

ON THE FRICTION OF SOLID BODIES¹

Leonhard Euler

I. It has been noted that in most machines, friction is so considerable that a good portion of the forces required to set the machine in motion is dedicated solely to overcoming this resistance; so that if it were possible to free machines from friction, a much smaller quantity of force would be sufficient to produce the same effect. All mechanicians agree that reducing friction is one of the principal items on which the ultimate perfection of machines depends, and it is in this view that they have tried for a long time to research the nature and quantity of friction, so they can discover means to diminish it, or to make it vanish altogether if possible.

II. Friction manifests itself whenever one body must slide against the surface of another body; for however polished the surfaces of the bodies that slide against each other may be, the motion still encounters some resistance which soon entirely destroys it, unless it is renewed by the reiterated action of new forces. Nonetheless, there isn't any doubt that friction becomes smaller the more the surfaces of the bodies sliding against each other become polished and smooth, so that there are no more tiny irregularities which can obstruct motion. It is for this reason that sleds glide fairly easily on ice, and that in machines one feels a considerable reduction in friction after greasing the surfaces that rub against each other, since the grease serves to make the surfaces more polished and smooth.

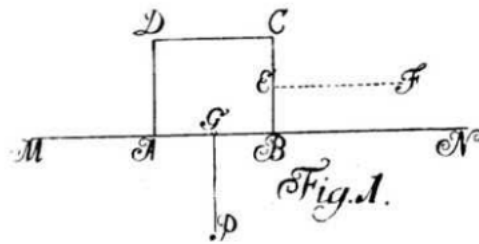
III. However, the materials used in the construction of machines, like woods and metals, are not susceptible to such a degree of polishing that friction is not still very considerable; and experimentation has shown that the resistance which opposes the motion of all these materials is almost the same, and equal to a quite considerable part of their entire weight. Mr. Amontons maintained that friction is always equal to a third of the weight of a body moving on a horizontal surface, or generally a third of the force with which the body is pressed against the surface on which it is sliding. Others have found slightly different values for the quantity of friction, and Mr. Bilfinger assigns friction only a quarter of pressure. As this

¹ Leonhard Euler, *Sur la friction des corps solides*, in *Opera Omnia*, vol. II.8, pp. 54-63, originally in *Memoires de l'academie des sciences de Berlin* 4, 1750, pp. 122-132. This article is numbered E143 in Eneström's index of Euler's work.

depends on the degree of polishing of the surfaces of the bodies, it is not surprising that experiments do not always yield the same quantity of friction.

IV. But a fairly remarkable circumstance, on which all those who have examined friction through experiments agree, is that the quantity of friction depends only on the weight, or on the force with which a body is pressed against the surface on which it is dragged; and that neither the shape of the body nor the size of its base enter in any way into the determination of friction. For if friction is caused by the tearing of little threads, or by the embedding of little bumps which are found on the surfaces sliding against each other, one would think that the larger the surfaces that touch, the greater friction would become. It may even be that this circumstance contributes something in thread-like materials, and others of a similar nature; but in woods and metals, with which experiments are principally done, we can agree that the size of the base serves neither to increase nor to reduce friction.

V. So if a body ABCD (Fig. 1) is pressed against a surface MN by some force GP, which is $= P$, whether this is the weight of body ABCD, if the surface MN is horizontal, or whether there is still another force with which the body is pushed against the surface; then in this case we require a certain force EF in order to move the body and pull it in direction BN.



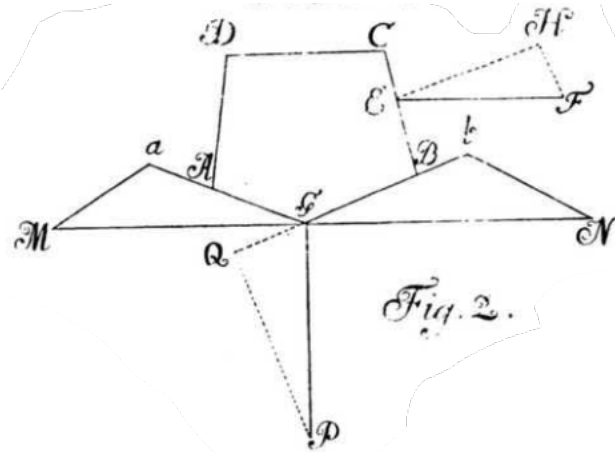
We know that if there were no friction whatsoever, the slightest force EF would be capable of setting the body in motion. But if friction is equal to one third of the force P , or if we set it as $= \frac{m}{n} P$, so as not to confine ourselves to a hypothesis which may be too specific; then as long as force EF is smaller than $\frac{m}{n} P$, the body will remain at rest, the same as if it had not been acted on by any force. As soon as we apply a force EF greater than $\frac{m}{n} P$, the body will presently be dragged in direction BN, but movement will be produced only by the excess of the force EF above the force of friction $\frac{m}{n} P$.

VI. Therefore friction must be regarded as a force $= \frac{m}{n} P$, by which the body is pulled backwards in direction AM, which is always opposite to that of the movement of the body and goes through the base AB. Yet it is quite different from other real forces that can act on the body, for it produces no effect unless the body is actually in motion, and it is only then that it has the same effect as if the body ABCD were effectively pushed backwards in direction AM. As long as the body is at rest, and it is pulled only by forces less than friction, its whole effect consists only of destroying those effects that these forces could have produced on their own. Thus, naming the force $EF = F$, the body receives no motion unless F exceeds the

value of friction $\frac{m}{n}P$; but once $F > \frac{m}{n}P$ the body receives an acceleration which corresponds to the excess $F - \frac{m}{n}P$, and it does not follow that if $F < \frac{m}{n}P$, the acceleration becomes negative.

VII. This may seem quite strange at first, and contrary to the Law of Continuity, so that nature seems to make a mistake here which never occurs in the action of other forces. However, one can represent the action of friction in a manner that will lift all doubt, and that will conform to the action of the other forces: for I will show than one can produce through the sole action of gravity an effect altogether similar to that of friction, by which we may even be able to discover the nature of friction, even though it would still not be known by experiment. This consideration will also serve to show what the true cause of friction consists of, and where this resistance which opposes motion comes from. For though it may be that the true cause of friction does not agree precisely with that which I am about to represent, the perfect resemblance that can be seen there will not leave a single doubt as to the possibility of these effects which seem so strange.

VIII. On the horizontal line MN (Fig. 2) let aG and bG be two equally inclined planes, which form at G the angle aGb, in which rests the body ABCD with pointed base AGB such that the angle AGB is exactly equal to aGb. In this situation, body ABCD will be not only in equilibrium, but also a small force EF applied horizontally will not be capable of setting it in motion, even if the faces of the body that touch the inclined planes are perfectly polished and no friction takes place. Because for the force QF to be able to move body ABCD, it must make it ascend the inclined plane Gb, and consequently it must be larger than the part of the body's weight which acts on it in the opposite direction GQ. In this way, body ABCD finds itself in a state quite similar to that of friction, since force EF is not capable of moving it when it is less than the amount required to overcome the slope of the inclined plane.

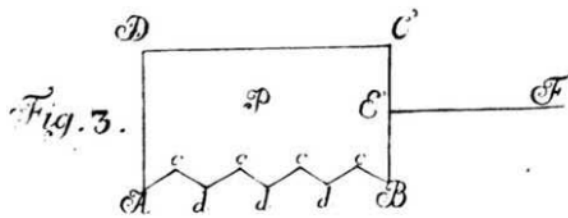


IX. The resemblance will appear greater still if we determine the quantity of the force EF required to set the body in motion. To that end, let angle MGa = NGb = α ; the weight of body ABCD = P , which acts on it from below in the vertical direction GP; and the force $EF = F$, which pulls the body in the horizontal direction EF. Since the body can only be set in motion in direction Gb, I decompose the force

EF = F into the direction EH parallel to Gb and FH which is perpendicular to it. As angle FEH is = NGb = α , the force EH will be = $F \cos \alpha$, and only this force is used to set the body in motion. As the motion is about to begin, the weight of the body, or the force GP = P, opposes it through the component GQ resulting from the decomposition along the directions GQ and PQ which is perpendicular to GQ. Thus, the angle GPQ being = α , the force GP will be = $P \sin \alpha$; from which we see that the body cannot be set in motion unless the force $F \cos \alpha$ is greater than $P \sin \alpha$.

X. Thus as long as $F \cos \alpha < P \sin \alpha$, body ABCD will remain at rest, and will not receive any motion from the action of force EF = F. But if $F \cos \alpha = P \sin \alpha$, or $F = P \tan \alpha$, the body will be, so to speak, in equilibrium, or completely ready to move as soon as force F becomes the slightest bit greater than $P \tan \alpha$. When it occurs that $F > P \tan \alpha$, the acceleration of the body in direction Gb will be produced by the excess of the force EH = $F \cos \alpha$ above $P \sin \alpha$, that is to say, by $F \cos \alpha - P \sin \alpha$. Consequently, the resistance that must be overcome in this case before the body can be moved will be = $P \sin \alpha$, which being equal to a portion of the weight of the body, and not depending whatsoever on the size of the base AGB along which the body touches the surface aGb, seems to entail a perfect enough resemblance between this case and that of friction. To make these cases identical, we have only to set $\sin \alpha = \frac{1}{3}$, in Amontons's hypothesis, or angles MGA and NGb = $19^\circ 29'$; but in Mr. Bilfinger's hypothesis these angles will be $14^\circ 28'$, because $\sin \alpha = \frac{1}{4}$.

XI. The same will be true if the base AB of body ABCD is formed (Fig. 3) of several obtuse angles AcdcdcB, all similar to AGB which we have just considered, and the surface MN is cut in a similar manner, so that the irregularities



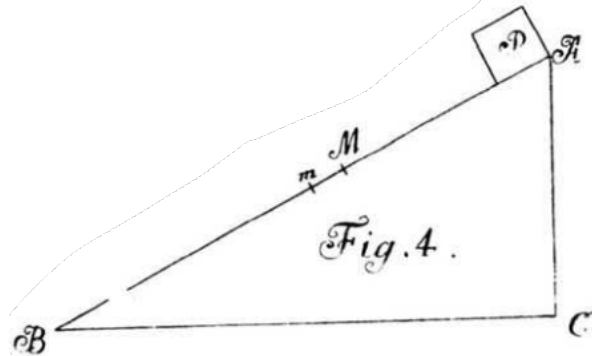
of the base and of the surface are in perfect agreement. For in this case, if each of the angles formed by the inclined planes cd with the horizontal line MN is = α , then body ABCD, whose weight is = P, will not be moved by the horizontal force EF = F unless $F \cos \alpha > P \sin \alpha$, or $F > P \tan \alpha$, and as long as force F is less than $P \tan \alpha$, the body will remain at rest. We can readily see that the same thing occurs however many bumps d, d , etc. there are, and it is not even necessary that all the inclines be equal to each other provided that none are greater than angle α , for even when there are some lesser angles, these do nothing to facilitate motion.

XII. If this is the case with friction, as seems quite probable, we can easily understand the phenomena of friction which I have described above and which

concern the difficulty of setting a body in motion. For this difficulty would lie only in the fact that for a body to move, it must effectively ascend an inclined plane. From there we see that once the body has begun to move, as the inclined planes dc , dc , etc. are extremely small, the body will rise and drop alternately. Consequently, since the descents occur on their own, while the body is moving, the difficulty of friction only makes itself felt in intervals, that is to say, in the moments when the body must ascend. From which it seems that we can draw this conclusion: that while the body is in motion, the effect of friction will only be half of what we feel before we are able to set the body in motion.

XIII. So in order for the force $EF = F$ to be able to impart motion on body $ABCD$, it must be larger than $P \tan \alpha$, but once the body is moving, the resistance of friction will be reduced by half. Consequently, to calculate the acceleration of the body, one has only to reduce the acting force by $\frac{1}{2}P \sin \alpha$, so that the acceleration will be proportional to $F \cos \alpha - \frac{1}{2}P \sin \alpha$, or maybe to $F - \frac{1}{2}P \tan \alpha$, since in the alternating descents, the acceleration is augmented by gravity. Those who have examined friction experimentally have confined themselves solely to discovering its quantity before the body is set in motion. Thus it would be highly desirable if we could also perform experiments from which we could determine the quantity of friction while the body is in motion; and I have almost no doubt that we will find it to be considerably less, because we know that to set a machine in motion the initial efforts must be larger than those which are then used to continue the motion.

XIV. One ordinarily uses an inclined plane to find the quantity of friction. Having (Fig. 4) set the body P on the plane AB , we successively tilt this plane from the horizontal position AC until body P comes to the point of descending. Then we measure the angle B of the slope of plane AB , or the sides of the right triangle ABC , from which we will solve for the value of the component of gravity which acts in direction AB , which will be $= P \sin B = \frac{AC}{AB} P$, and the friction of body P on plane AB is equal to this force. But as friction is proportional to the pressure with which body P is pressed to the plane, this pressure being $= P \cos B = \frac{BC}{AC} P$ [*sic*]², we can learn from this experiment that



² Translator's note: based on the scenario described and the conclusions that follow, it appears that this should read: $P \cos B = \frac{BC}{AC} P$.

friction is to pressure as $\sin B$ is to $\cos B$, or as AC is to BC . This ratio of friction to pressure will thus be as the tangent of the angle B is to 1. This will be the force of friction which must be overcome before body P can be set in motion.

XV. But to know if the friction which the body experiences while it is in motion is the same or not, one can determine the quantity of friction for the case of motion by means of the same inclined plane. We have only to tilt the plane AB a little more than in the preceding case, so that the body now slides to the bottom of the plane. Let the angle of inclination $B = \alpha$. The pressure of the body P on the plane will be $= P \cos \alpha$ and the force which acts on it in direction AB will be $= P \sin \alpha$. Let's suppose that during motion friction is to pressure as μ is to 1, so friction for the case that we consider will be $= \mu P \cos \alpha$. When this is deducted from the accelerative force $P \sin \alpha$, the body will still be pulled in the direction of its motion by the force $= P \sin \alpha - \mu P \cos \alpha = P(\sin \alpha - \mu \cos \alpha)$.

XVI. Say body P began its motion from rest at P , and that it arrived after a time t at M . Let the distance covered AM be $= s$, and the velocity at M , equal to that which a body acquires by falling from a height, $= v$, and the principles of mechanics furnish us with this equation:

$$P dv = P(\sin \alpha - \mu \cos \alpha) ds,$$

or by taking the integral,

$$v = (\sin \alpha - \mu \cos \alpha)s.$$

From there the equation for time will be

$$dt = \frac{ds}{\sqrt{v}} = \frac{ds}{\sqrt{(\sin \alpha - \mu \cos \alpha)s}},$$

the integral of which is

$$t = \frac{2\sqrt{s}}{\sqrt{(\sin \alpha - \mu \cos \alpha)}}.$$

If the distance traversed s is expressed in thousandths of a Rhine foot³, this expression will give us the time t expressed in seconds when we divide the expression by 250. Thus, if time t is in seconds and distance s is in thousandths of a Rhine foot, we will have this equation:

$$t = \frac{\sqrt{s}}{125\sqrt{(\sin \alpha - \mu \cos \alpha)}}.$$

XVII. Suppose now that we have precisely measured the time that body P has taken to descend the inclined plane AB , whose angle of elevation above the horizon,

³ Translator's note: The Rhine foot (*le pied de Rhin*) is a historic unit of measurement equal to around 314mm.

or angle B, is $= \alpha$. Let the length of plane AB = m thousandths of a Rhine foot, and the time of the descent of this plane = n seconds, and we have this equation:

$$n = \frac{\sqrt{m}}{125\sqrt{\sin \alpha - \mu \cos \alpha}},$$

where $15625n^2(\sin \alpha - \mu \cos \alpha) = m$, from which we can extract the value of the letter μ :

$$\mu = \tan \alpha - \frac{m}{15625n^2 \cos \alpha}.$$

Thus by means of a single experiment one will be able to determine the ratio of friction to pressure, which was designated as μ to 1, for the case of the motion of body P.

XVIII. From this formula it is first of all clear that if the angle α is equal to that for which the body P remains at rest, then the value of friction is precisely the same as what we would have found for rest. For since the body in this case receives no motion, it can be regarded as taking an infinite time to complete its descent. In this case, then, the time n becomes infinity, and the formula gives us $\mu = \tan \alpha$, or friction will be to pressure as the tangent of angle B is to 1, all as we have found. But once we tilt plane BA a little more, the body will now descend, and if we observe the time it takes to travel the distance AB, our formula will give us the value of μ corresponding to motion, and it will be, as would seem likely, smaller than the value in the preceding case of rest. We can convince ourselves still further on this matter if we successively give plane AB several different slopes, to see if each one will yield the same value for μ ; for in the case where we obtain different values, we would have to conclude that friction is not the same for all degrees of velocity, which does not seem probable.

XIX. In the case that the force of friction were smaller in motion than at rest, quite a strange phenomenon would arise, deserving of all possible attention. To explain it clearly, let α be the angle B of the inclined plane where weight P is supported still just barely at rest, so that should we increase this angle, the weight would at present descend on the inclined plane. Thus for the state of rest the value of friction will be $\mu = \frac{\sin \alpha}{\cos \alpha}$. But now supposing that friction becomes smaller while the body is in motion, let the value of friction for the state of motion be

$$\mu = v \frac{\sin \alpha}{\cos \alpha},$$

where v represents a fraction smaller than 1. Now let us increase angle B, so that motion occurs, and set angle B = φ , so that in the formula found above we have only to write φ for α and $v \frac{\sin \alpha}{\cos \alpha}$ for μ to find the time n'' in which body P will descend the inclined plane AB whose length is m thousandths of a Rhine foot. The time will thus be:

$$n = \frac{\sqrt{m}}{125 \sqrt{\sin \varphi - v \cos \varphi \frac{\sin \alpha}{\cos \alpha}}}.$$

At this time let's suppose that the angle $B = \varphi$ surpasses the angle of rest α only infinitesimally, and we should believe following the Law of Continuity that the motion of the body would be infinitely slow. But we will see to our surprise that this motion is suddenly completed in a finite and even fairly short time. For let $\varphi = \alpha + \omega$, where ω represents an infinitely small quantity, so that $\sin \varphi = \sin \alpha + \omega \cos \alpha$ and $\cos \varphi = \cos \alpha - \omega \sin \alpha$. Substituting these values, we will have

$$n = \frac{\sqrt{m}}{125 \sqrt{\sin \alpha + \omega \cos \alpha - v \sin \alpha + v \omega \sin \alpha \tan \alpha}} = \frac{1}{25} \sqrt{\frac{m}{(1-v) \sin \alpha}}.$$

XX. To better convey the phenomenon that this formula encapsulates, let the length of the inclined plane AB be exactly = 15625 thousandths of a Rhine foot, or let AB equal the height a body falls in one second. The time of the descent of body P down the inclined plane AB will be

$$n = \frac{1}{\sqrt{(1-v) \sin \alpha}}$$

seconds. In addition, let $\sin \alpha = \frac{1}{4}$ as Mr. Bilfinger found in his experiments, and the time will be

$$n = \frac{2}{\sqrt{(1-v)}}:$$

and if friction becomes twice as small in motion, which means $v = \frac{1}{2}$, this time will be $n = 2\sqrt{2}$ or almost $3''$. Thus it would not be possible to give plane AB an inclination such that the time of descent exceeds $3''$. For as long as angle B is $= \alpha$, body P does not descend at all; and once we tilt the plane the slightest bit from there, the descent suddenly becomes so rapid that the body only takes roughly 3 seconds to traverse the inclined plane AB of over 15 Rhine feet. Yet it is clear that if we tilt the plane even more, the time of descent will become even smaller. Experimentation seems to be rather in favor of this paradox than against it, for one will easily notice that it is not possible to give an inclined plane such an inclination that the descent occurs as slowly as one would like; the body either does not descend at all, or it descends fairly quickly. But to attain better success in these experiments, much care must be taken that the plane being used is equally polished everywhere, so that friction is the same everywhere, for there is no doubt that if friction is greater in one area of the plane than in another, one will be unable to draw a single well-assured conclusion from the experiment.