ON AN APPARENT CONTRADICTION IN THE THEORY OF CURVED LINES

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1. It is generally believed that Geometry distinguishes itself from other sciences, because all of its assertions are based on the most rigorous proofs and there is nothing in them that could give rise to controversy. Indeed, since in Geometry one does not accept any propositions other than those that are perfectly proved, it is difficult to understand how one could encounter inconsistencies. It would be even less possible that two solidly proven propositions could contradict one another, in view of the fact that the truths, far from being opposites, are still in perfect harmony with one another. And although it often happens in the other sciences that two truths appear contradictory, we are confident that they are but an apparent contradiction, originating for the most part from imprecise ideas or from a lack of sufficient knowledge of things, both of which are necessary to keep in mind. For this reason we will be all the more likely to believe that such apparent contradictions have no place in Geometry, since we are far from contenting ourselves with vague and insufficiently determined ideas.

2. Nevertheless I am going to put before you two propositions of Geometry, both of which are rigorously proved, that seem to lead to an open contradiction. This problem is found in the doctrine of curved lines, where for curved lines of a certain degree we know how many points are necessary for determination. Thus, a third degree line can be described by 9 given points, or 9 points determine a third degree line such that there is but one

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that can be traced through these nine points. But it is also shown that two third degree lines can intersect at nine points; so, it can happen that two third degree lines pass through nine given points, from whence it follows that nine points are not sufficient to determine a third degree line, which is contrary to what we just asserted. Before explaining and developing this apparent contradiction, it will be best to present it completely, to give a better idea of its significance. To this effect, I will start with an explanation of the two propositions, which seem to contain this contradiction.

PROPOSITION 1

3. Just as a first degree, or straight, line can be defined by any two given points, a second degree line, or conic section, will similarly be defined by 5 points, a third degree line by 9 points, a fourth degree by 14 points and in general a line of the nth degree can be defined by $\frac{(nn+3n)}{2}$ points.

The general equation of lines of this degree:

$$Ay^n + (B + Cx)y^{n-1} + (D + Ex + Fx^2)y^{n-2} \\ +(G + Hx + Ix^2 + Kx^3)y^{n-3} + \text{etc.} = 0$$

contains $\frac{(nn+3n)}{2} + 1$ arbitrary coefficients A, B, C, D etc. Now each given point, through which the curve must pass, provides given values for the x and y coordinates that, having been substituted, will give an equation. Therefore $\frac{(nn+3n)}{2}$ given points lead to the same number of equations, by which all the coefficients A, B, C etc. can be determined and consequently the curve itself. Although the coefficients number one more than the equations defined by the $\frac{(nn+3n)}{2}$ points, this is simply a matter of a mutual relationship between the coefficients. Only $\frac{(nn+3n)}{2}$ equations are necessary to determine all these relationships.

PROPOSITION 2

4. Two straight, or first degree, lines, can intersect at only one point; two conic sections, or second degree lines, cannot intersect at more than four points. Two third degree lines can intersect at 9 points; two fourth degree lines at 16 points and so on. And as a line of degree m can be intersected by a straight line at m points, by a second degree line at $2m$ points, by a third degree line at $3m$ points, we thus maintain that in general, a line of degree m can be intersected by a line of degree n at mn points. It must

be understood by this proposition that the number of intersections of two curved lines, one of which is of degree m, and the other of degree n, cannot be larger than mn, although it is often smaller: some points of intersection either extend to infinity or become imaginary. The proof of this proposition is not so straightforward and I will expand upon it amply in the following part of this treatise.

5. The truth of these two propositions being acknowledged, I will first describe the consequences that we draw from them and which appear to be contradictory. Next I will bring to light the flaw to be found within these consequences, that consists of a very subtle hastiness in reasoning, which not being easy to discover, should render us extremely cautious, principally in the other sciences, in order that we do not let ourselves be seduced by similar apparent contradictions. For, if we are subject to such remarkable difficulties in Geometry, where it is yet possible to refine all concepts to the highest attainable degree of exactitude, how much more can we be hindered in the other sciences, where it is not possible to attain such precise ideas and where it is infinitely more difficult to protect oneself from similar errors in reasoning? Finally I will fully bring to light the manner in which these two propositions should be understood by applying a particular necessary condition to them, which after having been noted, all the contradictions, however solid they may have seemed, will disappear suddenly and we will perceive a most beautiful harmony between these two propositions.

I. PROBLEM

6. The first contradiction seems initially to occur in the property of third degree lines. Depending on whether we consider either the one or the other of the two general propositions discusses, here are the conclusions that we draw:

I. *Since according to the first proposition 9 points are necessary to determine a third degree line, through 9 given points we can draw only one third degree line.*

II. *But, according to the second proposition, two third degree lines can intersect at 9 points, thus we will be able to identify 9 points, through which two third degree lines can pass.*

These two consequences openly contradict one another; for in the first it is asserted that given 9 points, we would know to describe only one third degree line that could pass through each of these 9 points. Nevertheless the second consequence shows us that there are an infinite number of cases where one can make two third degree lines pass through nine given points.

II. PROBLEM

7. The contradiction becomes even more apparent in lines of the fourth degree and higher. For the fourth degree the contradictory consequences are:

I. *According to the first proposition, 14 points are necessary to determine a fourth degree line; and consequently having 14 given points, we can describe only one line of that degree that passes through all of these points.*

II. *But the second proposition informs us that two fourth degree lines can intersect at 16 points; consequently in these cases it will be possible to make two fourth degree lines pass through 16 given points.*

The contradiction in these two consequences is obvious, for if it was not possible to describe more than one fourth degree line by 14 given points, it would be all the more impossible to describe two lines of this degree that both pass through the same 16 points.

III. PROBLEM

8. For fifth degree lines, our two general propositions provide us with these two even more contradictory consequences.

I. *As the first proposition shows that 20 points are sufficient to determine a fifth degree line, it follows that through 20 given points one could describe only a single fifth degree line.*

II. *Now the second proposition assures us that two fifth degree lines can intersect at 25 points. Thus it will be possible to identify 25 points, through which we will be able to make two fifth degree lines pass.*

Consequently there are cases where 25 given points are not sufficient to determine a fifth degree line and yet the first consequence seeks to persuade us that only 20 points are necessary to determine a fifth degree line. And it is clear that in curves of higher degree, the difference between the number of points that should be sufficient for determination becomes even larger.

9. These contradictions being completely evident, it is absolutely necessary that either one or the other of the two general propositions is false, or that the consequences that we have drawn from them are not true. For the second proposition, although to my knowledge, a rigorous enough proof

is nowhere to be found, it is yet very certain, as I will show later; and the consequences that have been drawn are so clear that not the least doubt lies in that direction. Because if, for example, two fourth degree lines intersect at 16 points, one will be absolutely forced to admit that it is possible to choose 16 points, through which we could trace not only one, but two fourth degree lines. Thus as the second proposition, as well as the consequences that have been drawn from it have proven correct, we are obliged to conclude that it is in the first proposition or in the manner that we draw the consequences labeled Number 1 that it is necessary to look for false reasoning.

10. The consequences that have been drawn from the first proposition are equally well founded; for if 9 points determine a third degree line, it follows that through 9 given points we could draw only one line of this degree, likewise that through two given points we would be able to draw only one straight line and through 5 given points a single conic section. Thus, if the general equation for lines of degree n is determined by $\frac{(nn+3n)}{2}$ points, through which the line must pass, it will not be possible that more than one line of this degree passes through as many given points as the formula $\frac{(nn+3n)}{2}$ contains units. It is thus not in the consequences that we have just deduced from the first proposition that we should look for the source of these contradictions and as a result there remains only the first proposition itself on which the suspicion of false reasoning can fall despite however well-founded it may appear.

11. Indeed, through careful reflection on the status of the first proposition, we notice that cases can arise where $\frac{(nn+3n)}{2}$ given points are not sufficient to determine the curve of degree n that can be drawn through these points or, similarly, that $\frac{(nn+3n)}{2}$ equations are not sufficient to determine the same number of coefficients or to determine the ratio between the $\frac{(nn+3n)}{2}+1$ coefficients, although these coefficients enter in each equation and occupy only one dimension. Moreover, this is the simplest case, where several unknowns must be determined by as many equations, since through successive elimination of these unknowns we remain in the first degree, so that for each unknown we never find more than one value, which in consequence will always be real. This circumstance seems further to confirm the truth of the first proposition and to keep it free from all exceptions.

12. However, once we consider the following remarks, there can no longer be any doubt that we must apply a particular restriction to this first proposition, without which it would not be true in general.

I. REMARK

First of all, let me start with the simplest of cases for which two equations can be insufficient to determine the values of two unknowns, although both appear in each equation and occupy only one dimension in each. For we only have to look at these two equations

$$3x - 2y = 5 \quad \text{and} \quad 4y = 6x - 10$$

and we will see that it is not possible to determine from them the two unknowns x and y, since by eliminating one x, the other goes away by itself and we obtain an identity from which we can determine nothing. The reason for this occurrence is immediately obvious, since the second equation changes into $6x - 4y = 10$, being nothing but the first $3x - 2y = 5$ doubled, with no difference. This is why, when we say that to determine two unknown quantities, it is enough to have two equations, it is necessary to add to this proposition the restriction that these two equations be different from one another or that one not be already included in the other, and it is only with this restriction that the said proposition can be accepted.

II. REMARK

13. Secondly, we readily recognize that three equations can be insufficient to determine three unknown quantities. For if the same case as in the preceding remark occurs, namely that one of the three equations is included in one of the other two, in which case the three equations are reduced to two, it will then be impossible to determine from them the three unknowns. Take for example these three equations:

$$\begin{aligned} 4x - 6y + 10z &= 16, \\ 3x - 5y + 7z &= 9, \\ 2x - 3y + 5z &= 8, \end{aligned}$$

it is clear that the first, like the third, contributes nothing to the determination of the three unknowns x, y, and z. But there are also cases where one of the three equations is contained jointly in the two others, as we have with these three equations:

$$\begin{aligned} 2x - 3y + 5z &= 8, \\ 3x - 5y + 7z &= 9, \\ x - y + 3z &= 7, \end{aligned}$$

where the sum of the first and the third produce the second.[1] In this case, we can omit whichever of these three equations we wish and it is as if there were only two equations. In this way, when we say that to determine three unknowns only three equations are required, it is necessary to add the restriction that these three equations must differ from each other such that none is already included in the others.

III. REMARK

14. The same is true for four equations, which are not sufficient to determine four unknown quantities unless they are all quite different from each other and none is already included in the others. For if one is already included in the three others, these four equations should be treated as if there were only three. It can even occur that two equations are already included in the other two and then there will be only two equations that remain in the calculation and consequently two unknowns will remain undetermined. For if one arrived at these four equations:

$$\begin{aligned} 5x + 7y - 4z + 3v - 24 &= 0, \\ 2x - 3y + 5z - 6v - 20 &= 0, \\ x + 13y - 14z + 15v + 16 &= 0, \\ 3x + 10y - 9z + 9v - 4 &= 0, \end{aligned}$$

they would reduce to only two. Because having drawn from the third the value of $x = -13y + 14z - 15v - 16$ and having substituted this into the second to produce:

$$y = \frac{33z - 36v - 52}{29} \quad \text{and } x = \frac{-23z + 33v + 212}{29}$$

the substitution of these two values of x and y into the first and fourth equations will bring about identities, so that the quantities z and v will remain undetermined.

IV. REMARK

15. The same circumstance can occur with any number of equations, so that even if we have as many equations as there are unknowns, the former would not be sufficient to determine the latter because one of these unknown

[1] where the sum of the second and the third is the first doubled. Corrected by A.S.

quantities will remain undetermined if one of the proposed equations is contained in the others. Moreover, two or more unknown quantities will remain undetermined if there are, amongst the equations, two or more that are already contained within the others and that consequently contribute nothing to the determination of the unknowns. This is why when we assert that to determine n unknown quantities it is sufficient to have n equations that express their mutual relationship, it is necessary to add the restriction that all the equations must be differ from each other or that there must not be any that are already contained in the others.

16. After these remarks we readily acknowledge the fact that for the determination of a curved line of a certain degree, the number of points, which according to the first proposition appear to be sufficient, can in certain cases become insufficient, since this determination reduces to the determination of a certain number of coefficients by as many equations; such equations, as we just saw, can sometimes become insufficient for this purpose. To understand these cases, I will first consider the general equation for straight, or first degree, lines:

$$\alpha y + \beta x + \gamma = 0,$$

and propose two points, through which a line of this degree must pass. Having chosen any straight line as an axis and letting a and b serve as the coordinates of the first point and c and d for the other, in other words for the first point we will have $x = a$, $y = b$ and for the other $x = c$, $y = d$, from which we will derive these two equations:

$$\alpha b + \beta a + \gamma = 0, \quad \alpha d + \beta c + \gamma = 0,$$

from whose difference we will have:

$$\frac{\alpha}{\beta} = \frac{c - a}{b - d} \, .$$

Thus the relation between α and β will always be determined, as long as we do not have $c = a$ and $d = b$, in which case the two points fall in the same place. Thus two points determine a straight line, provided that they are not coincident. This restriction is the same in the description of all other curved lines, since two points that fall in the same place, are treated as a single point.

17. The general equation for second degree lines is

$$\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \epsilon y + \zeta = 0,$$

so that through 5 proposed points we should describe one second degree line. To simplify the calculations, let us take a straight line that passes through two of these points as the axis and another line drawn through one of these two points and a third represent the slope of the coordinate values, since it does not matter if it is perpendicular to the axis. With these assumptions, take as the values of x and y for these 5 proposed points:

	I	II	III	IV	V
	$x = 0$	a	0	c	e
	$y = 0$	0	b	d	f

From here we will have the following 5 equations:

$$\begin{array}{rlllllll}
\text{I.} & & & & & & & \zeta = 0. \\
\text{II.} & & & & & \alpha\, a^2 & +\ \delta\, a & +\ \zeta = 0. \\
\text{III.} & & & & + & \gamma\, b^2 & +\ \epsilon\, b & +\ \zeta = 0. \\
\text{IV.} & \alpha\, c^2 & +\ \beta\, c\, d & +\ \gamma\, d^2 & + & \delta\, c & +\ \epsilon\, d & +\ \zeta = 0. \\
\text{V.} & \alpha\, e^2 & +\ \beta\, e\, f & +\ \gamma\, f^2 & + & \delta\, e & +\ \epsilon\, f & +\ \zeta = 0.
\end{array}$$

From the first three we will have first

$$\zeta = 0, \qquad \delta = -\alpha a \quad \text{and} \quad \epsilon = -\gamma b;$$

these values being substituted in the last two give the equations

$$\alpha c^2 + \beta cd + \gamma d^2 - \alpha ac - \gamma bd = 0,$$

$$\alpha e^2 + \beta ef + \gamma f^2 - \alpha ae - \gamma bf = 0,$$

that will determine the desired curve, as long as the equations are not equivalent. But this case arises when the two values of β that we draw from them are the same

$$\beta = \frac{\alpha c(a-c) + \gamma d(b-d)}{cd} = \frac{\alpha e(a-e) + \gamma f(b-f)}{ef},$$

that is to say

$$\frac{a-c}{d} = \frac{a-e}{f} \quad \text{and} \quad \frac{b-d}{c} = \frac{b-f}{e},$$

or when

$$a = \frac{cf - de}{f - d} \quad \text{and} \quad b = \frac{de - cf}{e - c}.$$

From this we see that the five given points can be arranged in such a way that a second degree line is not determined and, since a coefficient remains undetermined, it follows that through these five given points we can make an infinite number of second degree lines.

18. If we consider this case more closely, where the five given points can be insufficient to determine the second degree line, it is easy to see that this happens when four of these five given points lie on a straight line. This will become quite clear if from the equations

$$\frac{a-c}{d} = \frac{a-e}{f} \quad \text{and} \quad \frac{b-d}{c} = \frac{b-f}{e}$$

we draw the values

$$f = \frac{d(a-e)}{a-c} = b - \frac{e(b-d)}{c}$$

which give

$$(e-c)(ad-ab+bc) = 0.$$

When $e = c$, there will also be $f = d$ and two points will coincide; in which case one is excluded. Therefore take

$$ad - ab + bc = 0 \quad \text{or} \quad d = b - \frac{bc}{a}$$

and we will find

$$f = b - \frac{be}{a};$$

we now have only to look geometrically at these equations to assure ourselves that the four points lie on a straight line. It would not have been difficult to divine this case at the outset, for since the second degree includes also two straight lines placed in any manner, we recognize that if three of the five given points are on a straight line, the latter will be a part of the second degree line and the other combined straight line will be determined by the other two points. So if four points lie on a straight line, this will be a part of the desired line but the other part or the other straight line will pass through the fifth point, and having no other restrictions, we will be able to draw it as we wish. If all five points lay on a straight line, this same straight line along with any other would satisfy the requirements; therefore, this case will be even less determined than the preceding.

19. The general equation for third degree lines being:

$$\alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3 + \epsilon x^2 + \zeta x y + \eta y^2 + \theta x + \iota y + \kappa = 0$$

it is necessary to define the nine coefficients by a tenth so that the equation or the line of this degree is determined. Therefore, if there are nine given points through which this line must pass, each contributing an equation, the line will be determined, provided that these nine equations are all different from one another and that none is already included in the others. Now we can readily understand that there is an infinite number of cases where not only one, but also two or three of the nine equations can be contained in the others and consequently, in these cases, the curve will not be determined by the 9 proposed points, but we can still add a tenth and even an eleventh or a twelfth, in order to complete the determination. It is nevertheless quite difficult to define these cases generally as I have done for lines of the second degree since, due to the large number of points and coefficients, the calculations become too complicated. Nevertheless, it is not difficult to discover several specific cases in which the deficiency in determination takes place; from which we will conclude without difficulty that the number of such cases can be infinitely large, which suffices for my purposes.

20. We know that the general equation of the third degree line also includes, besides curved lines of this degree, either three straight lines or one straight line along with a conic section. So if of the nine given points, four are arranged in a straight line, since in curved lines of this degree, there are not four points in a straight line, the straight line drawn through these four points will constitute a part of the desired figure and the other five points will determine the other part, which will be either a conic section or two straight lines and we again can have the preceding case of lack of determination But let us suppose that we have five points situated in a straight line that constitutes a part of the figure and it is clear that the other 4 points are not sufficient to determine the other part. In this case then we will need ten points to determine the figure, but if, of the proposed points, 6 are situated on a straight line, it will be necessary to have 5 more and 11 points in total will be needed to determine the equation entirely. And hence it is evident that it can arise that however many points there are, they would not be sufficient to determine a third degree line that passes through all the points.

21. As these cases happen only in straight lines that are included in the general equation of third degree lines, we may doubt that the same thing can

happen in curved third degree lines or that nine points can be insufficient to determine a curved line of that degree. To prove this I will consider but one case, where the nine given points are arranged in a square

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$$

whose axis is drawn through the points $d\ e\ f$ and for which the point e is the beginning of the abscissa. Naming the interval of two points $= a$, we will have for the abscissa $x = 0$ three values for the applied y, that are 0, $+a$ and $-a$, and these same three values will also match the abscissa $x = a$ and $x = -a$. The following equation will match these values

$$my(yy - aa) = nx(xx - aa),$$

with any relation between the coefficients m and n so that we can describe an infinite number of third degree lines that pass through all the given points.

22. It is certainly true that this equation also contains straight lines and conic sections; for if $n = 0$, we will have three straight lines ac, df, and gi, if $m = 0$, we will have three straight lines ag, bh, and ci, if $m = n$, we will have a straight line aei and an ellipse drawn through the points $bcfdgh$ and if $m = -n$, the third degree line will be composed of a straight line ceg and an ellipse that passes through the points $abdfhi$. But in all the other relations we formulate between m and n, we will always have a real curved line that passes through the nine given points. And hence we will readily understand whenever two third degree lines intersect at 9 points, these points will be such that they do not completely determine the third degree line, that in the general equation, after we have applied it to these nine points, there will remain one undetermined coefficient. In these cases then, there will not only be two third degree lines, but an infinitude of lines of this degree that can all be described by these nine points.

23. When two fourth degree lines intersect at 16 points, since 14 points, when leading to as many different equations are sufficient to determine a line of this degree, these 16 points will be always such that three or more of the resulting equations are already included in the others. In this way, these 16 points will determine only as much as if there were 13 or 12 or even less and consequently to completely determine the curve we will still need to add one

or two to these 16 points. The same thing will happen if two fifth degree lines intersect each other at 25 points, which not being sufficient to determine the curve, will be worth no more than 19, or even 18, in such a way that 6 or 7 points are superfluous, and in consequence these 25 points will always be arranged so that as soon as the curve passes through 19 of the points it will pass automatically through the others, or it will be impossible for it to pass through 19 points without also passing through all 25. These remarks being well understood, we will readily resolve all the other difficulties that could be borne of the comparison of the two general propositions that I brought up in the beginning of this treatise.