Translation with notes of Leonhard Euler's

METHODVS VNIVERSALIS SERIERVM CONVERGENTIVM SVMMAS QUAM PROXIME INVENIENDI

A general method for finding approximations to the sums of convergent series (E46)

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Introduction to the translation

This translation is the pleasant collaborative effort of student and professor. Walter Jacob is a senior undergraduate mathematics major and Tom Osler has been a professor of mathematics for the past 45 years. We found making this translation a delightful challenge.

The modern reader might find this paper particularly difficult to read because of the primitive mathematical notation available to Euler. He did not have our notations for summation, functions or definite integrals. Without the notation $f(x)$ it becomes especially difficult to explain these results.(In later papers we find Euler using $f:(x)$.)

For this reason we have rewritten many of Euler's expressions in modern notation in the Notes that follow this translation and in our Synopsis of E46. We recommend that the reader consult both of these while studying this translation.

We tried to imagine how Euler would express himself if he was fluent in modern English. He sometimes wrote in very long sentences which we converted into several smaller ones. In the body of the translation we did not change any of Euler's notations, but we did use modern notation in our Notes and Synopsis. We did not find any typographical errors in Euler's paper, thus any errors that the reader finds here are probably ours.

1. In the past, using integration, I found approximations to the sums of certain series and very easily applied the method to series whose sums converge. I also gave some examples of this method by differentiating the harmonic series. It is now time to perfect this method and also to look forward to deriving many interesting examples involving the approximate sum. In this method the calculation of

the approximate sum is reduced to integration. Let n be a positive number indexing the terms of the series, which sums to s. As n is a discrete variable, the sum s changes with each unit increase in n. The term x becomes very small as n becomes very large. The instantaneous rate of growth of n is 1 and the corresponding instantaneous rate of growth of s is x, therefore $dn : ds = 1 : x$. Thus we have $ds = xdn$ and $s = \int xdn$. In this the unknown constant of integration needs to be added. On the other hand, if the sum s is from an infinite series that converges, this constant of integration vanishes. Accordingly by this method we obtain a sum from the term x starting at any point in the infinite sum. When we add this to the (previously determined) finite sum s we obtain the total sum of the series.

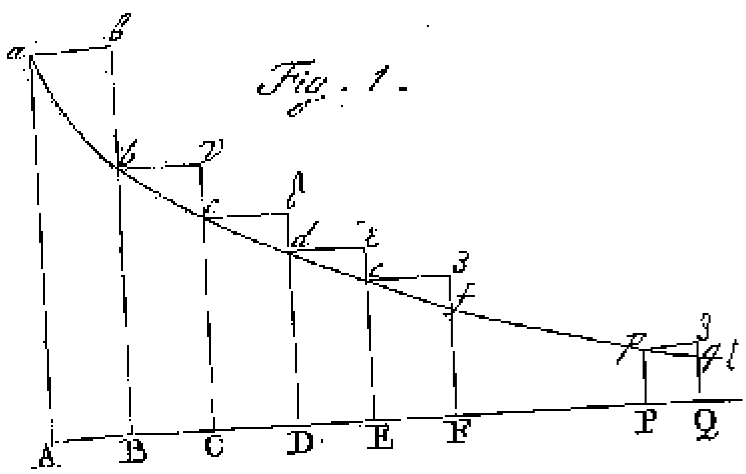


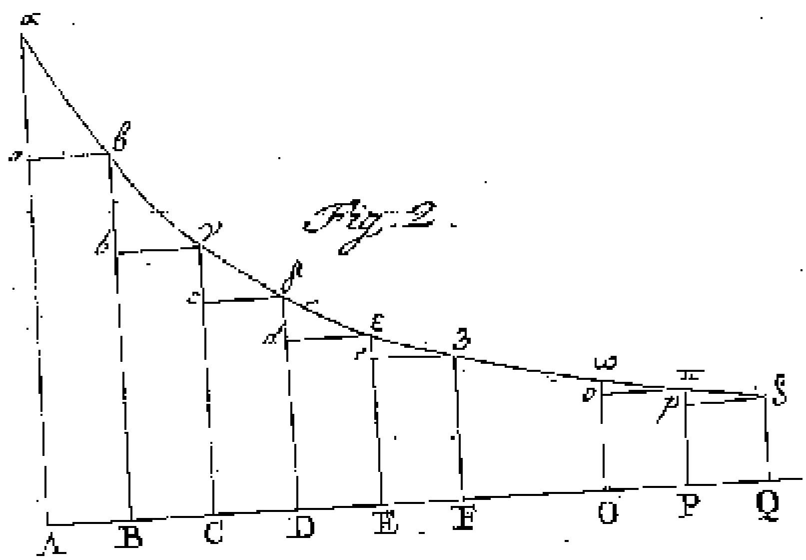
Fig. 1.

2. On the other hand this method always sums the series with some small error. By using different comparisons, we determine values between which the true sum exists. In this work great light is shed on the unknown, if, before performing the calculation, we consider a continuous curve appropriate for our series. By examining the graph, we imagine a method of discovery that can later be applied to some remarkable examples. We proceed now to examine the figure in which the summation of series is compared the integration under the curve.

3. In the series, whose sum we investigate, $a+b+c+d+e+etc.$ the index is *n* and the terms *x* are known functions of *n*. We sum over the interval *AP* partitioned into *AB, BC, CD, DE* etc. which denotes equal sections of length 1. On the points *A, B, C, D* etc. we erect ordinates *Aa, Bb, Cc, Dd* etc. which equal respectively the terms of the series *a, b, c, d* etc. At $AP = n-1$ we have the ordinate $Pp = x$. To these ordinates, we extend lines of length =1 to form the rectangles *Aβ, Bγ, Cδ, Dε* etc up to *Pρ*. Therefore the sum of these rectangles equals the sum of the series $a+b+c+d+...+x$. Hence to discover the sum of this series, we must determine an approximation to the area of these rectangles.

4. Through the points *a, b, c, d, e, f, ...p* we extend the curved line *abcdef...p*, as is natural, to the point where the abscissa is $AP=n-1=t$ and the ordinate is $Pp=x$. The term *x* is composed of *n* and constants, if in place of *n* we write $t+1$ we will have an equation for the curve *x* in terms of *t* .
If the next ordinate after *x* is *y*, it will have index $n+1$ and the equation will have abscissa *n* and ordinate *y* . We naturally get the curve *abc...pq*. This series will give *y* in terms of *n*, from which we find the equation between *n* and *y* which naturally describes the curve.

5. The area under this curve with base *AQ* between the ordinates *Aa* to *Qq* is slightly less than the sum of all the areas of the rectangles $A\beta+B\gamma+...+P\rho$, and the difference is the sum of all the curvilinear triangles *abβ, bcγ, ... pqρ*. With the above sum of the rectangles equal to the series $a+b+c+...+x$, this sum is larger than the area *AaqQ*. The area under the curve is $=\int ydn$ and when this integral is evaluated, with lower limit $n=0$, we get the inequality $a+b+c....+x > \int ydn$.



6. Therefore we have discovered a lower limit, such that the sum of the series is larger, and we now find the upper limit by a similar method. Suppose (Fig. 2) again we partition the interval AP into increments AB, BC, CD, ... PQ, which are of unit length, and ordinates are erected, such that $B\beta=a$, $C\gamma=b$, $D\delta=c$, and at $AP=n$ we erect two final ordinates $P\pi=x$ and $Q\zeta=y$. Now y is the final term of the sequence, just after x, and α is the first term which is represented by the ordinate $A\alpha$. In fact the total of the rectangles $Ba+Cb+Dc+...Po$ is equal to the sum of the series $a+b+c+d+...+x$.

7. Now in a similar way we draw through the points α, β, γ, ... π the curved line $\alpha\beta\gamma...\pi$, which is shown, and we see that at the end of the segment $AP=n$ the ordinate is $P\pi=x$. Therefore the area, which is between the curve and the segment AP, is $=\int xdn$, where this integral is defined such that lower limit of integration is $n=0$. In truth this area is larger than the area of the sum of the rectangles Ba, Cb, ... Po, which we describe by $a+b+c....+x<\int xdn$. Therefore $\int xdn$ and $\int ydn$ are the upper and lower limits of the series.

8. So far we have obtained close approximations to the sums of these series by neglecting the small triangles in each figure. In the first figure it is necessary to add these triangles to the area of the curve $\int ydn$, to obtain the sum of the series. In truth these triangles are curvilinear as well as larger, than if they were rectilinear, because of the concavity of the curve. The sum of these rectilinear triangles is equal to $(Aa - Qq)AB : 2$ or $\frac{a-y}{2}$ Hence if $\int ydn$ is added to $\frac{a-y}{2}$ it will not be sufficient to complete the area. Thus we have $a + b + c .... + x > \int ydn + \frac{a-y}{2}$.

9. In the second figure, from the area $A\alpha\pi P$, which is $=\int xdn$, we subtract the area of the triangles $\alpha a\beta$, $\beta b\gamma$, ... $\omega o\pi$ to obtain the true sum of the rectangles. We suppose the area of the curvilinear triangle $\alpha a\beta$ is less than the rectilinear triangle, which is $= \frac{a-b}{2}$ and the sum of all of these rectilinear triangles is $\frac{a-y}{2}$ In truth $\frac{a-y}{2}$ is less than the sum of all of the curvilinear triangles, so if $\frac{a-y}{2}$ is subtracted from $\int xdn$, we get $a + b + c .... + x < \int xdn - \frac{a-y}{2}$.

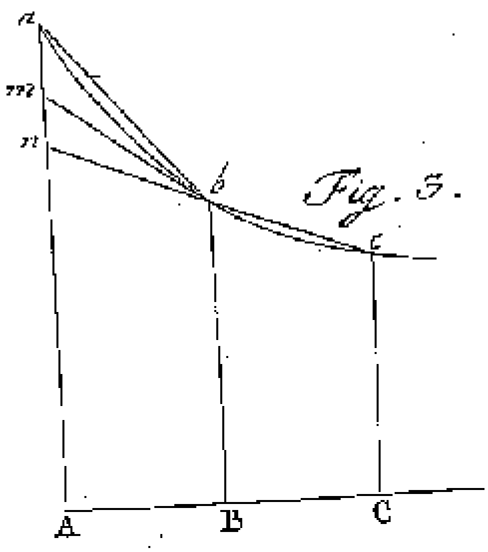


Fig. 3.

10. The value of the former of these two new limits is closer than the latter. The former $\int ydn + \frac{a-y}{2}$ is slightly less than the sum of $a+b+c+...+x$; the difference being the total of all of the segments bound by the arcs *ab*, *bc* etc. and the corresponding chords *ab*, *bc* etc. A value close to the area of these segments is estimated by the extending the chord *cb* to the left until it intersects the ordinate *Aa* at *n*, We bisect the segment *na* by *m* and notice that the line *bm* is an approximation to the tangent of the curve at *b*. The area of the segment *aba* is nearly one third the area of the triangle *abm* and consequently one sixth the area of the triangle *abn*.

11. Therefore if $Aa=a$, $Bb=b$, and $Cc=c$, $an=a-2b+c$ and since $AB=1$ the triangle $abn = \frac{a-2b+c}{2}$ Therefore this sixth part equaling $\frac{a-2b+c}{12}$ is the first curvilinear segment *aba*. Similarly the second segment $= \frac{b-2c+d}{12}$ and the furthest $= \frac{x-2y+z}{12}$, denoting by *z* the term indexed by $n+2$. Therefore the result of the sum of all of the segments $= \frac{a-b}{12} - \frac{y-z}{12}$ which add to the greater sum gives us

$$a+b+c+.......+x=\int ydn+\frac{a}{2}-\frac{y}{2}+\frac{(a-b)}{12}-\frac{(y-z)}{12}.$$

12. The value of this expression is extremely close to the true sum of the series, which it is indeed when we do not neglect (*any part*) of the segments. If the series converges, this formula gives approximate sums as close to the true value as you wish. This is made possible, by first finding the exact sum of the initial terms of the series and finally appending the sequence of points *a, b, c* etc. The greater the number of initial terms added exactly, the better the approximation we can obtain. Also if we have an infinite series, then the final terms x and y disappear, and in $\int ydn$ let the upper limit be $n=\infty$, so that the sum of the infinite series $=\int ydn+\frac{7a}{12}-\frac{b}{12}$.

13. We seek the sum to the millionth term of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}etc.$, and we begin by adding the first ten terms exactly and get 2.928968. The sum of the remainder of the terms $\frac{1}{11}+\frac{1}{12}+\frac{1}{13}etc.......\frac{1}{1000000}$ is estimated by our method is $a=\frac{1}{11}$, $b=\frac{1}{12}$, $x=\frac{1}{n+10}$, $y=\frac{1}{n+11}$ and $z=\frac{1}{n+12}$ also $\int ydn=\log\frac{n+11}{11}$. With the value $n=999990$ the desired sum is

$$\log\frac{1000001}{11}+\frac{1}{22}+\frac{1}{132}-\frac{1}{2000002}-\frac{1}{12000012}+\frac{1}{12000024}+2.928968 \text{ or } = 14.392669,$$

approximately.

14. We now examine the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}etc.$, whose sum to infinity is desired. If the first ten terms are added, we get 1.549768. The values needed for the remainder are $a=\frac{1}{121}$, $b=\frac{1}{144}$,

$x = \frac{1}{(n+10)^2}$, $y = \frac{1}{(n+11)^2}$. From these we get $\int ydn = \frac{1}{11} - \frac{1}{n+11}$ and at $n=\infty$ this becomes

$\int ydn = \frac{1}{11}$. Therefore the series summed to infinity is $\frac{1}{11} + \frac{1}{12.121} - \frac{1}{12.144} + 1.549768$. When

expressed in decimal form this is given by 1.644920.

Notes for E46

Euler uses primitive notations for describing functions, series, and definite integrals. To help clarify his mathematics we will use modern notation in these notes. The following table compares Euler's variables and notation with the modern notation we will use in these notes.

Description	Euler's Notation	Our Modern Notation
Series to be summed	$a+b+c+\cdots+p$ Also $a+b+c+\cdots+x$	$f(\alpha)+f(\alpha+1)+f(\alpha+2)+\cdots+f(\beta-1)$
Specific Values	x and p y and q	$f(\beta-1)$ $f(\beta)$
Curve shown in Figure 1	$abcde\cdots pq$	$y=f(x)$
Curve shown in Figure 2	$\alpha\beta\gamma\delta\cdots\pi\zeta$	$y=f(x-1)$
Integral for Lower Bound of Series	$\int ydn$	$\int_\alpha^\beta f(x)dx$
Integral for Upper Bound of Series	$\int xdn$	$\int_\alpha^\beta f(x-1)dx$

Euler's figures 1 and 2 can be combined to give us the following modern version.

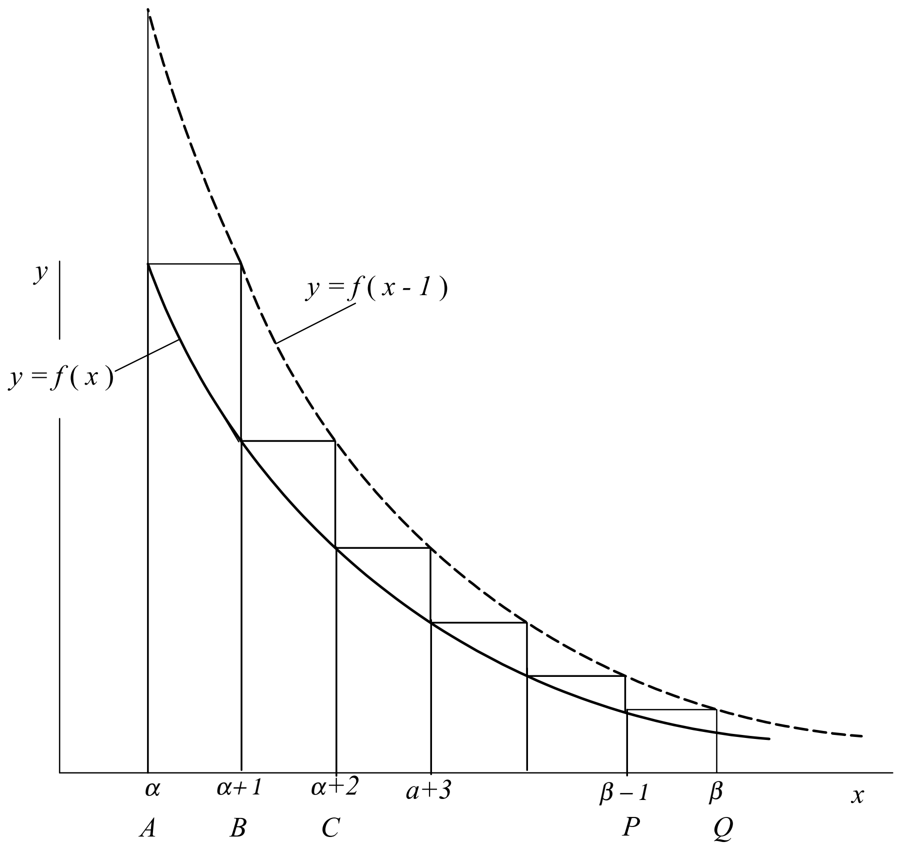


Figure 1: Sum bound by two curves

Section 1. Euler states that he will find approximations to series. The end result will be

$$\sum_{n=\alpha}^{\beta} f(n) \approx \int_{\alpha}^{\beta} f(x)dx + \frac{f(\beta)+f(\alpha)}{2} + \frac{f(\alpha)-f(\alpha+1)}{12} - \frac{f(\beta)-f(\beta+1)}{12}.$$

Sections 2 to 5. Euler uses the curve $y = f(x)$ to obtain the lower bound

(2) $$\int_{\alpha}^{\beta} f(x)dx < \sum_{n=\alpha}^{\beta-1} f(n).$$

Sections 6 and 7. Euler uses the curve $y = f(x-1)$ to obtain the upper bound

(3) $$\sum_{n=\alpha}^{\beta-1} f(n) < \int_{\alpha}^{\beta} f(x-1)dx.$$

Section 8. The estimate given by (2) can be improver by adding to the integral the sum of the areas of all the small curvilinear triangles *STQ*, one at the top of each rectangle. See figure 2.

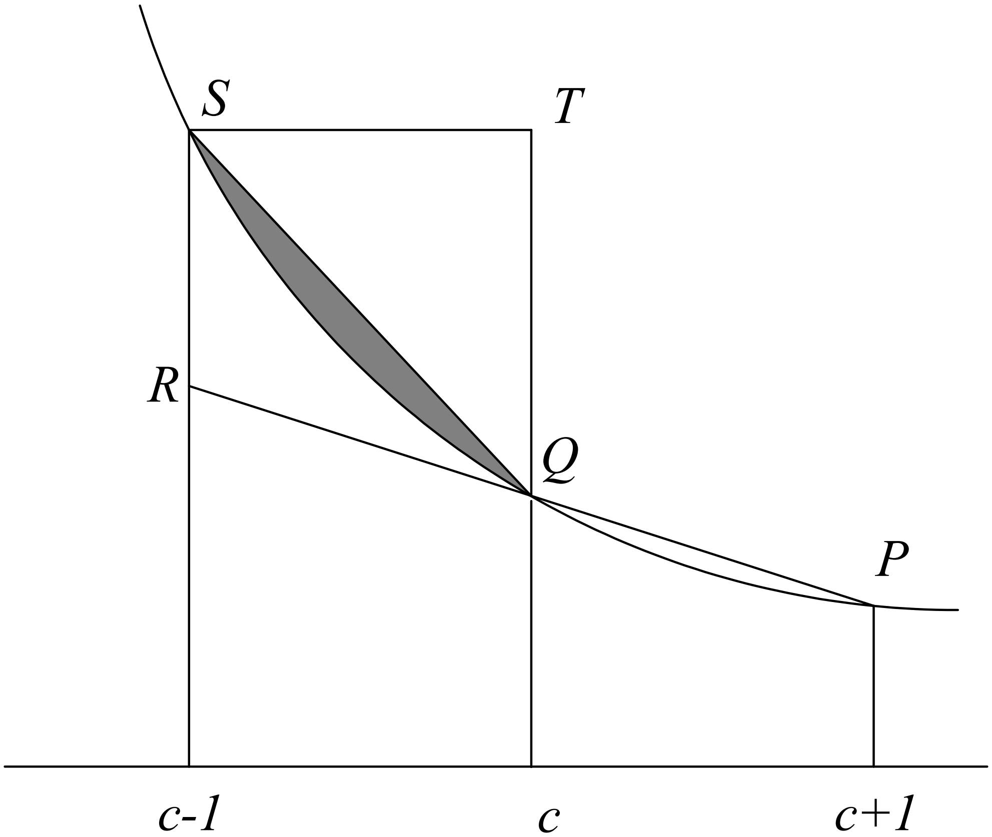


Figure 2: Triangular and curvilinear areas

In this section Euler sums the areas of the *rectilinear triangles* and gets the improved lower bound

(4) $$\int_{\alpha}^{\beta} f(x)dx + \frac{f(\alpha)-f(\beta)}{2} < \sum_{n=\alpha}^{\beta-1} f(n).$$

Section 9. Applying the ideas used in the previous section Euler improves the upper bound

(5) $$\sum_{n=\alpha}^{\beta-1} f(n) < \int_{\alpha}^{\beta} f(x-1)dx - \frac{f(\alpha)-f(\beta)}{2}.$$

Section 10. In this section Euler begins examining the curvilinear segment of area *SQS* shown shaded in figure 2. He assumes that his readers are familiar with the following lemma which we prove now:

Lemma: Let the arc *PQS* be a segment of a parabola, and let *PQR* be a chord as shown in Figure 2. Then the area of the curvilinear segment *QSQ* equals one sixth the area of the triangle *QSR*.

Proof: For the moment, assume that the curve shown in figure 2 is a parabola. Since all parabolas are similar, we can use the equation $y = f(x) = x^2$ to describe this curve. In the following we give our results using the notation $f(x)$ for the curve as well as the specific value x^2. We do this because we will approximate a general function by a parabola in the next section and will need the results in terms of $f(x)$. The slope of the chord *PQR* is

$$\frac{f(c+1)-f(c)}{1} = \frac{(c+1)^2 - c^2}{1} = 2c+1.$$

The equation for this chord is then

$$y = f(c) + \frac{f(c+1)-f(c)}{1}(x-c) = c^2 + (2c+1)(x-c).$$

The ordinate of the point *R* is then

$$\begin{aligned} y &= f(c) + \frac{f(c+1)-f(c)}{1}((c-1)-c) \\ &= 2f(c) - f(c+1) \\ &= c^2 - 2c - 1 \end{aligned}$$

and thus the length of the vertical segment *RS* is

$$f(c-1) - (2f(c) - f(c+1)) = (c-1)^2 - (c^2 - 2c - 1) = 2.$$

The area of the triangle *RSQ* is then

$$\frac{\left(f(c-1) - 2f(c) + f(c+1)\right)}{2}1 = 1. \tag{6}$$

The area of the curvilinear segment *QSQ* is given by the area of the quadrilateral under the chord *QS* minus the area under the parabola from $c-h$ to c. This is given by

$$\frac{(c-1)^2 + c^2}{2}1 - \int_{c-1}^{c} x^2 dx = \frac{(c-1)^2 + c^2}{2}1 - \left(\frac{c^3}{3} - \frac{(c-1)^3}{3}\right)$$

which simplifies to $\frac{1}{6}$. Comparing this with (6) we see that the lemma is proved.

Section 11. Using the lemma just proved, Euler approximates the short segment of the curve $y = f(x)$

above each rectangle by a parabola and thus can estimate the areas of the curvilinear segments like *SQS* shown in figure 2. When these segments are added and combined with (4) we get

$$\sum_{n=\alpha}^{\beta-1} f(n) \approx \int_{\alpha}^{\beta} f(x)dx + \frac{f(\alpha)-f(\beta)}{2} + \frac{f(\alpha)-f(\alpha+1)}{12} - \frac{f(\beta)-f(\beta+1)}{12}.$$

Adding $f(\beta)$ to both sides we get the main result

$$(7) \qquad \sum_{n=\alpha}^{\beta} f(n) \approx \int_{\alpha}^{\beta} f(x)dx + \frac{f(\beta)+f(\alpha)}{2} + \frac{f(\alpha)-f(\alpha+1)}{12} - \frac{f(\beta)-f(\beta+1)}{12}.$$

<u>Section 12.</u> If our series in infinite, then $\beta = \infty$ and (1) becomes

$$(8) \qquad \sum_{n=\alpha}^{\infty} f(n) \approx \int_{\alpha}^{\infty} f(x)dx + \frac{7f(\alpha)}{12} - \frac{f(\alpha+1)}{12}..$$

We can also compare (7) with the Euler-Maclaurin summation formula (see [1]),

$$\sum_{n=\alpha}^{\beta} f(n) = \int_{\alpha}^{\beta} f(x)dx + \frac{f(\beta)+f(\alpha)}{2} + \sum_{k=2}^{\infty} \frac{B_k}{k!}\left(f^{(k-1)}(\beta) - f^{k-1)}(\alpha)\right)h^{k-1}.$$

(Most likely Euler did not know this result at the time he wrote this paper.) The first term in the summation is $\frac{B_2}{2!}\left(f'(\beta)-f'(\alpha)\right)$, and since $B_2 = \frac{1}{6}$ this term becomes $\frac{f'(\beta)-f'(\alpha)}{12}$.

Approximating the derivatives by $f'(\alpha) \approx \frac{f(\alpha+1)-f(\alpha)}{1}$ and $f'(\beta) \approx \frac{f(\beta+1)-f(\beta)}{1}$ we get (7).

<u>Section 13</u>. Euler seeks the sum $\sum_{n=1}^{1000000} \frac{1}{n}$. For convenience we use (7) to introduce the notation

$$(9) \qquad S(a,b) = \int_{a}^{b} f(x)dx + \frac{f(b)+f(a)}{2} + \frac{f(a)-f(a+1)}{12} - \frac{f(b)-f(b+1)}{12}.$$

Euler does not simply calculate $\sum_{n=1}^{1000000} \frac{1}{n} \approx S(1,1000000)$, rather, he first finds the sum of the first ten terms exactly, and then adds the estimate from (9). We repeated his calculations using Mathematica to get

$$\sum_{n=1}^{1000000} \frac{1}{n} \approx \sum_{n=1}^{10} \frac{1}{n} + S(11,1000000)$$

$= 2.9289682539682539683 + 11.463701643751678813$

$= 14.392669897719932781.$

If we sum more terms exactly, the approximation should improve. Consider the computations in the following table

***a* = number of terms in the exact part of the sum**	$\sum_{n=1}^{1000000} \frac{1}{n} \approx \sum_{n=1}^{a} \frac{1}{n} + S(a+1,1000000)$
10	**14.39**2669897719932781
100	**14.392726**642856255069
1000	**14.392726722**782731017
10000	**14.392726722865**640332
100000	14.392726722865723548

<u>Section 14.</u>: In his second example Euler approximates $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since the series is infinite he can use (8).

As before we define

$$S(a,\infty) = \int_a^{\infty} f(x)dx + \frac{7f(a) - f(a+1)}{12}.$$

Euler makes his estimate using a two part calculation, and we repeat it using Mathematica

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx \sum_{n=1}^{10} \frac{1}{n^2} + S(11,\infty)$$

$= 1.5497677311665406904 + 0.0951513238445056626$8

$= 1.6449190550110463530.$

The following table shows the effect of using more terms in the exact sum

***a* = number terms in the exact part of the sum**	$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx \sum_{n=1}^{a} \frac{1}{n} + S(a+1,\infty)$
10	**1.6449**190550110463530
100	**1.64493406**44802745199
1000	**1.644934066847**9777984
10000	**1.64493406684822264**115

The exact value of this sum is $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} =$ 1.6449340668482264365. While Euler became famous for discovering this exact value, we assume this paper was written before his discovery since he never compares his approximate value with it.

References

[1] Knopp, Konrad., *Theory and Application of Infinite Series*, Dover Publications, New York, 1990. (A translation by R. C. H. Young of the 4th German addition of 1947.) ISBN: 0486661652