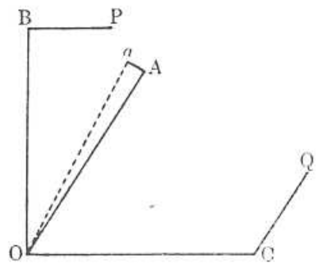

Fig. 5

$$\int dM(xx+yy) = Mff$$
$$\int dM(xx+zz) = Mgg$$
$$\int dM(yy+zz) = Mhh$$
$$\int xydM = Mll$$
$$\int xzdM = Mmm$$
$$\int yzdM = Mnn$$

Here we assume that the body already has an arbitrary motion about an axis Oz, where $Ox = \nu u$, $x\eta = \mu u$, $\eta z = \lambda u$, and that the rotational speed about that axis is $= \sqrt{(\lambda\lambda + \mu\mu + \nu\nu)}$.

55. In this situation, let the body be subject to given forces, and in order to find the change which will thereby be produced in the motion of the body, we need only consider the moments of these forces with respect to the three axes OA, OB, OC; thus let the moment which results from these forces[1)]

For the axis OA in the sense	$BC = Pa$.
The moment for the axis OB in the sense	$CA = Qa$.
The moment for the axis OC in the sense	$CO = Ra$.

Now equating these moments to those which we have found above (paragraph 49), we will obtain the following three equations

$$\text{I.}\quad \frac{Pa}{2M} = \frac{ffd\lambda}{dt} - \frac{nnd\mu}{dt} - \frac{mmd\nu}{dt} + \lambda\nu nn - \lambda\nu mm - (\mu\mu - \nu\nu)ll + \mu\nu(hh - gg),$$
$$\text{II.}\quad \frac{Qa}{2M} = \frac{ggd\mu}{dt} - \frac{lld\nu}{dt} - \frac{nnd\lambda}{dt} + \lambda\mu ll - \mu\nu nn - (\nu\nu - \lambda\lambda)mm + \lambda\nu(ff - hh),$$
$$\text{III.}\quad \frac{Ra}{2M} = \frac{hhd\nu}{dt} - \frac{mmd\lambda}{dt} - \frac{lld\mu}{dt} + \mu\nu mm - \lambda\nu ll - (\lambda\lambda - \mu\mu)nn + \lambda\mu(gg - ff),$$

from which we can determine the infinitely small changes $d\lambda$, $d\mu$ and $d\nu$ which will be produced in the element of time dt.

56. However, since the solution of these equations would lead us to formulas which are too long, suppose, as we have done previously, that the body is rotating at the present instant about the axis OA in the sense BC with an angular speed $= v$, and that during the time $= dt$, the axis of rotation changes, so that it then makes an angle $= d\zeta$ with the axis OA, an angle $= 90° + d\eta$ with the axis OB and an angle $= 90° + d\vartheta$ with the axis OC; and that the angular speed at that time becomes $= v + dv$, and we have seen that $d\zeta^2 = d\eta^2 + d\vartheta^2$. This being so, we will have the following three equations

$$\text{I.}\quad \frac{Pa}{2M} = \frac{ffdv}{dt} + \frac{nnvd\eta}{dt} + \frac{mmvd\vartheta}{dt},$$
$$\text{II.}\quad \frac{Qa}{2M} = -\frac{nndv}{dt} - \frac{ggvd\eta}{dt} + \frac{llvd\vartheta}{dt} + mmvv,$$
$$\text{III.}\quad \frac{Ra}{2M} = -\frac{mmdv}{dt} + \frac{llvd\eta}{dt} - \frac{hhvd\vartheta}{dt} - nnvv.$$

[1)] For the axis OC the sense should be AB. *Tr.*

57. Now the solution of these three equations will provide us with the following values for dv, $d\eta$ and $d\vartheta$:

$$\frac{dv}{dt} = \frac{Pa(gghh-l^4)+Qa(hhnn+llmm)+Ra(ggmm+llnn)-2Mvv\big(mmnn(hh-gg)+ll(m^4-n^4)\big)}{2M(ffgghh-ffl^4-ggm^4-hhn^4-2llmmnn)},$$

$$-\frac{vd\eta}{dt} = \frac{Pa(hhnn+llmm)+Qa(ffhh-m^4)+Ra(ffll+mmnn)-2Mvv\big(ffhhmm-ffllnn-mm(m^4+n^4)\big)}{2M(ffgghh-ffl^4-ggm^4-hhn^4-2llmmnn)},$$

$$-\frac{vd\vartheta}{dt} = \frac{Pa(ggmm+llnn)+Qa(ffll+mmnn)+Ra(ffgg-n^4)+2Mvv\big(ffggnn-ffllmm-nn(m^4+n^4)\big)}{2M(ffgghh-ffl^4-ggm^4-hhn^4-2llmmnn)}.$$

Thus, from these formulas, we will know for each instant the elementary change which will occur, both in the position of the axis of rotation, and in the angular speed. Now it is necessary for each instant to change the position of the three axes OA, OB, OC in order that OA always coincide with the axis of rotation, and then we will be obliged to calculate anew for each instant the values ll, mm, nn, ff, gg, hh, because they will vary continually as a result of the change in the position of the body with respect to the three axes.

58. It will thus be these three formulas which contain the new principles of Mechanics which we need in order to determine the motion of solid bodies, when the axis of rotation about which they rotate does not remain immobile, or directed toward the same part of the Heavens, or of absolute space. And it is clear that these new principles are sufficient for all conceivable cases of the motions to which solid bodies are susceptible. But up to now it has only been possible to solve the very special case in which, for the body, $m = 0$ and $n = 0$, and, in addition, for the forces acting on it, $Q = 0$ and $R = 0$. But for that case we have

$$\frac{dv}{dt} = \frac{Pa}{2Mff}$$

and $d\eta = 0$ and $d\vartheta = 0$, whence we see how limited the study of Mechanics is, without the help of the new principles which I have been able to deduce from the general axiom on which all of Mechanics is founded.

59. But because the formulas which contain these principles are too involved to be able to see their nature clearly, it will be appropriate to apply them to a certain type of body, for which the formulas become quite simple. Suppose thus that the solid body whose motion is to be determined is a globe formed of homogeneous material, or at least of concentric spherical shells, each of which is homogeneous. In that case it is clear that the moments of inertia Mff, Mgg, Mhh with respect to each axis will all be equal, so that $gg = hh = ff$; furthermore, we will always have $ll = 0$, $mm = 0$ and $nn = 0$. Thus the three formulas which give the change of motion produced by the three moments Pa, Qa, Ra will be:

$$\frac{dv}{dt} = \frac{Pa}{2Mff}, \quad \frac{vd\eta}{dt} = -\frac{Qa}{2Mgg} \quad \text{and} \quad \frac{vd\vartheta}{dt} = -\frac{Ra}{2Mhh},$$

the explanation of which will not be very difficult to understand.

60. The two axes OB and OC being arbitrary, provided that they lie in the plane perpendicular to the axis of rotation OA, we can always set them up in such a way that the moments of forces with respect to one of them cancel out. If we therefore let the axes OB and OC be chosen in such a way that the moment with respect to OC vanishes, or that $R = 0$, we will have

$$\frac{dv}{dt} = \frac{Pa}{2Mff}, \quad \frac{vd\eta}{dt} = -\frac{Qa}{2Mgg}$$

and $d\vartheta = 0$, whence we see that the axis OA will approach the axis OB by an infinitely small angle

$$-d\eta = \frac{Qadt}{2Mggv}$$

in the time dt, for since $d\vartheta = 0$, this indicates that the axis of rotation remains perpendicular to the axis OC. It is therefore the moment of forces which tends to turn the body about the axis OB in the sense CA, by which the inclination of the axis of rotation OA toward the axis OB is produced, while the body rotates about OA in the sense BC, and that inclination is inversely proportional to the rotational speed of the body about its axis of rotation.

61. In order to represent this more clearly, let OA, OB, OC be the three axes, about the first of which OA the spherical body which I am considering here rotates in the sense BC with an angular speed $= v$. Suppose that the body is subject to the two forces P and Q, of which the first P is applied to it at B in the direction BP parallel to OC, and the other Q at C in the direction CQ parallel to OA, so that the distances are $OB = a$ and $OC = a$. This being so, the first force will have a moment only with respect to the axis OA, and that moment will be $= Pa$ in the sense BC; now the other force $CQ = Q$ will have a moment only with respect to the axis OB, and that moment will be $= Qa$ in the sense CA. Now if the moment of inertia of the body with respect to any axis passing through the center of gravity is taken $= Mff$, the effect of the first force P will consist in the acceleration of the rotational motion, giving $dv = \frac{Padt}{2Mff}$; and the effect of the other force Q will be to incline the axis of rotation OA toward OB and to carry it to Oa, so that the angle $AOa = \frac{Qadt}{2Mffv}$.

62. We will have a better understanding of the origin of these effects, if we consider (Fig. 6) the spherical surface BCD of the body, whose radius is

$$OB = OC = OA = a$$

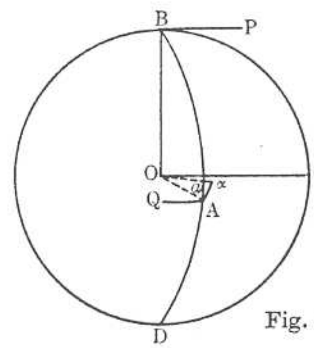

Fig. 6

and which rotates about the axis OA in the sense BC with rotational speed $= v$. Let BP be the force P which accelerates the rotational motion about the axis OA, and we see that that force produces the same effect, as if the other force Q did not act on the body at all; thus this effect is known by the ordinary principles, according to which it must be

$$dv = \frac{Padt}{2Mff}.$$

Let us consider the other force Q applied at A in the direction AQ parallel to CO, for it is clear that this force will produce the same moment, as if it were applied at C in a direction parallel to OA. Thus the effect of this force will consist in carrying the axis of rotation OA to $O\alpha$ through the angle $AO\alpha = \frac{Qadt}{2Mffv}$.

63. In order to give an explanation of this effect, consider to begin with the rotational motion only, by which the point a will be carried about A to α through the angle $AO\alpha$ in the time dt; the angular speed being $= v$, the angle $AO\alpha$ will be $= vdt$, so that the space $a\alpha = a \cdot vdt$. Next we can abstract from the rotational motion and consider the body as if it were subject only to the force $AQ = Q$, which will impart to it a rotational motion about the axis BOD, in the sense CA or αa, and the rotational speed generated in the time dt will be $= \frac{Qadt}{2Mff}$; thus in the time dt the motion will cover an angle $= \frac{Qadt^2}{2Mff}$. By this motion the point α will be drawn back toward a by a space $= \frac{Qaadt^2}{2Mff}$, and it is clear that the point α must be precisely returned to a, in order that it will be the point a and the axis OA which remain at rest. From this we will have

$$a\alpha = a \cdot vdt = \frac{Qaadt^2}{2Mff}$$

and consequently

$$\frac{A\alpha}{AO} = \text{angle } AO\alpha = \frac{Qadt}{2Mffv},$$

which is the same expression which has been deduced from our principles.

Translator's note: This last paragraph does not seem to make sense as it stands. I think that the text here is not only confused but in error. In the case of a homogeneous ball, the rotational momentum is a scalar multiple of the angular velocity, the scalar being the moment of inertia. If there is an external torque, then the angular velocity vector will begin to move in the direction of the torque. In Figure 6, therefore, the point α should actually lie on the great circle BAD, between A and B. In §63, EULER is trying, I think, to find the angle through which the direction of the angular velocity vector —which is the direction of the instantaneous axis of rotation— moves in the infinitesimal time dt. To do this, let the point α, on the arc

AB, mark the location of the instantaneous axis of rotation after time dt. I think that this is what EULER means by the "point a" —confusingly, since a also stands for the radius of the ball. Now, ignoring the external torque, the rotation of the ball about the axis OA will carry the point α, in time dt, to α' (which EULER calls "α"). Because the angular speed is v, the angle $\alpha A \alpha'$ (which EULER mistakenly labels "$AO\alpha$") will be $v\,dt$. Now I think that EULER *also* uses the letter 'a' for the arc between A and (what I have labelled) α; let us instead call this arc γ. Thus length of the arc between α and α' (what EULER calls the "space $a\alpha$") will be $\gamma \cdot v\,dt$.

Next, EULER looks at the effect of the force Q, or the torque Qa. The rate of change of the angular velocity vector will be equal to this torque divided by the moment of inertia; so, as EULER writes, "the rotational speed generated in the time dt will be $= \frac{Qadt}{2Mff}$". But then the angle traversed in that time will be $\frac{1}{2}\frac{Qa\,dt^2}{2Mff}$; EULER is missing the factor of $\frac{1}{2}$. Now since, by assumption, the point α marks the location of the instantaneous axis of rotation (after time dt), this second rotation must, according to EULER, bring the point α' right back to α. By equating the two arcs involved, EULER pretends to calculate the angle $AO\alpha$.

The reason that EULER is able to obtain the correct result despite dropping the factor of $\frac{1}{2}$ is that his method is wrong. He wants to find the point on the sphere which is instantaneously at rest at the end of the interval of time dt. Instead, his method (correctly applied) would find the point which, at the end of the interval dt, has returned to its initial position. These are not the same points. (To see this in a simpler example, suppose that we put the points of the real line in motion in such a way that the position x' after time t of the point which was initially at x will be $x' = x + xt - \frac{1}{2}t^2$. We can ask, which point is instantaneously at rest at time t? The answer, clearly, is $x = t$. On the other hand, we can ask, which point, at time t, occupies the same position it had initially? The answer to this second question is $x = \frac{1}{2}t$.)

The correct method would balance *velocities*. Thus, the point α has a horizontal velocity γv coming from the sphere's initial rotation. On the other hand, the torque Qa generates, after time dt, a velocity $\frac{Qaa\,dt}{2Mff}$ in the other direction. Equating these velocities, we find that $\gamma = \frac{Qaa\,dt}{2Mffv}$, so that angle $AO\alpha = \frac{Qa\,dt}{2Mffv}$.