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PROOF OF THE NUMBER OF POINTS AT WHICH TWO LINES OF ANY DEGREE CAN INTERSECT

1. In the preceding piece, I used without proof this proposition, *that two algebraic curved lines, of which one is of degree m and the other of degree n, can intersect at mn points*. All Geometers recognize the truth of this proposition although we must admit that we do not find a rigorous enough proof anywhere. There are general truths that our mind is ready to embrace as soon as it recognizes their accuracy in several specific cases: and it is in this category of truth that we can justifiably place the proposition that I have just mentioned, since we find it to be true not only in some or several cases, but also in an infinitude of different cases. However we will readily agree that all these infinite proofs are not capable of sheltering this proposition from every objection that an adversary can raise, and that it is absolutely necessary to have a rigorous proof to silence him.

2. Before undertaking the proof of this proposition, it is necessary to define it carefully. First, it is of note that the number of intersections of two lines, of which one is of degree m, the other of degree n, is not necessarily $= mn$, but that it can very often be smaller. Thus it can happen that two straight lines do not intersect at all when they are parallel, that a straight line intersects a parabola only in one point and that two conic sections intersect each other in only two points, or not at all. In this way the meaning of our proposition is that the number of intersections can never be larger than mn, although it is often smaller; and so we consider that either several intersections extend to infinity, or that they become imaginary. In this way, by counting the intersections at infinity and the imaginary intersections as well as the real, we can say that the number of intersections is always $= mn$.

3. Yet there can be cases where the number of intersections is infinite, if we want to consider the coincidence of two equal and similar lines for an infinitude of intersections. This case will occur if the two equations that express the two lines are themselves the same, or if they have equal factors. But as perfect coincidence can not properly be regarded as an infinitude of intersections, since it is rather a continual touching, the contents of the proposition suffer no real exception from this direction; and if the inquiry relates to the number of intersections of two curved lines, we always assume that they are neither coincident nor that they have parts such that one falls

perfectly on the other. In this way we will be able to state the proposition in question in this manner: two curved lines, one of degree m and the other of degree n, of which the equations are neither the same, nor have a common divisor, can never intersect at more than mn points, though the number of intersections can very often be smaller.

4. We will readily recognize the truth of this general proposition in an infinitude of different cases, which would be able to serve as a proof itself if we did not pride ourselves on only advancing theories in Geometry that are equipped with a rigorous proof. However, as these particular proofs are of no little importance to a better understanding of the proposition itself and an appreciation of its significance, I will start with an explanation of these proofs before undertaking the general proof. And first, the truth of this proposition is recognized in the case where one of the two intersecting lines is straight, or of first degree, that is if $m = 1$, since then it is easy to show that the number of intersections of a line of nth degree is equal to or less than n. For the general equation of lines of degree n being

$$\alpha y^n + (\beta + \gamma x)y^{n-1} + (\delta + \epsilon x + \zeta x^2)y^{n-2} + \text{ etc.} = 0,$$

if from the equation for any straight line

$$ay + bx + c = 0$$

we substitute the value of $y = -\frac{bx+c}{a}$, we attain an equation where the unknown x is at most only of the nth degree. Therefore since each intersection is identified by a root of x of this equation, it is clear the number of intersections is equal to the number of roots of this equation and consequently there cannot be larger than n. Moreover, we will see that the number of intersections is $= n$ if all the roots are real and that it will be smaller if some of these roots are imaginary. Now if the terms of highest degree of x mutually cancel and the equation, after the elimination of y, is reduced to an inferior degree, it signals that some points of intersection extend to infinity.

5. Given $m = 2$ and a second degree line that is composed of two straight lines, which happens when the equation is factorable into two parts such as

$$(ay + bx + c)(dy + ex + f) = 0.$$

Now let the other line be any curve of degree n, whose nature is expressed by the equation

$$\alpha y^n + (\beta + \gamma x)y^{n-1} + (\delta + \epsilon x + \zeta x^2)y^{n-2} + \text{ etc.} = 0.$$

In this case it is clear, since this line of degree n can be intersected only by a straight line at n points, that two straight lines regarded as a single second degree line will be able to intersect it at $2n$ points, when each straight line intersects it at n points, which conforms to the terms of the proposition since mn in this case becomes $= 2n$.

6. If one of the two proposed lines is of third degree but composed of three straight lines, and the other remaining any curve of degree n, it is clear that the number of intersections will be $= 3n$ or less, as the proposition requires. And it will be the same for a line of any degree m if it consists of m straight lines, or its equation is factorable into as many simple equations of the form $ay + bx + c = 0$, for since each of these straight lines can intersect the other proposed line of degree n at n points, the number of all intersections will be at most mn, in accordance with the terms of the proposition. Consequently, we already have an infinitude of cases where the truth of this proposition is firmly established. But in all these cases, one of the two proposed lines is not really a curved line, but rather a collection of several straight lines according to the degree to which it belongs.

7. But there are also an infinitude of curved lines where the truth shines forth with as much clarity. For, given that one of the two lines is a parabola expressed by the equation

$$y = axx + bx + c$$

and consequently $m = 2$, the other curve is expressed by the general equation of degree n

$$\alpha y^n + (\beta + \gamma x)y^{n-1} + (\delta + \epsilon x + \zeta xx)y^{n-2} + \text{ etc.} = 0,$$

it is evident that if we substitute here for all y the value $axx + bx + c$, the equation will be at most of degree $2n$, and the root x will be able to have that many roots, indicating all of the intersections, thus it will be possible that the line of degree n is intersected by the parabola at $2n$ points and although the number of intersections can often be smaller, we can however see that there is never be greater than $2n$.

8. The same thing arises if one of the two lines is a parabolic curve of any degree

$$y = ax^m + bx^{m-1} + cx^{m-2} + \text{ etc.}$$

For if we substitute this value for y into the equation for the other curve of degree n, we will see without difficulty that in the resulting equation the letter x will obtain the mnth degree, that will identify that many roots and consequently that many intersections, all as the proposition asserts. Hence we will also conclude that, since the axis of the two curves is arbitrary, even when one of the two curves cannot be expressed by an equation such as

$$y = ax^m + bx^{m-1} + cx^{m-2} + \text{ etc.},$$

so long as by changing the axis, or even the incline of the coordinates, the equation can be reduced to this form, the number of intersections will likewise be $= mn$, the equation that indicates the intersections will always go up to this degree, or a lower one, and never to a higher one.

9. These particular cases taken together lead us to a much more general case, where the truth of the proposition is confirmed. For every time that the equation of the first line, that I suppose to be of degree m, can be decomposed into factors, that express either straight lines or parabolic curves, this equation being

$$(y - P)(y - Q)(y - R)(y - S) \text{ etc.} = 0,$$

where P, Q, R, S, etc. are rational functions of x and the first factor $y - P = 0$ identifies a line of degree p, the second $y - Q = 0$ one of degree q, the third one of degree r etc., such that $p + q + r + s + \text{ etc.} = m$, this line will be composed of all these straight or curved lines together and the other curve, which I suppose to be of degree n, will be able to be intersected by the part of the first, which is expressed by the factor $y - P = 0$, at pn points, by the part included in $y - Q = 0$ at qn points, by the part included in $y - R = 0$ at rn points etc. Consequently the line of degree n can be intersected by all the parts of the first line of degree m at $pn + qn + rn + sn + \text{ etc.}$ points, that is at mn points, since $p + q + r + s + \text{etc.} = m$.

10. Although these cases go on infinitely, we will nevertheless acknowledge the fact that we are still a long way from proving the truth of the proposition in its entirety. And to arrive at such a proof, we must show that for two equations of any proposed degree, such as

$$\begin{array}{lclclcl} a\,y^m & + & (b + c\,x)\,y^{m-1} & + & (d + e\,x + f\,x\,x)\,y^{m-2} & + & \text{etc.} = 0, \\ \alpha\,y^n & + & (\beta + \gamma\,x)\,y^{n-1} & + & (\delta + \epsilon\,x + \zeta\,x\,x)\,y^{n-2} & + & \text{etc.} = 0, \end{array}$$

if we eliminate either of the two variables x and y, the other variable, after elimination, only goes up to the mnth degree. It is certainly true that it would be impossible to complete this elimination in general and at this point in time to make seen what degree the other variable could reach and even in most cases, if we use ordinary methods of elimination, we will arrive at an equation of degree higher than mn; such that by employing this technique, we should rather believe that the proposition is false. For although the equation at which we arrive by this method has divisors, there is reason for doubt, if we can ignore these divisors and whether they contain roots that identify intersections.

11. To make this difficulty even more evident, I will eliminate, according to the ordinary method, the quantity y from these two equations

$$\begin{array}{rl} \text{I.} & Py^3 + Qy^2 + Ry + S = 0, \\ \text{II.} & py^3 + qy^2 + ry + s = 0, \end{array}$$

where P, Q, R, S, p, q, r, s are any functions of the other variable quantity x. Multiplying the first by s, and the second by S, and dividing the difference by y will give:

$$\text{III.}\quad (Ps - pS)\,y^2 + (Qs - qS)\,y + Rs - rS = 0.$$

Next, multiplying the first by p and the second by P, and taking the difference will give:

$$\text{IV.}\quad (Qp - qP)\,y^2 + (Rp - rP)\,y + Sp - sP = 0.$$

In the same manner, from these two second degree equations, we will draw two of the first degree of y:

$$\begin{array}{ll} \text{V.} & ((Ps - pS)\;(Sp - sP) - (Qp - qP)\;(Rs - rS))\;y \\ & + (Qs - qS)\;(Sp - sP) - (Rp - rP)\;(Rs - rS) = 0. \end{array}$$

$$\begin{array}{ll} \text{VI.} & ((Qs - qS)\;(Qp - qP) - (Rp - rP)\;(Ps - pS))\;y \\ & + (Rs - rS)\;(Qp - qP) - (Sp - sP)\;(Ps - pS) = 0. \end{array}$$

And from there we will draw this equation, in which the quantity y no longer exists:

$$\begin{aligned}\text{VII.}\quad & (Ps - pS)(Sp - sP)(Rs - rS)(Qp - qP) - (Ps - pS)^2(Sp - sP)^2 \\ - \quad & (Qp - qP)^2(Rs - rS)^2 + (Qp - qP)(Rs - rS)(Sp - sP)(Ps - pS) \\ = \; & (Qs - qS)^2(Qp - qP)(Sp - sP) - (Qs - qS)(Qp - qP)(Rp - rP)(Rs - rS) \\ - \; & (Rp - rP)(Ps - pS)(Qs - qS)(Sp - sP) + (Rp - rP)^2(Ps - pS)(Rs - rS)\end{aligned}$$

which changes into this:

$$\begin{aligned}0 = \; & (Ps - pS)^4 \\ + \; & 2(Qp - qP) \;\; (Rs - rS)(Ps - pS)^2 + (Rp - rP)(Qs - qS)(Ps - pS)^2 \\ - \; & (Qp - qP) \;\; (Qs - qS)^2(Ps - pS) + (Rs - rS)(Rp - rP)^2(Ps - pS) \\ + \; & (Qp - qP)^2 \;\; (Rs - rS)^2 - (Qp - qP)(Qs - qS)(Rp - rP)(Rs - rS)\end{aligned}$$

But the last terms that do not contain the factor $(Ps - pS)$ reduce to

$$(Qp - qP)(Rs - rS)((Qp - qP)(Rs - rS) - (Qs - qS)(Rp - rP)),$$

which is

$$(Qp - qP)(Rs - rS)(Ps - pS)(Qr - qR)$$

consequently the whole equation will be divisible by $Ps - pS$ given

$$0 = (Ps - pS)^4 \quad \left.\begin{aligned} + \; & 2(Qp - qP)(Rs - rs) \\ + \; & (Rp - rP)(Qs - qS)\end{aligned}\right\} (Ps - pS)^2$$

$$\left.\begin{aligned} - \; & (Qp - qP)(Qs - qS)^2 \\ + \; & (Rs - rs)(Rp - rP)^2\end{aligned}\right\} (Ps - pS) + (Qp - qP)(Qr - qR)(Rs - rS)(Ps - pS).$$

12. It is clear enough that in this case the factor $Ps - pS$, being set $= 0$, cannot identify an intersection and that consequently the intersections of the two proposed curves will be contained in this equation:

$$\begin{aligned}(Ps - pS)^3 \; + \; & 2(Qp - qP)(Rs - rS)(Ps - pS) - (Qp - qP)(Qs - qS)^2 \\ + \; & (Rp - rP)(Qs - qS)(Ps - pS) + (Rs - rS)(Rp - rP)^2 \\ + \; & (Qp - qP)(Qr - qR)(Rs - rS) = 0.\end{aligned}$$

Thus in the case of two curved lines of third degree, the coefficients P and p will be constants, Q and q will be functions of x of one degree like $\alpha + \beta x$, R and r will be second degree functions of x like $\alpha + \beta x + \gamma x^2$, and S and

s will be third degree functions of x like $\alpha + \beta x + \gamma x^2 + \delta x^3$. Consequently, the factors that are found in this equation will be functions of x:

$Ps - pS$	of deg. 3	$Qs - qS$	of deg. 4
$Qp - qP$	of deg. 1	$Qr - qR$	of deg. 3
$Rs - rS$	of deg. 5	$Rp - rP$	of deg. 2

whence it is evident that the equation that indicates the intersections will be of 9 dimensions and that consequently two lines of third degree can in general intersect at 9 points.

13. These same given equations:

$$\begin{aligned} P\,y^3 + Q\,y^2 + R\,y + S = 0 \\ p\,y^3 + q\,y^2 + r\,y + s = 0 \end{aligned}$$

can show the number of intersections in an infinitude of other cases as well. For if the first equation represents a line of degree m and the second equation a line of degree n, which will occur when the coefficients are entirely rational functions of x, namely

P	of deg.	$m-3$	p	of deg.	$n-3$
Q	of deg.	$m-2$	q	of deg.	$n-2$
R	of deg.	$m-1$	r	of deg.	$n-1$
S	of deg.	m	s	of deg.	n

Then the factors that constitute the equation that no longer contains the variable y will be functions of x:

$Ps - pS$	of deg.	$m+n-3$	$Qs - qS$	of deg.	$m+n-4$
$Qp - qP$	of deg.	$m+n-5$	$Qr - qR$	of deg.	$m+n-3$
$Rs - rS$	of deg.	$m+n-1$	$Rp - rP$	of deg.	$m+n-2$

Consequently the number of intersections of these two curves will be $= 3m + 3n - 9$; which is always smaller than mn, if m and n are larger than 3. For if $m = 3 + \alpha$ and $n = 3 + \beta$, the number of intersections will be $= 9 + 3(\alpha + \beta)$, instead of $mn = 9 + 3(\alpha + \beta) + \alpha\beta$. But we clearly see that this reduction in the number of intersections comes from the fact that the chosen equations do not in general represent curves of degrees m and n, but only certain types of curves of these degrees; whence it is not surprising that the number of intersections has been found to be smaller, as the proposition requires.

14. As the elimination of the unknown y from the two cubic equations, of which I have done the calculation, led to an equation that was of too high a degree, that was only actually brought to the right degree by the division by a factor, which we can clearly see contains no intersections: thus in the equations where y has more degrees, we will arrive by the elimination of y at an equation of even higher degree, which in truth will necessitate a divisor, but this method that would, moreover, be impractical in equations of higher degree, will not be assuredly correct, if we will always find there such a divisor, which does not at all concern intersections, and even less so, if after division the equation will indeed be of as many degrees as indicated by the general proposition; that is to say if the number of degrees will never be larger than mn, if the two proposed equations have been of degrees m and n. This circumstance thus proves the even greater necessity of proving the general proposition to its fullest extent since without this, we would have good reason to doubt its truth.

15. It is thus principally on the work of elimination that the proof of our general proposition depends, where it is necessary to take care that through elimination we do not attain an equation containing useless roots. For two equations having been proposed, each of which contains the same variable y that needs to be eliminated, we clearly see that the elimination can be done in an infinitude of different ways, according to the arbitrary quantity by which we multiply both equations. It is thus a matter of precisely defining the idea of elimination and of directing this operation so that the equation at which we arrive does not contain any roots other than those that indicate intersections and that we can be assured that it does not contain superfluous factors which might or might not indicate intersections.

16. Given any two equations:

$$\begin{array}{lllllllllll} y^m & - & Py^{m-1} & + & Qy^{m-2} & - & Ry^{m-3} & + & Sy^{m-4} & - & \text{etc.} & =0, \\ y^n & - & py^{n-1} & + & qy^{n-2} & - & ry^{n-3} & + & sy^{n-4} & - & \text{etc.} & =0, \end{array}$$

that must necessarily be combined so that the resulting equation no longer contains the letter y. But first, we see that the value for y that results from one of these equations must be equal to the value for y that results from the other. Thus if both equations give several values for y, the two proposed equations will be able to coexist, if any value for y of one will be equal to any value for y of the other. Let us suppose that all the roots of the first equation are:

$$A, B, C, D, E, F, G \text{ etc.}$$

and the roots of the other equation are:

$$a, b, c, d, e, f, g \text{ etc.}$$

This being posited, it is clear that each of the two proposed equations will occur in all cases where one of the roots of the first equation is equal to one of the roots of the other.

17. The number of roots, A, B, C, D etc. of the first equation will be $= m$ and the number of roots of the other equation will be $= n$; thus the proposed equations could be represented in these forms:

$$\begin{aligned}
&(y-A)(y-B)(y-C)(y-D)(y-E) \text{ etc. } &= 0,\\
&(y-a)(y-b)(y-c)(y-d)(y-e) \text{ etc. } &= 0.
\end{aligned}$$

At present, it is clear that if $A = a$, the value $y = A = a$ will satisfy both equations; the same thing will happen if $A = b$ or $A = c$ or $A = d$ or $A = e$ etc.; moreover the value $y = B$ will satisfy both if $B = a$ or $B = b$ or $B = c$ or $B = d$ or $B = e$ etc., and the value $y = C$ will satisfy both equations if $C = a$ or $C = b$ or $C = c$ or $C = d$ or $C = e$ etc. and in a similar fashion for the others. And it is evident that all these combinations together represent all the possible cases where the two proposed equations could coexist.

18. Thus since the equation that we are looking for by elimination should contain all the possible cases where the same value assigned to y will satisfy both equations at the same time, it is clear that it will have to contain all the indicated cases, and consequently it will be composed of all these factors

$$\left.\begin{array}{ll}
(A-a)(A-b)(A-c)(A-d)(A-e) & \text{etc.}\\
(B-a)(B-b)(B-c)(B-d)(B-e) & \text{etc.}\\
(C-a)(C-b)(C-c)(C-d)(C-e) & \text{etc.}\\
(D-a)(D-b)(D-c)(D-d)(D-e) & \text{etc.}\\
(E-a)(E-b)(E-c)(E-d)(E-e) & \text{etc.}\\
\text{etc.} &
\end{array}\right\} = 0.$$

Thus, since the quantity y is no longer in this equation, it will be the same that we look for by elimination and that shows all cases where the two proposed equations can have the same root. But as the roots A, B, C, D etc.

a, b, c, d, etc. are often impossible to assign, it is a matter of expressing this equation by the coefficients P, Q, R, S etc. p, q, r, s etc., for which the relation between the roots is known.

19. Because, as we have seen, the product of all these factors

$$(y-a)(y-b)(y-c)(y-d)(y-e) \text{ etc.}$$

is equal to the expression

$$y^n - py^{n-1} + qy^{n-2} - ry^{n-3} + sy^{n-4} - \text{ etc. },$$

if we substitute in turn the values A, B ,C, D, etc. for y, the equation resulting from the elimination will be composed of these factors:

$$\left.\begin{array}{l} (A^n - pA^{n-1} + qA^{n-2} - rA^{n-3} + sA^{n-4} - \text{ etc.}) \\ (B^n - pB^{n-1} + qB^{n-2} - rB^{n-3} + sB^{n-4} - \text{ etc.}) \\ (C^n - pC^{n-1} + qC^{n-2} - rC^{n-3} + sC^{n-4} - \text{ etc.}) \\ (D^n - pD^{n-1} + qD^{n-2} - rD^{n-3} + sD^{n-4} - \text{ etc.}) \\ (E^n - pE^{n-1} + qE^{n-2} - rE^{n-3} + sE^{n-4} - \text{ etc.}) \end{array}\right\} = 0.$$
$$\text{etc.}$$

where the number of these factors is $= m$, according to the number of roots of the first equation. Whence it is evident as well that changing the equations, the equation that results from elimination can be also represented in this form:

$$\left.\begin{array}{l} (a^m - Pa^{m-1} + Qa^{m-2} - Ra^{m-3} + Sa^{m-4} - \text{ etc.}) \\ (b^m - Pb^{m-1} + Qb^{m-2} - Rb^{m-3} + Sb^{m-4} - \text{ etc.}) \\ (c^m - Pc^{m-1} + Qc^{m-2} - Rc^{m-3} + Sc^{m-4} - \text{ etc.}) \\ (d^m - Pd^{m-1} + Qd^{m-2} - Rd^{m-3} + Sd^{m-4} - \text{ etc.}) \\ (e^m - Pe^{m-1} + Qe^{m-2} - Re^{m-3} + Se^{m-4} - \text{ etc.}) \end{array}\right\} = 0.$$
$$\text{etc.}$$

where the number of factors is $= n$.

20. Although the expressions of the roots A, B, C, D etc. and a, b, c, d etc. are for the most part very irrational and often such that we cannot assign them, we nevertheless know that

$$\begin{array}{ll} \text{the sum of all the roots } A, B, C, D \text{ etc. is} & = P, \\ \text{the sum of the products of two and two} & = Q, \\ \text{the sum of the products of three and three} & = R, \\ \text{the sum of the products of four and four} & = S \\ \text{etc.} & \end{array}$$

And with these values for P, Q, R, S etc. we are in a position to express all the equations into which all the roots enter equally by rational formulas composed of P, Q, R, S etc. Now we clearly see that if we multiply the currently mentioned factors, we always arrive at similar expressions that contain all the roots equally, and instead of which we can put rational functions of the coefficients P, Q, R, S etc. and p, q, r, s etc. This is also clear from the double form of this equation in the preceding paragraph. For if some irrationality remained in the first form, it would be an irrationality of the first equation, but from the second form we see that there cannot be an irrationality in the first equation. Whence it follows that both forms should lead to the same rational expression that contains only the coefficients P, Q, R, S etc and p, q, r, s etc.

21. If we now reflect upon the proposed equations

$$\begin{array}{lllllllllll} y^m & - & Py^{m-1} & + & Qy^{m-2} & - & Ry^{m-3} & + & Sy^{m-4} & - & \text{etc.} & =0, \\ y^n & - & py^{n-1} & + & qy^{n-2} & - & ry^{n-3} & + & sy^{n-4} & - & \text{etc.} & =0, \end{array}$$

insofar as they describe lines of degrees m and n, the coefficients P and p indicate functions of the first degree of x like $\alpha+\beta x$, the coefficients Q and q functions of the second degree $\alpha+\beta x+\gamma xx$, the coefficients R and r functions of the third degree $\alpha+\beta x+\gamma x^2+\delta x^3$ etc., thus the sum of the roots A, B, C, D etc. or a, b, c, d, e etc will be expressed by a function of x of the first degree, the sum of the products of two with two of these roots by a function of the second degree, the sum of the products three with three roots by a function of the third degree and so forth. This is why in the composition of all the roots in the first form (paragraph 18) we will be able to look at each root as a function of a degree of x, consequently this form being composed of mn simple factors will go up to mn degrees of x and will consequently designate mn intersections of the two proposed curves.

22. If there is still any lack of clarity in this proof, it comes from its overarching universality, and all doubts that we might have will disappear completely, as soon as we apply it to several particular cases in which we will first recognize that all that I have just propounded on the degrees of each part must occur, not only in these cases, but also in general. I will start with the two second degree equations

$$\begin{array}{ll|c} & & \text{The roots} \\ yy-Py+Q & =0 & A, B \\ yy-py+q & =0 & a, b \end{array}$$

thus, since $m = 2$ and $n = 2$, the equation to which elimination must lead will be:

$$(A^2 - pA + q)(B^2 - pB + q) = 0 \ ,$$

which being developed will give:

$$A^2B^2 - pAB(A+B) + q(A^2+B^2) + ppAB - pq(A+B) + qq = 0 \ .$$

Now having $AB = Q$ and $A + B = P$, will give $AA + BB = PP - 2Q$, consequently the desired equation will be

$$Q^2 - pPQ + qPP - 2Qq + ppQ - pqP + qq = 0 \ ,$$

of which each term will be of the fourth degree of x, provided that P and p contain one degree of x and Q and q two.

23. Given two third degree equations:

$$\begin{array}{ll|c} & & \text{The roots being} \\ y^3 - Py^2 + Qy - R & = 0 & A, B, C \text{ and } m = 3 \\ y^3 - py^2 + qy - r & = 0 & a, b, c \text{ and } n = 3 \end{array}$$

Thus the equation sought by the elimination of y will be:

$$(A^3 - pA^2 + qA - r)(B^3 - pB^2 + qB - r)(C^3 - pC^2 + qC - r) = 0$$

which will, through development, become:

$$\begin{array}{l} A^3B^3C^3 - pA^2B^2C^2(AB + AC + BC) + qABC(A^2B^2 + A^2C^2 + B^2C^2) \\ -r(A^3B^3 + A^3C^3 + B^3C^3) + p^2A^2B^2C^2(A + B + C) \\ -pqABC(A^2B + AB^2 + A^2C + AC^2 + B^2C + BC^2) - p^3A^2B^2C^2 \\ +q^2ABC(A^2 + B^2 + C^2) + pr(A^3B^2 + A^2B^3 + A^3C^2 + A^2C^3 + B^3C^2 + B^2C^3) \\ +q^3ABC + r^2(A^3 + B^3 + C^3) - qr(A^3B + AB^3 + A^3C + AC^3 + B^3C + BC^3) \\ -r^3 + p^2qABC(AB + AC + BC) \\ +pqr(A^2B + AB^2 + A^2C + AC^2 + B^2C + BC^2) \\ -p^2r(A^2B^2 + A^2C^2 + B^2C^2) - q^2r(AB + AC + BC) \\ -pq^2ABC(A + B + C) + qr^2(A + B + C) \\ -pr^2(A^2 + B^2 + C^2) = 0 \ , \end{array}$$

where it must be noted that

$$\begin{array}{rl} A + B + C = P & \text{of one degree of } x \\ AB + AC + BC = Q & \text{of two degrees} \\ ABC = R & \text{of three degrees.} \end{array}$$

24. For the other expressions, we will find them to be formed of the coefficients P, Q, R such that:

$A^2+B^2+C^2=P^2-2Q$ of deg. 2
$A^2B+AB^2+A^2C+AC^2+B^2C+BC^2=PQ-3R$ of deg. 3
$A^3+B^3+C^3=P^3-3PQ+3R$ of deg. 3
$A^3B+AB^3+A^3C+AC^3+B^3C+B^3C+BC^3=P^2Q-PR-2Q^2$ of deg. 4
$A^2B^2+A^2C^2+B^2C^2=Q^2-2PR$ of deg. 4
$A^3B^2+A^2B^3+A^3C^2+A^2C^3+B^3C^2+B^2C^3=PQ^2-2P^2R-QR$ of deg. 5
$A^3B^3+A^3C^3+B^3C^3=Q^3-3PQR+3RR$ of deg. 6,

whence we clearly see, since p, q and r are functions of one, 2 and 3 degrees of x, that all the terms contain the same number of degrees of x and that this number is $=9$, as the terms of the proposition require. Now this substitution will give the following equation by the elimination of the variable y:

$$\begin{aligned}
&+R^3-pQR^2+pQ^2R-2qPR^2-rQ^3+3rPQR-3rR^2\\
&-r^3+qr^2P-q^2rQ+2pr^2Q+q^3R-3pqrR+3r^2R\\
&+p^2PR^2-pqPQR+3pqRR+prPQ^2-2prP^2R-prQR+q^2P^2R\\
&-pr^2P^2+pqrPQ-3r^2PQ-pq^2PR+2p^2rPR+qrPR-p^2rQ^2\\
&+rrP^3-2qqQR-qrP^2Q-p^3R^2+2qrQQ+ppqQR=0\ .
\end{aligned}$$

25. This example will serve to convince us in general that if the two proposed equations are:

$$y^m-Py^{m-1}+Qy^{m-2}-Ry^{m-3}+Sy^{m-4}-\cdots\pm V=0\ ,$$

$$y^n-py^{n-1}+qy^{n-2}-ry^{n-3}+sy^{n-4}-\cdots\pm v=0\ ,$$

where P and p are function of one degree of x, Q and q of two, R and r of three etc. and the last terms V of m degrees and v of n degrees, then the first term, which the equation resulting from the elimination given in paragraph 19 will provide, will be $A^n\,B^n\,C^n\,D^n$ etc. $=V^n$, and consequently, of mn degrees of x, and since we clearly see also that all the other terms able to be expressed by the letters P, Q, R etc. and p, q, r etc should contain the same number of degrees of x, it is incontestably proved that the equation at which we arrive by the elimination of the letter y will be of mn degrees of x, just as the general proposition states.