On
The line of the fastest descent
in whichever resistant medium
By the author
L. Euler.

§. 1. The curves, to be exposed to some certain motion in vacuum, are found without much work. The same curves in a resistant medium, are not only found with much more work; but they also require more skill and caution. It also repeatedly occurs, that many problems in the hypothesis for a resistant medium either utterly obstruct a solution, or only allow a solution for particular cases. The problem about tautochrones, of which I rather very much have doubts about whether hypotheses for friction can be solved, for other frictions, that are more than simple and a ratio of multiplied speeds, is of this kind.

§. 2. It also reaches out here to the problem of the brachistochrone lines or the fastest descents, which is proposed by the celebrated Johannes Bernoulli in his hypothesis for empty geometries, soon after he encountered multiple different solutions, which one may see in the *Actis Lipsiensibus, Transact. Angl. Comment. Parisinis*, and many more other books. However, I proposed the same to be solved problem of the hypothesis for a resistant media first in *Actis Lipf. A. 1726.*[1], and it doesn't leave anyone uncertain, both for its not to be despised elegance, and its unique foresight, which is necessary to use in its solution.

§. 3. Moreover, after I had proposed this problem, the celebrated Hermans deemed it a worthy problem, of which he included the solution in the dissertation on various motions in *Tom. II. Comment.*[2]. He very carefully examined a copy of the matters sufficiently, however, which he investigated in that dissertation, as it was seen by the most perspicacious man, that that problem, which had only been mentioned by few, did not allow a solution, and he carefully examined the found solution. From this it was argued, that the curves, assigned by this problem, are neither convenient, nor do they possess the brachistochrone property. I have also advised the man with good memory about this matter by letter, and I have sent him my solution, that disagreed with his, so that he searched for the cause of the discrepancy, about which he responded me, both that he began

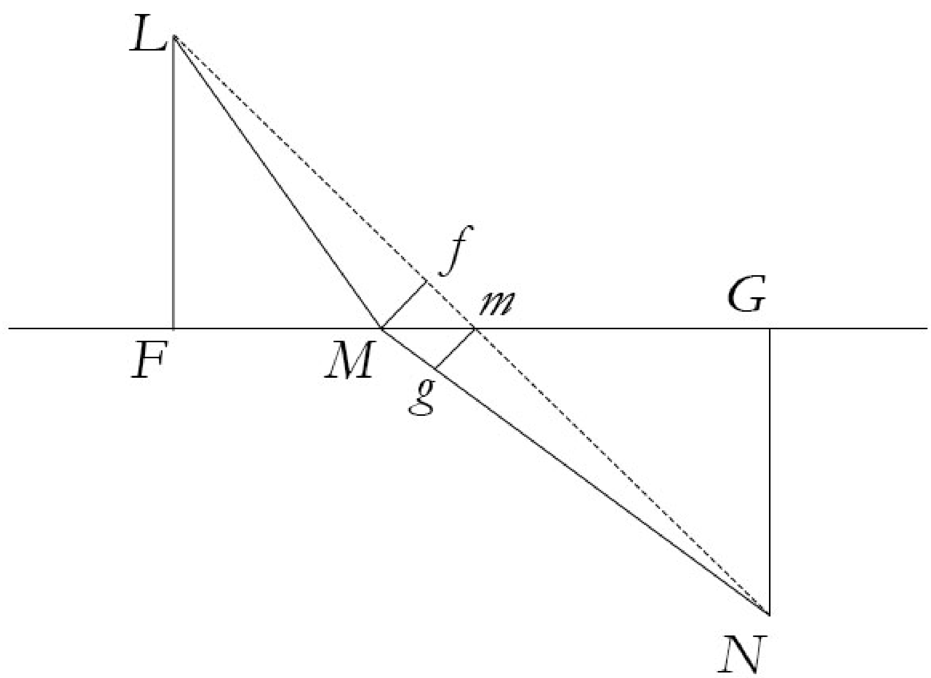
[1]Consult the dissertation E 1 of this edition, vol. II, 6

[2]Consult the commentaries of the academy of sciences of Saint Petersburg 2 (1727), 1729, p. 139, and also the preface of the preceding volume

to doubt his solution, how the troubles, that were about to be resolved, were first, and that he wanted to perfect his amendment for its remarkable soundness, which we would already certainly have had too, if death didn't intervene.

§. 4. Because he himself would do it, if he stayed alive, I therefore don't judge anyone about to reluctantly report, if I will do the same and I will yet correct his solution. I don't only not settle this injustice, but I do also believe myself to be obliged to do it, in order that in the future others, which remarkable men are responsible for the diminishing of his fading fame and reputation, don't accidentally take his fame. I will however show both how much foresight is required to be applied in order to avoid errors of this kind, and each one will forgive those errors, rather easily made by the deceased man, and also not find fault with my intention, to which end I decided to resolve the problem, proposed by me, with a genuine method.

§. 5. A particular sought lemma, to which we have to attend in the solution to his problem, is from the nature of maxima and minima, through which the direction of two contiguous segments of a sought curve is determined, over which a body would move, to be descended in less time, than whichever other segments, posed within the very same boundaries. Such a proposition, demonstrated by Huygens, is considered, and Hermans used it in his solution: as soon will be clear though, it yielded more, than was necessary, and not enough attention was paid to the restriction, that this proposition requires. On that account I will yet bring forth both the Huygenian Lemma and another utility, wider, than in whichever case in an appropriate medium.



§. 6. Let it therefore be necessary to define a point, M, on the straight line FG from which at given ends, L and M, the drawn lines LM and MN are traversed by a descending body in the fastest time: let moreover the speed of the body above FG be m and below be M on FG. After posing this, $\frac{LM}{m} + \frac{MN}{n}$ then consequently has to be minimal, because the time is assigned to LMN by this quantity. Although, the letter is already used, m is to be chosen close to point M, and by drawing Lm and mN, LMN and LmN are to be passed during equal times. Thus, henceforth $\frac{LM}{m} + \frac{MN}{n} = \frac{Lm}{m} + \frac{mN}{n}$ will hold, from which, by having drawn the arcs Mf and mg with L and N as centres, this equation holds, $\frac{mf}{m} = \frac{Mg}{n}$, or this analogy, $mf : Mg = m : n$. Truly, mf relates to Mg as the cosine of the angle LMF relates to the cosine of the angle GMN. Therefore the cosine of the angles, that the two segments should establish with the line FG, are proportional to the speeds, with which those segments are passed. That is the Huygenian lemma, which Hermans used to reach his solution to the problem.

§. 7. Where it's however seen, how widely this lemma is accessible and in which cases it can be evoked, attention must be had for that, what this lemma is used for; I freed every segment below the line FG from their assumed speed n. On which account, if the bodies in all these here segments, wherever the point M

is assumed, don't possess this same speed, this lemma is incorrectly applied, and leads to the wrong solution. That however happens in a resistant medium, and it was used so, although the celebrated Hermans, after he had used it abundantly in this lemma on the discovery of the brachistochrone in vacuum, for resistant media was tempted by this very lemma in a proper way.

§. 8. In vacuum a matter also still must be built upon in this way, in order that the line FG is everywhere perpendicular to the direction of the disturbing force field. Truly, when this, which is required, holds, and the body itself, descending to any point on the line FG from L, always gains an increase in speed, in order that thus single segments, situated within FG, are traversed with equal speed. Therefore, the curve in these cases, naturally in vacuum, will be a brachistochrone, if the speed of the body on whichever segment, would have been proportional to the sines of the angles, that this segment establishes in the direction of the disturbing force field. On this account, the curve of the fastest descent in vacuum will be able to have been discovered with the help of this rule, whichever law of the disturbing force field will exist.

§. 9. From this it's already seen plentily that the given rule for finding a brachistochrone in a resistant medium can't be adapted. For indeed the growths in speed, that the body in descent from L to the points of the line FG acquires, aren't mutually equal, if not only the line FG were perpendicular to the direction of the disturbing force field; but in addition they go down by the incline of the segments that will be passed, as is easily evident from the nature of friction. For these cases it's furthermore necessary for a peculiar lemma to be established, in which the speeds through the lower segments are ordained to be variables, in which the point M on FG is accepted for diverse loci.

§. 10. Then consequently, with like beforehand both the points M and m, being assumed close, and the segments LM, MN and also Lm and mN being drawn, let the speed through the segments LM and Lm is q, the speed through $MN = q + dt$, but on the segment mN it's $q + dt + dd\theta$. The growth of speed, acquired through LM, is of course called dt, and the growth, that is acquired through Lm, is called $dt + dd\theta$. For therefore the time through LMN becomes a minimum, it's necessary that it becomes equal in time to LmN. From this it's obtained that $\frac{LM}{q} + \frac{MN}{q+dt} = \frac{Lm}{q} + \frac{mN}{q+dt+dd\theta}$, but from this it comes forth that $\frac{mf}{q} = \frac{Mg}{q+dt} + \frac{Mndd\theta}{(q+dt)(q+dt+dd\theta)}$ or $\left(q^2 + 2qdt + dt^2 + qdd\theta + dtdd\theta\right) mf = \left(q^2 + qdt + qdd\theta\right) Mg + qmNdd\theta$. It truly holds that $mf = \frac{FM \cdot Mm}{LM}$ and $Mg = \frac{MG \cdot Mm}{LM}$. By having substituted those and neglected what had to be neglected $q\left(\frac{MG}{LM}-\frac{FM}{LM}\right)=\frac{FMdt}{LM}-\frac{LMdd\theta}{Mm}$ arises. Because $dd\theta$ is always in this way determined by Mm, in order that it is of the form $Z\cdot Mm$, it won't involve other quantities, if they won't depend on the point M.

§. 11. If the segments LF and NG are set to be equal, and they're called dx, and also FM becomes dy, $LM=ds$, MG will be $dy+ddy$ and $MN=ds+dds$. With these substitutions, the aforementioned formula passes over to $\frac{qdsddy-qdydds}{ds^2}=\frac{dydt}{ds}-\frac{dsdd\theta}{Mm}$, or because $dsdds=dyddy$, with dx being fixed constant, to $\frac{qdx^2ddy}{ds^3}=\frac{dydt}{ds}-\frac{dsdd\theta}{Mm}$. And this is the lemma, which, instead of the Huygenian, we should use to to find brachistochrones in a resistant medium.

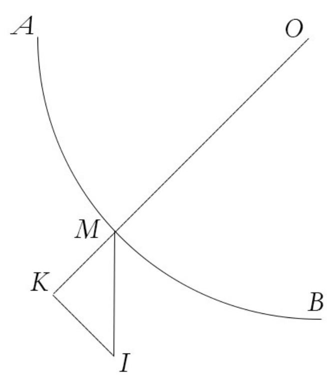
§. 12. Let there now be whichever disturbing force field, and its direction specifically, as before, perpendicular to the line FG. Let the force field $=p$, posing the force of gravity $=1$, be evoked, driving a body, describing the segment LM or Lm. Let it further resist the medium in whichever multiplied ratio of the speeds, of which the exponent is $2n$, and this friction therefore maintains itself, as it is equal to the force of gravity, 1, if the speed of the body had size c. Let the speed of the body in L already be as much, as is acquired by sliding the weight through the height, v. By having posed these things, the force of friction, which retards the motion of the body, that proceeds from L through FG, will be $=\frac{v^n}{c^n}$.

§. 13. The body descends because of the force field p, either through LM or through Lm and receives the same increase in speed, because FG is situated perpendicular to the direction of the force field. The size of v also will take on an increase $=pdx$. Moreover, the friction will thus delay the body, descending through LM, so that in a decrease of size of v is $=\frac{v^n}{c^n}LM$. But if the body is placed to move through Lm, the decrease of the size of v will be $=\frac{v^n}{c^n}Lm$. By which means the speed, with which the segments LM and Lm are passed over, is owed to the size of v; in particular the speed through MN to the size $v+pdx-\frac{v^n}{c^n}LM$ and the speed through mN to the size $v+pdx-\frac{v^n}{c^n}Lm$.

§. 14. After having compared this with our lemma, we have $q=\sqrt{v}$, $q+dt=\sqrt{v+pdx-\frac{v^n}{c^n}LM}=\sqrt{v}+\frac{pdx-\frac{v^n}{c^n}LM}{2\sqrt{v}}$, from which $dt=\frac{pdx-\frac{v^n}{c^n}LM}{2\sqrt{v}}$. And also $q+dt+dd\theta=\sqrt{v+pdx-\frac{v^n}{c^n}Lm}=\sqrt{v}+\frac{pdx-\frac{v^n}{c^n}Lm}{2\sqrt{v}}$. From this thus becomes $dd\theta=\frac{v^n(LM-Lm)}{2c^n\sqrt{v}}=-\frac{v^nFM\cdot Mm}{2c^nLM\sqrt{v}}$, consequently $\frac{dd\theta}{Mm}=-\frac{v^ndy}{2c^nds\sqrt{v}}$. The following equation will then consequently come up from multiplying everything by $2\sqrt{v}$, $\frac{2vdx^2ddy}{ds^3}=\frac{pdxdy}{ds}-\frac{v^ndy}{c^n}+\frac{v^ndy}{c^n}$ or $2vdxddy=pdyds^2$. A brachistochrone curve should therefore have this here property, which is $v = \frac{pdyds^2}{2dxddy}$, from which it will be easy to discover it.

§. 15. Because the boundaries, in which the friction $\frac{v^n}{c^n}$ begins, mutually cancel each other, this most extensive lemma is attainable and this lemma can be adapted to whichever friction, without any change. Then this is the universal property of all brachistochrones, both in vacuum, as in whichever resistant medium. To the end that it's possible to remember that lemma easier by memory, though, we induce it in another form.

§. 16. If the found equation, $2vdxddy = pdyds^2$ is divided by ds^3, it transforms into this $\frac{2vdxddy}{ds^3} = \frac{pdy}{ds}$, in which solution the disturbing force field p gives rise to the risen perpendicular force $\frac{pdy}{ds}$. In the other portion, $\frac{2vdxddy}{ds^3}$, $-\frac{ds^3}{dxddy}$ means the radius of curvature of the curve LMN, following the region extended from F. However, because the curve is convex towards F, the radius of the origin will be directed to the opposing intersection G, and for this reason it has a negative value. Its length will thus be $\frac{ds^3}{dxddy}$. Therefore, by posing the radius of the origin $= r$, and the perpendicular force $= N$ this equation will be had $\frac{2v}{r} = N$. Moreover, $\frac{2v}{r}$ marks the centrifugal force, by which a body, to which extent it can't advance in a straight line, pursues a curve, on which it moves. On account of this matter every brachistochrone has that property, so that the perpendicular force is equal to the centrifugal force.

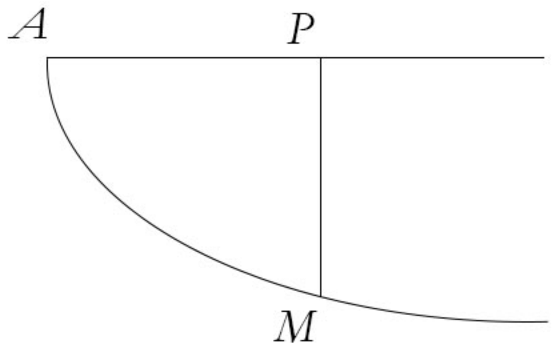


§. 17. It must however be noted that every body, that moves either in vacuum or in a resistant medium because of any disturbing force over the concave part of a certain curve AMB, that the curve pursues by two forces, naturally the force, perpendicular to the original disturbing force field, and its centrifugal force. Let there be a force field along MI, disturbing a body on M; this is usually decomposed in two others, MK and KI, of which the direction of MK is perpendicular to the curve and therefore this force is called perpendicular, and of the other, KI, the direction along the tangent of the curve and is called tangential. Therefore it's evident that the body only presses on the curve with the perpendicular component. In addition, the curve AMB is pressed in M by the centrifugal force, which maintains itself through the the gravitational force, so that its size generates the speed v, towards half the radius of curvature, MO.

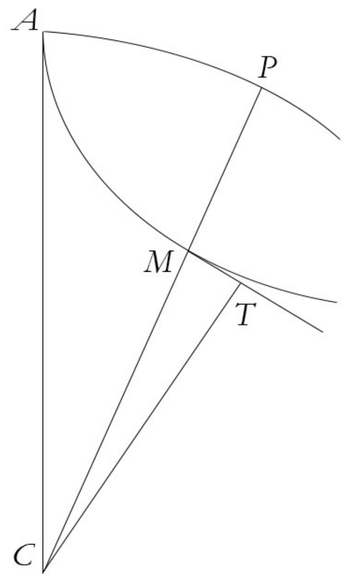
§. 18. If thus the curve AMB were in this way compared, either in vacuum or in whichever resistant medium, as both forces on the body, that descends over the curve, by which the curve is naturally pressed perpendicular and centrifugal, will be mutually equal, the curve will always be a brachistochrone, or the body descends over it from A to M in less time, than passing over whichever other line through A and M. Therefore this equality between the perpendicular force and centrifugal force is the true and universal law of all brachistochrone curves, and its benefit in whichever hypothesis for both a disturbing force field and friction will be that brachistochrone curves are easily determined.

§. 19. Because according to the Huygenian Theorem the speed in vacuum has to be proportional to the sine of the angle, that the curve establishes with the direction of the force field, i.e. to $\frac{MK}{MI}$ itself, $\frac{MK^2}{MI^2 \cdot MO}$ will be proportional to MK itself or $\frac{MK}{MI}$ to $MI \cdot MO$ itself. Therefore all brachistochrones in vacuum possess this property, that the sine of the angle, which the direction of the force field makes with the curve, is everywhere proportional to the radius of curvature and jointly the disturbing force field. Hence with help of this rule without determining the speed all brachistochrones in vacuum can be found.

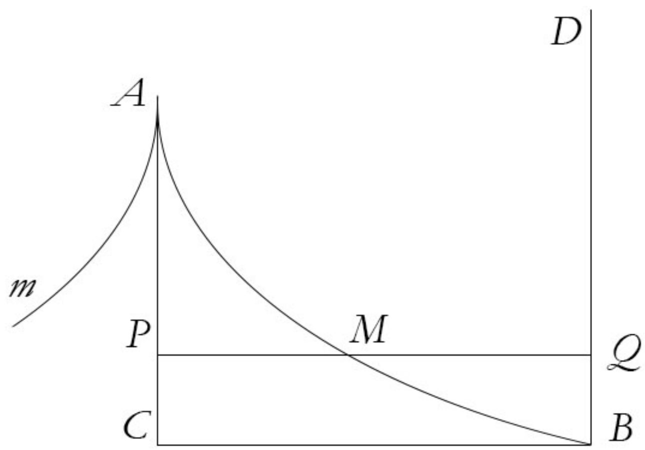
§. 20. The beginning of the curve is however always in A, on which all descents should happen from rest, the locus, on which the tangent of the curve coincides with the direction of the force field. Let in this locus in vacuum the speed itself of the body be equal to 0, because the angle of the curve, with the same direction as the force field, also becomes 0. In a resistant medium, however, that beginning of motion doesn't differ from vacuum, and on account of this matter in this case too, the tangent of the beginning of the curve should coincide with the direction of the force field. Truly, this reasoning must be held in the addition of constant quantities, when we integrate the differentio-differential equation of the brachistochrone, and we should work it out, so that the curve has a given beginning and moves through a given point.



§. **21.** Let's illustrate the rule, given in §. 19. for finding brachistochrones in vacuum with examples, and let the disturbing force field be constant $= g$, its direction vertical along PM. Let the sought brachistochrone truly be AM and let the abscissae on the horizontal line, passing through the beginning of the curve, be AP. These things being done, let $AP = y$, $PM = x$, $AM = s$ and the sine of the angle, that PM makes with the curve, be $= \frac{dy}{ds}$, and the radius of curvature $= \frac{ds^3}{dxddy}$, posing dx a constant, which should be proportional to that $\frac{dy}{ds}$ due to the constant force field. Therefore $\frac{ds^3}{dxddy}$ becomes $\frac{ady}{ds}$ or because $ddy = \frac{dsdds}{dy}$ in this way $ds^3 = adxdds$. By dividing by ds^2 and integrating, $s = C - \frac{adx}{ds}$ comes forth. Because by making $s = 0$, dx should become ds, C will be a and for that reason $sds = ads - adx$, which further yields $s^2 = 2as - 2ax$, the equation for a cycloid, after integrating, just as agreed upon.



§. 22. Let onwards C be the center of the forces, that attracts in whichever multiplied ratio of distances, of which the exponent is m. Let the curve AM be a brachistochrone for a body that is moving in vacuum. Let be declared that $CA = a$, $CM = y$ and that the perpendicular CT on the tangent MT, drawn from C, $= z$. The force in M along the line MC, pulling the body, will thus be as y^m, the sine of the angle of the curve with that direction will be $= \frac{z}{y}$, and the radius of curvature will be $-\frac{ydy}{dz}$. Therefore, since the rule must be enforced, $\frac{z}{y}$ will be as $\frac{y^{m+1}dy}{dz}$ or $Azdz = y^{m+2}dy$, of which the integral is $C + Az^2 = y^{m+3}$. Because, if $y = a$, $z = 0$ will hold, C will be a^{m+3}, and consequently $Az^2 = a^{m+3} - y^{m+3}$, with A arbitrarily being taken negative. And this equation involves all brachistochrones, which exists around a center of forces.



§. 23. Let's however go back to a medium that resists in whichever multiplied ratio of the speeds, of which the exponent is $2n$. Let the disturbing force field be posed constant, specifically $= g$, and having a vertical direction everywhere, parallel to that AP. Let AMB be the to be found curve of the fastest descent, on which we pose $AP = x$, $PM = y$ and $AM = s$. Let the speed on M further be owed to the size v, whereby the friction in M will be $= \frac{v^n}{c^n}$. Hence from the disturbance of the force field and the effect of the friction will simultaneously, $dv = gdx - \frac{v^n ds}{c^n}$ will be had. The brachistochronism truly yields $2vdxddy = gdyds^2$, having posed dx constant (§. 14.). From these connected equations, the equation for the sought brachistochrone curve will advance, after getting rid of the letter v.

§. 24. Because dx will be constant, $ddy = \frac{dsdds}{dy}$ and therefore $v = \frac{gdsdy^2}{2dxdds}$. Thus $dv = \frac{gdy^2dds^2+2gds^2dds^2-gdsdy^2d^3s}{2dxdds^2}$. By substituting these values in the equation $dv = gdx - \frac{v^n ds}{c^n}$, $\frac{gdsdy^2d^3s-3gdy^2dds^2}{2dxdds^2} = \frac{g^n nds^{n+1}dy^{2n}}{2^n c^n dx^n dds^n}$ will be had, or $dsd^3s - 3dds^2 = \frac{g^{n-1}ds^{n+1}dy^{2n-2}}{2^{n-1}c^n dx^{n-1}dds^{n-2}}$. This equation, if the resistant medium is an exotic infinite or it transforms into vacuum, in which case $c = \infty$ holds, transforms into $dsd^3s = 3dds^2$, of which the integral is $adxdds = ds^3$. We showed this in §. 21., which constitutes a cycloid.

§. 25. However, to construct a general equation, I pose $ds = pdx$, so that $dds = dpdx$ holds and $d^3s = dxddp$. Hence $dy = dx\sqrt{p^2-1}$ and $v = \frac{gpdx(p^2-1)}{2dp}$ will hold. However, that equation will transform into this $pddp - 3dp^2 = \frac{g^{n-1}p^{n+1}dx^n(p^2-1)^{n-1}}{2^{n-1}c^n dp^{n-2}}$. Further $dx = qdp$ is posed, and $ddp = -\frac{dpdq}{q}$

will hold. By substituting this, $-\frac{pdq-3qdp}{q^{n+1}} = \frac{g^{n-1}p^{n+1}\left(p^2-1\right)^{n-1}dp}{2^{n-1}c^n}$ will come forth. This equation is multiplied by np^{-3n-1}; by having done this,

$$-\frac{np^{-3n}dq - 3np^{-3n-1}qdp}{q^{n-1}} = \frac{ng^{n-1}p^{-2n}\left(p^2-1\right)^{n-1}dp}{2^{n-1}c^n}$$

will be had. Of this, the integral is

$$\frac{2^{n-1}c^n}{ng^{n-1}p^{3n}q^n} = \int \frac{\left(p^2-1\right)^{n-1}dp}{p^{2n}}$$

Let for the grace of brevity be $\frac{ng^{n-1}}{2^{n-1}c^n}\int\frac{\left(p^2-1\right)^{n-1}}{p^{2n}} = P^{-n}$, which quantity, if the integration doesn't succeed, can always be showed by allowing quadratures. Having posed this, $p^3q = P$ will thus hold, and on account of $q = \frac{dx}{dp}$, dx becomes $\frac{Pdp}{p^3}$. Consequently $x = \int\frac{Pdp}{p^3}$, $s = \int\frac{Pdp}{p^2}$ and $y = \int\frac{Pdp\sqrt{p^2-1}}{p^3}$. Therefore the brachistochrone hypothesis in whichever resistant medium will be possible to be constructed in this way.

§. 26. If the friction of the medium is as a square of the speed, $n = 1$ will hold, and for that reason $P^{-1} = \frac{1}{c}\int\frac{dp}{p^2} = \frac{1}{ac} - \frac{1}{cp} = \frac{p-a}{acp}$. Hence P becomes $\frac{acp}{p-a}$; and $p^3q = \frac{ac}{p-a} = \frac{p^2dx}{dp}$, or $dx = \frac{acdp}{p^2(p-a)}$, of which the integral is $x = b + \frac{c}{p} + \frac{c}{a}l\frac{p-a}{p} = b + \frac{cdx}{ds} + \frac{c}{a}l\frac{ds-adx}{ds}$ [l denotes the natural logarithm]. In this equation, because, after making $x = 0$, ds should be dx, becomes $b = -c - \frac{c}{a}l\left(1-a\right)$. For the sought curve this equation will thus be had $x = \frac{c(dx-ds)}{ds} + \frac{c}{a}l\frac{ds-adx}{ds-ads}$. Or if the equation is wanted free of logarithms; this differentio-differential, $acdxdds = ds^3 - adxds^2$, having posed dx constant. After this is arranged in another way, it transforms into $\frac{acdxdds}{ds^2} = ds - adx$, of which the integral is $s - ax = ac - \frac{acdx}{ds}$ or $sds - axds = acds - acdx$. Integrating this yields

$$s = cl\frac{s - ax - ac + c}{c - ac}$$

or

$$e^{\frac{s}{c}}\left(c - ac\right) = s - ax + c - ac$$

The infimum point of this curve, B, will be there, where $s = a\left(x + c\right)$ holds. Therefore in this case AB will be $cl\frac{1}{1-a}$ and $AC = \frac{c}{a}l\frac{1}{1-a} - c$.

§. 27. If however the Huygenian Theorem was used, just as in that suitable case, we would thence immediately have had this equation $v = \frac{ady^2}{ds^2}$. Hence $dv = \frac{2adx^2ddy}{ds^3} = gdx - \frac{a^n dy^{2n}}{c^n ds^{2n-1}}$ or $2adx^2ddy = gdxds^3 - \frac{a^n dy^{2n}}{c^n ds^{2n-4}}$. This, after having posed $ds = pdx$, transforms into $\frac{2apdp}{\sqrt{p^2-1}} = gpx^2dx - \frac{a^n dx\left(p^2-1\right)^n}{c^n p^{2n-2}}$, which is already separated by itself, and therefore it can be constructed. If $n = 1$ is posed, so that the brachistochrone for a medium, resistant in a ratio of squared speeds, appears , $2acdx^2ddy = cgdxds^3 - ady^2ds^2$ will hold or $2acdx^2dds = cgdxdyds^2 - ady^3ds$. This equation, if it furthermore rests on a simpler lemma, is yet much more ordered and intricate, than our found brachistochrone; it is usually often a criterium of truth by itself, especially if the more painstaking calculus deduces it.

§. 28. Where it moreover appears, along what figure our brachistochrone in a resistant, with the square of the speed, medium must be had, we sum this equation $e^{\frac{s}{c}}(c - ac) = s - ax + c - ac$. This, after converting $e^{\frac{s}{c}}$ into a series, transforms into

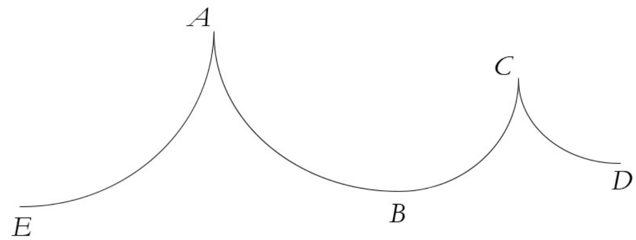
$$(c-ac)\left(1+\frac{s}{c}+\frac{s^2}{1\cdot 2\cdot c^2}+\frac{s^3}{1\cdot 2\cdot 3\cdot c^3}+\frac{s^4}{1\cdot 2\cdot 3\cdot 4\cdot c^4}+etc.\right) = s-ax+c-ac$$

from which, after posing $\frac{1-a}{a} = k$, this equation is discovered

$$x = s - \frac{ks^2}{1\cdot 2\cdot c} - \frac{ks^3}{1\cdot 2\cdot 3\cdot c^2} - \frac{ks^4}{1\cdot 2\cdot 3\cdot 4\cdot c^3} - etc.$$

It's thus perceived that k should necessarily be a positive number, because otherwise x is made $> s$, which can't happen; thus $a = \frac{1}{1+k}$ will hold. From this series, because it exceedingly easily converges for whichever value of that s, the answer for that x will be found. Besides, it is understood that this curve beyond A is continued in Am, which is similar to that AM.

§. 29. In what way the curve beyond B verily is extended, I investigate in this reasoning. By having drawn a vertical axis BD from B and applying MQ in it, let $BQ = PC = u$, and the arc $BM = t$. Having posed this, $s = cl\frac{1}{1-a} - t$ will hold, and $x = \frac{c}{a}l\frac{1}{1-a} - c - u$, by substituting which, this equation emerges $ce^{-\frac{t}{c}} = au - t + c$ or this differential, $tdt - audt = acdu$. Through the series $au = \frac{t^2}{1\cdot 2\cdot c} - \frac{t^3}{1\cdot 2\cdot 3\cdot c^2} + \frac{t^4}{1\cdot 2\cdot 3\cdot 4\cdot c^3} - etc.$ will verily be had. This equation directly corresponds with that, which I invented in *A. 1729.* for the tautochrone in that rising hypothesis on friction. Therefore another relation of the permitted curve beyond the axis BD will be the tautochrone, pertaining to the descent.



Thus the brachistochrone curve will have the form of this kind $EABCD$, provided with infinite cusps $A, Cetc.$, of which some are higher than A, some are lower than C. The branches from each of two parts of each one cusp are mutually equal and similar. The elevation of higher cusps is $\frac{c}{a}l\frac{1}{1-a} - c$, and of lower ones in particular $c - \frac{c}{a}l(1+a)$. Those higher branches specifically AB or AE are $= cl\frac{1}{1-a}$ and the length of the more depressed CB, CD is $= cl(1+a)$.That convenience among the tautochrone and brachistochrone is certainly deserved to be inspected beyond vacuum and too especially in this hypothesis on friction, and the investigation rests, what similar analogy perhaps maintains the locus in the remaining hypotheses of friction? That, which renders the most difficult invention of the tautochrones the easiest.