On Various Ways of Approximating the Quadrature of a Circle by Numbers.[1]

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§1. *Archimedes* and those who followed him investigated the approximate ratio in numbers of the diameter to the circumference by regular polygons both inscribed and circumscribed to the circle. Since indeed the perimeter of an inscribed polygon is less than, and that indeed the perimeter of a circumscribed polygon is greater than the circumference of that circle, they hence deduced appropriately enough that the circumference would be contained between these limits, as they must be defined; especially since these limits approach themselves more closely the more sides the polygons have. So when the radius of the circle is set $=1$, a side of the inscribed polygon of 96 sides will be

$$=\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}},$$

and truly the side of a circumscribed polygon with the same number of sides will be

$$\frac{2\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}};$$

the following limits will appear, between which the total circumference of the circle is contained, the lesser being clearly

$$96\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$$

and the greater being

$$\frac{192\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}.$$

§2. However it is seen from this single example how difficult and laborious it is to produce these limits in rational numbers, due to so very many repeated extractions of square roots: which work turns out to be even greater, if polygons having more sides than those above are considered: so much so that an extremely precise ratio of the diameter to the circumference should not be expected through this method, [a ratio] which is now indeed well-known, and which has been produced in

[1] *Comm. Acad. Sci. Imp. Petropol.* **9**, 1737/1744, pp. 222-236. [E74]

decimal fractions to 127 figures: after assuredly the diameter is set $= 1$, the circumference will be expressed by the following decimal fraction.

$$\begin{aligned}3,\quad &14159265358979323846264338327950288419\\ &71693993751058209749445923078164062862\\ &08998628034825342117067982148086513272\\ &3066470938446+\end{aligned}$$

of which fraction the first hundred digits are due to *the most brilliant Machin*[2], in fact *the most brilliant Lagny*[3] has calculated all the digits by his own related method.[4]

§3. Thus to the preceding method of Archimedes by inscribed and circumscribed polygons a new method deservedly must be preferred, honored above all at this time, in which the circumference of the circle is customarily expressed by a convergent infinite series. If indeed a series of this type converges rapidly, and moreover if the terms in that series are able to be converted easily into decimal fractions, then the ratio of the diameter to the circumference will be able to be expressed in approximate rational numbers with much less work, than by that other method, which requires so many extractions of roots. Now because of this reason, in order that the calculation may be brought to an end, appropriate series must be selected for this purpose, which should have the two following requirements, to be useful and meaningful. The first, of course, is that the series should be converging rapidly, or of such a type, that any term is much smaller than the preceding, so that having not taken very many terms, a ratio close enough to the true one may be obtained. The fewer the terms that differ very little from the true value, the more suited the estimating series will be for discerning the true ratio of the diameter to the circumference.

§4. The other requirement demands that single terms of the series are not completely composed, or consist of simple numbers. Indeed the more complicated that single terms become, the more work with which it is converted into a decimal fraction, so perhaps more work is required to collect ten terms, than one thousand terms of a different simpler series that is much more slowly converging. Then indeed to render a simpler calculation, each term thus should be compared, so that after the preceding term has already been changed into a decimal fraction, the next term ought to be found easily from it; which property occurs principally in geometric series and those related, in which any term may be obtained from the preceding term with a single division. For this reason from among the series, by which circular arcs are wont to be expressed, those will be especially suitable to this use, which define from a given tangent the corresponding angle; these indeed differ from geometric series only in this way, that single terms have been divided by unequal terms above, whence few troubles arise in calculation.

§5. Thus after rejecting other series, by which the arc is defined either from the sine or the chord, as less suitable to our purpose, we will study principally this series, which from a given tangent is determined the corresponding circular arc. Now after setting the radius of the circle $= 1$, the arc

[2] Machin's work was published in *Synopsis palmorium mathescos* by W. Jones in 1706.

[3] DeLagny's work was published in his *Mémoire sur la quadrature du circle, et sur la mesure de tout arc, tout secteur et tout segment donneé* in 1721.

[4] The approximation reported in this paragraph is incorrect at its 113^{th} place, which should be an 8 rather than a 7. This error in DeLagny's work was noted and corrected by Vega in 1794 in his *Thesaurus logarithmorum completus*.

corresponding to the tangent x is $= \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} -$ etc. to infinity; from which is understood, the smaller the tangent x is taken to be, the more easily the corresponding arc will be able to be found. After setting $x = \frac{1}{10}$ for example, by easy work the arc corresponding to the tangent $\frac{1}{10}$ may be found in a decimal fraction even to a thousand figures; with even less work the arc may be found which corresponds to a tangent of $\frac{1}{100}$ or $\frac{1}{1000}$ etc. But from this not even the least help follows to find the ratio, which the diameter has to the entire circumference; although all arcs of that type, whose tangents are $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$ or of such type, render a rapidly coverging series simultaneously summable with light effort, they are incommensurate with the entire circumference, and the ratio between them and the circumference cannot be wholly found.

§6. So that the ratio which the diameter has to the circumference can be discovered with the aid of this series, a tangent of such a type ought to be substituted for x, whose corresponding arc would have a known ratio to the total circumference. Now of the arcs commensurable with the total circumference only one is given, which has a rational tangent, and it is the $45°$ arc, its tangent is of course equal to 1, the radius of the circle. After therefore setting $x = 1$, an eighth part of the entire circumference will be produced

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \text{ etc.}$$

which is that series of Leibniz, so that the ratio of the diameter to the circumference appears as 1 to

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \text{ etc.}\right).$$

Now this series converges so slowly, that more than 10^{50} terms ought to be collected, that the decimal fraction would be extended to one hundred such figures; which work would not be able to be overcome entirely until eternity. Indeed many shortcuts are used, by which that summation is able to be rendered more easily, and moreover since this series is transformed into others by these shortcuts, I will rather more fully investigate other converging series, by which the intended target may be obtained immediately.

§7. And thus other help seems not to remain, unless such an arc may be sought, whose tangent is indeed irrational, but which nevertheless corresponds to a single term; if indeed a more composed irrational quantity were substituted for x, then insuperable work to collect the terms would result, even if the series converged most rapidly. However, two arcs of just this type exist, one $60°$ and the other $30°$, of which the first tangent is $= \sqrt{3}$ and of the other is $\frac{1}{\sqrt{3}}$. We therefore set $x = \frac{1}{\sqrt{3}}$, for it is not fitting for $\sqrt{3}$ to be substituted for x, because a divergent series will arise; and the twelfth part of the entire circumference of the circle will be

$$= \frac{1}{1 \cdot \sqrt{3}} - \frac{1}{3 \cdot 3 \cdot \sqrt{3}} + \frac{1}{5 \cdot 3^2 \cdot \sqrt{3}} - \frac{1}{7 \cdot 3^3 \cdot \sqrt{3}} + \frac{1}{9 \cdot 3^4 \cdot \sqrt{3}} - \text{etc.}$$

whence the ratio of the diameter to the circumference appears as

$$1 \text{ to } \frac{2\sqrt{3}}{1} - \frac{2\sqrt{3}}{3 \cdot 3} + \frac{2\sqrt{3}}{5 \cdot 3^2} - \frac{2\sqrt{3}}{7 \cdot 3^3} + \text{ etc.}$$

which series now converges sufficiently, as each term is more than three times smaller than the preceding term. Moreover by collecting around 210 terms, the exact ratio will be obtained in a decimal fraction to one hundred figures, which work now would be achievable.

§8. By the use of this very series from geometric angles, the exact ratio of the diameter to the circumference has been determined in decimal fractions as far as to 74 figures; and the entire calculation appears in mathematical tables published by *Sharp* and others[5]. However, the greatest difficulty of this calculation consists in this, because before everything the square root of 3 ought to be extracted in decimal fractions to as many figures, as the exact ratio should be desired. Now having found the decimal fraction correct to 100 figures (for example), which is equal to $2\sqrt{3}$ or $\sqrt{12}$, then this fraction must be divided by 3 successively, by which are found the terms

$$\frac{2\sqrt{3}}{1}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3^2}, \frac{2\sqrt{3}}{3^3}, \text{ etc.}$$

Having done this, those successive terms must be divided by the odd integers $1, 3, 5, 7$, etc., so that the terms of this series

$$\frac{2\sqrt{3}}{1}, \frac{2\sqrt{3}}{3\cdot 3}, \frac{2\sqrt{3}}{5\cdot 3^2}, \frac{2\sqrt{3}}{7\cdot 3^3}, \text{ etc.}$$

may be produced. Finally the sum of the terms of even order is subtracted from the sum of those of odd order, and the resulting value will give the expressed circumference of the circle, whose diameter is $= 1$.

§9. But before I explain how an approximate ratio of the diameter to the circumference can be defined very easily and exactly by the use of this series by which the arc is expressed from a given tangent, it will be appropriate to demonstrate some shortcut, by the help of which the sum of such series will be able to be found with much less effort. Clearly the arc corresponding to the tangent $\frac{1}{p}$ is

$$= \frac{1}{p} - \frac{1}{3p^3} + \frac{1}{5p^5} - \frac{1}{7p^7} + \text{ etc.}$$

We will now regard n terms of this series collected into one sum, with n appearing as an even number, and the found sum $= S$, I state that the sum of the entire series continued to infinity is going to be [6]

$$\begin{aligned} = \quad & S + \frac{1}{p^{2n+1}}\left(\frac{1}{(1+p^2)(2n+1)} - \frac{2p^2}{(1+p^2)^2(2n+1)^2}\right. \\ + \quad & \left.\frac{2^2(p^4-p^2)}{(1+p^2)^3(2n+1)^3} - \frac{2^3(p^6-4p^4+p^2)}{(1+pp)^4(2n+1)^4} + \text{ etc.}\right) \end{aligned}$$

Therefore the summation of the remaining terms is reduced to the summation of another series, in which each term is less than the preceding by around the reciprocal of[7] $2n+1$; so that as more terms are collected in that act, the more quickly will the new series be converging.

[5]In *Mathematical tables* by Briggs, Wallis, Halley, and Sharp, 1705

[6]The original paper has the following formula stated incorrectly as $S + \frac{1}{p^{2n-1}}\left(\frac{1}{(1+p^2)(2n-1)} - \frac{2p^2}{(1+p^2)^2(2n-1)^2} + \frac{2^2(p^4-p^2)}{(1+p^2)^3(2n-1)^3} - \frac{2^3(p^6-4p^4+p^2)}{(1+pp)^4(2n-1)^4} + \text{ etc.}\right)$. The equation was corrected in the *Opera Omnia*.

[7]Corrected in *Opera Omnia* from $2n-1$ in the original paper.

§10. However rapidly this new series that encompasses the sum of all the remaining terms of the earlier series converges, new shortcuts can nevertheless still be applied to the finding of its sum. That is, after setting the sum

$$\overset{1}{\frac{1}{p}} - \overset{2}{\frac{1}{3p^3}} + \overset{3}{\frac{1}{5p^5}} - \cdots - \overset{n}{\frac{1}{(2n-1)p^{2n-1}}} = S,$$

the arc whose tangent is $\frac{1}{p}$ will be

$$= S + \frac{1}{p^{2n-1}(2n(1+pp)+p^2-1)} \text{ approximately,}$$

the value of which will be more exact, as more terms are collected by the action, or as the number n becomes larger, for which I have instructed to use an even number. And if $n = p^\mu$ then these forms will render a correct decimal fraction to as many figures as $(2n+3+3\mu)lp$ expresses[8]. Having set[9]

$$\frac{2}{(1+pp)(2n+1)} = q$$

for the sake of brevity, the true sum of the series[10]

$$\frac{1}{p} - \frac{1}{3p^3} + \frac{1}{5p^5} - \text{ etc. continued to infinity}$$

$$= S + \frac{1}{2p^{2n-1}}\left(\frac{q}{1-q+q^2p^2-q^3(p^4-2p^2)+q^4(p^6-8p^4+4p^2)-\text{ etc.}}\right)$$

or $\frac{1}{2p^{2n-1}}$ again due to be divided by[11]

$$\frac{1}{q} - 1 + qp^2 - q^2(p^4-2p^2) + q^3(p^6-8p^4+4p^2) - \text{ etc.}$$

and adding the resulting quotient to S will give the arc whose tangent $= \frac{1}{p}$.

§11. Having explained these aids, which follow from my method for summing a related series elsewhere[12], I go on to another much easier way that ought to be made known, by which with the help of this same series expressing the arc from a given tangent the ratio of the diameter to the circumference will be able to be defined as exactly as desired with lighter work, without any tedious extraction of roots. Namely I decompose the arc whose tangent is $= 1$ into two or more arcs, the tangents of which are rational. Certainly when the tangents of these arcs are less than one, the arcs themselves may be determined easily from these through the general series. Even if these

[8]In the preceding equation, $lp \equiv \log_{10} p$.

[9]Corrected in *Opera Omnia* from $\frac{2}{(1+pp)(2n-1)} = q$ in the original paper.

[10]Corrected in *Opera Omnia* from $S + \frac{1}{2p^{2n-1}}\left(\frac{q}{1+qp^2+q^2p^2-q^3(2p^4-p^2)+q^4(4p^6-8p^4+p^2)-\text{ etc.}}\right)$ in the original paper.

[11]Corrected in *Opera Omnia* from $\frac{1}{q} + p^2 + qp^2 - q^2(2p^4-p^2) + q^3(4p^6-8p^4+p^2) -$ etc. in the original paper.

[12]The formula at the bottom of the preceding page may be found by transforming a series given in §16 of *Comm. Acad. Sci. Imp. Petropol.* **8**,1741, pp. 147-158. [E55], substituting $\frac{\sqrt{-1}}{p}$ for n and $2n+1$ for x.

arcs considered by themselves are incommensurable with the total circumference, nevertheless because when added together they are equal to an arc of 45 degrees whose tangent $= 1$; the sum of these will yield an eighth part of the total circumference, from which the desired ratio of the diameter to the circumference flows readily. I now set $\alpha =$ the arc whose tangent is $= 1$, and the diameter to the circumference will be as 1 to 4α.

§12. Therefore we set $At1 = At\dfrac{1}{a} + At\dfrac{1}{b}$, where it ought to be that $1 = \dfrac{a+b}{ab-1}$; whence $ab - 1 = a + b$ and $b = \dfrac{a+1}{a-1}$. As moreover a and b are integers, which is required to make the calculation easier, I set $a = 2$, and thus $b = 3$. Therefore the arc whose tangent $= 1$, which I have set $= \alpha$ is equal to the sum of the arcs whose tangents are $\frac{1}{2}$ and $\frac{1}{3}$. Wherefore the arc α will be equal to the sum of the two following series

$$+\frac{1}{1\cdot 2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \text{ etc.}$$
$$+\frac{1}{1\cdot 3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \text{ etc.}$$

each of which converges more [rapidly], than that one above deduced from the tangent $\frac{1}{\sqrt{3}}$, and is not hindered by any extraction of roots. Therefore with the aid of these two series the ratio of the diameter to the circumference will be able to be defined to many more exact figures with lighter trouble, than was allowed to be done by that single series, especially if the indicated helps are applied.

§13. Now if the sum of the series

$$\frac{1}{1\cdot 2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \text{ etc.}$$

is desired in decimal fractions correct to one hundred figures, then 154 terms ought to be collected, and to their sum should be added $\frac{1}{2^{307}\cdot 1543}$, by which the desired sum may be obtained; I am clearly using a single help mentioned in §10, by which the total sum of the series was

$$= S + \frac{1}{p^{2n-1}(2n(1+pp)+pp-1)}.$$

But however if the sum to 200 figures is desired then 318 terms ought to be collected. Truly the other series

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \text{ etc.}$$

may be reduced to a decimal fraction, so that it would not be mistaken through at least the one hundredth figure by collecting 98 terms[13]; moreover so that it may be obtained exact to two hundred figures, 202 terms must be collected[14]. Therefore to find the ratio of the diameter to the circumference in a decimal fraction correct to 100 figures 252 terms should be added together[15], while to obtain the same from the single series

$$\frac{1}{1\sqrt{3}} - \frac{1}{3\cdot 3\sqrt{3}} + \frac{1}{5\cdot 3^2\sqrt{3}} - \text{etc.}$$

[13]The original paper and *Opera Omnia* have an incorrect value of 96 terms.
[14]The original paper and *Opera Omnia* have an incorrect value of 200 terms.
[15]The original paper and *Opera Omnia* have an incorrect value of 250 terms.

more than 200 terms should be added.

§14. Moreover, by taking these steps it will be easy to decompose the arc α whose tangent $= 1$ into two or more arcs in an infinite number of other ways, which will produce series converging much more quickly. Since for instance

$$At\frac{1}{p} = At\frac{1}{p+q} + At\frac{q}{p^2+pq+1},$$

it will result that

$$At\frac{1}{2} = At\frac{1}{3} + At\frac{1}{7}.$$

Wherefore since

$$\alpha = At\frac{1}{2} + At\frac{1}{3},$$

it will now be the case that

$$\alpha = 2At\frac{1}{3} + At\frac{1}{7},$$

and α again will equal these two series joined together

$$+\frac{2}{1\cdot 3} - \frac{2}{3\cdot 3^3} + \frac{2}{5\cdot 3^5} - \frac{2}{7\cdot 3^7} + \text{ etc.}$$

$$+\frac{1}{1\cdot 7} - \frac{1}{3\cdot 7^3} + \frac{1}{5\cdot 7^5} - \frac{1}{7\cdot 7^7} + \text{ etc.}$$

which will converge much more quickly that the previous. As it happens the most suitable decomposition will be

$$\alpha = 4At\frac{1}{5} - At\frac{1}{239}$$

or

$$\alpha = 4At\frac{1}{5} - At\frac{1}{70} + At\frac{1}{99},$$

which arcs most certainly are able to be defined with the aid of the most rapidly converging series. But each one, who is pleased to undertake a calculation of this type, selects for himself or herself the most suitable decomposition.

§15. Other series, by which also an arc from a given tangent is defined, can also be used with not less success, as will be seen; now these series, which can be called into use suitably, are especially the following.

$$\begin{aligned}
At.\frac{p}{p^2-1} &= \frac{1}{p} + \frac{2}{3p^3} + \frac{1}{5p^5} - \frac{1}{7p^7} - \frac{2}{9p^9} - \frac{1}{11p^{11}} + \text{ etc.}\\
At.\frac{2p}{2p^2-1} &= \frac{1}{p} + \frac{1}{3\cdot\ 2p^3} + \frac{1}{5\cdot 2^2\,p^5} - \frac{1}{7\cdot 2^3\,p^7} - \frac{1}{9\cdot 2^4\,p^9} + \text{ etc.}\\
At.\frac{3p}{3p^2-1} &= \frac{1}{p} - \frac{1}{5\cdot 3^2\cdot p^5} + \frac{1}{7\cdot 3^3\cdot p^7} - \frac{1}{11\cdot 3^5\cdot p^{11}} + \frac{1}{13\cdot 3^6\cdot p^{13}} - \text{ etc.}\\
At.\frac{3p(pp-1)}{p^4-4pp+1} &= \frac{3}{1\cdot p} + \frac{3}{5\cdot p^5} - \frac{3}{7\cdot p^7} - \frac{3}{11\cdot p^{11}} + \frac{3}{13\cdot p^{13}} + \text{ etc.}
\end{aligned}$$

From this last series p is set equal to 2

$$At\ 18 = 3\left(\frac{1}{1\cdot 2} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} - \frac{1}{11\cdot 2^{11}} + \frac{1}{13\cdot 2^{13}} + \frac{1}{17\cdot 2^{17}} - \text{ etc.}\right)$$

if to this arc is added $At\ \frac{1}{18}$, which may be produced through common series, the fourth part of the circumference or 2α appears. In a similar way from the second series appears $2\alpha =$

$$1 + \frac{1}{3\cdot 2} - \frac{1}{5\cdot 2^2} - \frac{1}{7\cdot 2^3} + \frac{1}{9\cdot 2^4} + \frac{1}{11\cdot 2^5} - \text{ etc.}$$
$$+\frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \frac{1}{9\cdot 2^9} - \frac{1}{11\cdot 2^{11}} + \text{ etc.}$$

§16. When however in this work the arc, whose tangent is indeed small, but its numerator is not $= 1$, can be found by Leibnitz's series, then it may be difficult to evaluate single terms of the series. Therefore when these occur, it will be convenient to decompose the arc into two others, whose tangents have one for a numerator, which will be able to be done quite often. Namely an arc is to be investigated whose tangent is $\frac{a}{b}$; set $At\frac{a}{b} = At\frac{1}{m} + At\frac{1}{n}$; then it will follow that $\frac{m+n}{mn-1} = \frac{a}{b}$. Hence $(ma-b)(na-b) = a^2+b^2$ will follow. For this reason it must be sought whether a^2+b^2 can be decomposed into two factors, both of which when augmented by the denominator b are divisible by the numerator a. Now when this happens the values which will have arisen from those divisions must be substituted for m and n. Thus if an arc whose tangent $= \frac{7}{9}$ must be found, since $7^2+9^2 = 130 = 5\cdot 26$ the values of m and n will be $\frac{5+9}{7}$ and $\frac{26+9}{7}$ or 2 and 5. And so it will follow that $At\frac{7}{9} = At\frac{1}{2} + At\frac{1}{5}$, from whence $At\frac{7}{9}$ would be recovered with no difficulty.

§17. More often however the arc cannot be decomposed into two other arcs of this kind since the sum of the squares does not have factors of this nature. Therefore when these occur, the given arc should be decomposed into three or more arcs, which will be accomplished in the following way. Let the given arc whose tangent is $\frac{x}{y}$ be [16]

$$At\frac{x}{y} = At\frac{ax-y}{ay+x} + At\frac{1}{a}.$$

If now an integer value for a cannot be found, so that $ax-y$ is a divisor for $ay+x$ itself, then at least it is sought as a fraction, and for $At\frac{1}{a}$ is set $At\frac{b-a}{ab+1} + At\frac{1}{b}$; and again it may be clearly seen, whether a whole number may be given which when substituted for b gives $b-a$ as a divisor of $ab+1$ itself. Therefore by continuing in this way, the following formulae arise

I. $At\frac{x}{y} = At\frac{ax-y}{ay+x} + At\frac{1}{a}$.

[16] The original paper reads x/a for x/y here, and states the next equation incorrectly as $At\frac{x}{y} = At\frac{ax-y}{ay+x} + At\frac{1}{b}$. These were corrected in the *Opera Omnia*.

II. $At\frac{x}{y} = At\frac{ax-y}{ay+x} + At\frac{b-a}{ab+1} + At\frac{1}{b}$

III. $At\frac{x}{y} = At\frac{ax-y}{ay+x} + At\frac{b-a}{ab+1} + At\frac{c-b}{bc+1} + At\frac{1}{c}$

IV. $At\frac{x}{y} = At\frac{ax-y}{ay+x} + At\frac{b-a}{ab+1} + At\frac{c-b}{bc+1} + At\frac{d-c}{cd+1} + At\frac{1}{d}$

§18. If therefore a, b, c, d, etc. is any progression of numbers eventually increasing to infinity, we will have an infinite series of arcs, which added all together equals the given arc. Namely it will be

$$At\frac{x}{y} = At\frac{ax-y}{ay+x} + At\frac{b-a}{ab+1} + At\frac{c-b}{bc+1} + At\frac{d-c}{cd+1} + At\frac{e-d}{de+1} + \text{ etc.}$$

However, it is necessary that the limit of the progression a, b, c, d, etc. should be infinitely large, because the arc whose cotangent is [infintitely large] is disregarded, hence follow highly esteemed summable series of arcs; as having set $\frac{x}{y} = 1$ and the series of odd numbers 3, 5, 7, 9, etc. for a, b, c, d, etc.

$$At1 = At\frac{1}{2} + At\frac{1}{8} + At\frac{1}{18} + At\frac{1}{32} + \text{ etc.}$$

will result, in which the denominators of the tangents are twice the squares of the natural numbers. By a similar method, it will follow that

$$At1 = At\frac{1}{3} + At\frac{1}{7} + At\frac{1}{13} + At\frac{1}{21} + At\frac{1}{31} + \text{ etc.}$$

§19. As a crown, I will attach in this place a not inelegant theorem, which can serve to examine the nature of the circle more thoroughly. Namely in the circle whose radius or total sine $= 1$, any arc A is equal to this value

$$\frac{\sin.A}{\cos.\frac{1}{2}A.\ \cos\frac{1}{4}A.\ \cos\frac{1}{8}A.\ \cos\frac{1}{16}A.\ \text{etc.}}$$

Or rather, this is likewise expressed through secants as

$$A = \sin.A.\ \sec.\frac{1}{2}A.\ \sec\frac{1}{4}A.\ \sec\frac{1}{8}A.\ \sec\frac{1}{16}A.\ \text{etc.}$$

which expression can be properly applied to finding the logarithm of whatever arc from given logarithms of the sines and secants: namely

$$\text{l}.A = \text{l.sin}.A + \text{l.sec}.\tfrac{1}{2}A + \text{l.sec}.\tfrac{1}{4}A + \text{l.sec}.\tfrac{1}{8}A + \text{etc.}$$

where it must be noted that, if we use the standard table of logarithms, the logarithm of the sine of the total ought to be taken away from whatever logarithm you have. Thus if the logarithm of 1 degree is sought

$$
\begin{array}{lcr}
\text{log.sin. } 1^\circ & = & (-2)\,,2418553 \\
\text{log.sec. } 30' & = & 0\,,0000165 \\
\text{log.sec. } 15' & = & 0\,,0000041 \\
\text{log.sec. } 7\tfrac{1}{2}' & = & 0\,,0000010 \\
\text{log.sec. } 3\tfrac{1}{4}' & = & 0\,,0000003 \\ \hline
\text{log. Arc. } 1^\circ & = & (-2)\,,2418772
\end{array}
$$

$$
\begin{array}{lcr}
\text{log. } 180 & = & 2\,,2552725 \\
\text{l. A. } 1^\circ & = & (-2)\,,2418762 \\ \hline
\text{l. A. } 180^\circ & = & 0\,,4971497
\end{array}
$$

to this logarithm corresponds the number $3,14159$.[17]

§20. The demonstration of this theorem depends upon a reciprocal relation of the sines and the cosines of angles which have a double ratio between themselves. When indeed the sine of any angle is multiplied by its cosine, it produces half of the sine of the doubled angle, so the sine of any angle divided by the cosine of the halved angle will equal twice the sine of the halved angle, thus

$$\frac{\sin.A}{\text{cosin}.\frac{1}{2}A} = 2\sin.\frac{1}{2}A.$$

By like reason

$$\frac{\sin.A}{\text{cosin}.\frac{1}{2}A.\ \text{cosin}.\frac{1}{4}A} = \frac{2\sin.\frac{1}{2}A.}{\text{cosin}.\frac{1}{4}A}.$$

and by the same property

$$\frac{\sin.A}{\text{cosin}.\frac{1}{2}A.\ \text{cosin}.\frac{1}{4}A} = 4\sin.\frac{1}{4}A.$$

And proceeding farther will be had

$$\frac{\sin.A}{\text{cosin}.\frac{1}{2}A.\ \text{cosin}.\frac{1}{4}A.\ \text{cosin}.\frac{1}{8}A} = 8\sin.\frac{1}{8}A$$

From this it is concluded that, if the progression of cosines were continued to infinity, it is going to be

$$\frac{\sin.A}{\text{cosin}.\frac{1}{2}A.\ \text{cosin}.\frac{1}{4}A.\ \text{cosin}.\frac{1}{8}A.\ \text{cosin}.\frac{1}{16}A\ \text{ etc.}} = \infty\sin.\frac{1}{\infty}A.$$

which is equal to the arc A itself. Q. E. D.

[17] The sum leading to the value of log. Arc. 1° was reported incorrectly as -2.2418762, in the original paper, and thus the value of l. A. 180° was incorrectly reported as 0.4971487. The errors were corrected in the *Opera Omnia*.