Determination of the Orbit Around the Sun (an English translation of *Orbitae Solaris Determinatio*)

Leonhard Euler

All footnotes are comments by the translator, Patrick Headley[1].

1. The power of the method described in the previous article *(see footnote*[2]*)* is that it will be easy to describe the annual motion of the Earth around the Sun, or, equivalently, to determine the solar orbit, given three positions of the Sun as seen from Earth, together with the times between them. In order to define the solar orbit as accurately as possible, it is necessary that we choose from astronomical observations three positions of the Sun that allow for a minimum of uncertainty. Likewise, as it is necessary to derive the location of the Sun in the ecliptic, or the Sun's path, from altitudes at the meridian, it leads us to employ the observations of this kind that are made near the equinoxes, since the smallest errors in longitude are made at these times. Therefore, I reject observations made near the solstices, for, at these times, even the most accurate observations of the position of the Sun can scarcely be certain to within several minutes.

2. Therefore, to this end I searched through Flamsteed's celestial history *(see footnote*[3]*)*, and I made an effort to compute the Sun's position in the ecliptic from its altitude at the meridian. However, not only has the effect of refraction not been removed from these observations, but I am also of the opinion that they are not all equally well measured; I hardly dared to draw any conclusions about the

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[2]The article is *De Motu Planetarum et Orbitarum Determinatione*, which is E037 in the Euler Archive.

[3]This is the *Historia Coelestis Britannica* (1725), a collection of the observations of John Flamsteed (1646-1719), the first director of the Royal Observatory at Greenwich, England.

solar orbit from them. When I read through the preface to the third volume of this work, I discovered that Flamsteed had devoted it to the same problem, but I can little recommend the method by which he tried to determine the apogee and eccentricity of the orbit. For first he claims that his tables of the mean motion of the Sun are the most accurate, which all of those working in this area have had to doubt; then he changes the position of the apogee for a short while without taking the eccentricity into account, in order that one or another observation be satisfied. It is easily perceived that this method can scarcely be able to contribute anything.

3. Yet, at the same time, Flamsteed declared the observations he made use of in this work to be the most certain and exact, and I do not think to question their superiority, since without doubt he selected those of which he was most certain. I do not doubt that these same observations can be used for the purpose of determining the solar orbit through the means of my method, and from these I will choose such observations that were made near the equinoxes, since these have the most certainty. Therefore, I decided to employ the following three observations in this work, made in the year 1690 *(see footnote[4])*. On March 7 the position of the Sun at midday was ♓, $27°, 21', 47''$. Next, at midday on the 14th of the same month the Sun, was at ♈, $4°, 17', 18''$. And, for the third, on September 15, the Sun was at ♎, $2°, 45', 37''$ *(see footnote[5])*. These positions were derived by Flamsteed himself from precise observations of the Sun's altitude at the meridian.

4. However, the Sun was located at these positions at the instant of true noon, when it crossed the meridian. Therefore, in order to correctly derive the mean anomalies from these times, it is necessary to correct them with the assistance of the equation of time. The results will be as follows.

	Mean Time	*Position of Sun*
March 7	$12h., 8', 24''$	$11S, 27°, 21', 27''$
March 14	$12h., 6', 15''$	$0S, 4°, 17', 18''$
Sept. 15	$11h., 51', 27''$	$6S, 2°, 45', 37''$

Therefore, I take the first of these observations as a starting point, and I let z be its true anomaly, and x be its mean anomaly. Next, I will let $1 : v$ denote the ratio of the mean distance to the eccentricity, so these three quantities z, x, and v

[4]Flamsteed's dates are according to the Julian calendar, still in use in England at this time.

[5]Euler uses the traditional division of the ecliptic into 12 equal arcs of 30° represented by the signs of the zodiac. The symbols here are for the signs Pisces, Aries, and Libra respectively.

may be determined from the three observations. The true anomaly of the second observation is set equal to $z+f$, and the mean to $x+m$; the true anomaly of the third is set to $z+g$, and the mean to $x+n$. The quantities f, g, m, and n are immediately determined from the observations.

5. The difference in time between the first and second observations is thus $6d, 23h, 57', 51''$, to which time the corresponding mean motion is $6°, 53', 52''$. But, as the equinoxes have precessed by nearly $1''$ between them, the difference between the mean anomalies of the first and second observations is $m = 6°, 53', 51''$. The difference between the solar positions is $6°, 55', 31''$, *(see footnote[6])* which, similarly adjusted by a second due to precession of the equinoxes, gives $f = 6°, 55', 30''$ between the true anomalies of the first and second observations. Next, the difference in time between the first and third observations is $191d, 23h, 43', 3''$, to which the corresponding mean motion is $189°, 12', 0''$; because of the motion of the equinoxes $26''$ is subtracted, leaving $n = 189°, 11', 34''$ as the difference between the mean anomalies of the first and third observations. Finally, $26''$ is subtracted from the difference in the solar positions between the first and third observations, giving a difference of $g = 185°, 23', 24''$ between the true anomalies of the first and third observations.

6. With these provided, we next explain, so that everything is easily understood, what is required in order to be able to determine the value of a certain angle P, which is the eccentric anomaly. In my previous article I showed that, approximately *(see footnote[7])*,

$$\tan P = \frac{\sin\frac{m+f}{2} - \left(\frac{m-f}{n-g}\right)\sin\frac{n+g}{2}}{\operatorname{versin}\frac{m+f}{2} - \left(\frac{m-f}{n-g}\right)\operatorname{versin}\frac{n+g}{2}},$$

assuming that the whole sine *(see footnote[8])* is 1. In order for this expression to be calculated, I carry out the work in the following way.

[6] This doesn't work out, but perhaps the number of seconds in the March 7 position is 47, not 27.

[7] The *versine*, or versed sine, appearing in this formula is given by $\operatorname{versin}\theta = 1 - \cos\theta$. Euler actually uses $\int$ for sin, and $\int v.$ for versin throughout this article.

[8] The whole sine, or *sinus totus*, is a value chosen as a radius, so that Euler's *sine* is what would now be considered the radius multiplied by the sine. The tables that Euler used had varying choices for the whole sine. The logarithms of sine values that Euler uses in this article assume a whole sine of 10^{10}, but the trigonometric values assume a whole sine of 10^7. As a result, Euler's use of

$$m = 6°, 53', 51'' \qquad f = 6°, 55', 30''$$
$$n = 189°, 11', 34'' \quad g = 185°, 23', 24''$$

From these will be *(see footnote[9])*

$$m - f = -1', 39'' = -99''$$

$$n - g = 8°, 48', 10'' = 13690''$$

and also
$$-\left(\frac{m-f}{n-g}\right) = 0.0072315.$$

Moreover,
$$\frac{m+f}{2} = 6°, 54', 40''$$

and
$$\frac{n+g}{2} = 187°, 17', 29''$$

Now we have
$$\sin\frac{m+f}{2} = 1203393$$

and also
$$\sin\frac{n+g}{2} = -1269155$$

which multiplied by $-\left(\frac{m-f}{n-g}\right) = 0.0072315$ gives -9178. Therefore, the numerator of the fraction equal to which $\tan P$ is equal is 1194115. Moreover, versin $\frac{m+f}{2} = 72661$ and versin $\frac{n+g}{2} = 19919130$, which, when multiplied by $-\left(\frac{m-f}{n-g}\right) = 0.0072315$ gives $-\left(\frac{m-f}{n-g}\right)$ versin $\frac{n+g}{2} = 144045$. Thus, the denominator is 216706. The result from the numerator in the whole sine is now divided by the denominator, producing $\tan P = 55103000$. The angle is thus found to be $P = 79°, 42', 51''$, but this value is only approximate, and should be corrected in the following way.

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the decimal point (actually a comma in the original text) seems somewhat erratic to the modern reader. For more information on the *sinus totus* see González-Velasco, Enrique A. *Journey through Mathematics*. New York, Springer, 2011.

[9]The error in the expression for $n - g$ is in the original text. However, the number of seconds is correct.

7. The letters Q and R can be found from the value of P; indeed, with $P = 79°, 42', 51''$, it will be that $Q = P + \frac{m+f}{2} = 86°, 37', 31''$, and also $R = P + \frac{g+n}{2} = 267°, 0', 20''$. From these values v will be $\frac{\frac{m-f}{2}}{\sin Q - \sin P} = \frac{\frac{n-g}{2}}{\sin R - \sin P}$, from which it is clear that the value of v will be negative, indicating that the distance from the first observation to the perigee, and not the apogee, is to be found. Therefore, $\log -v = \log \frac{f-m}{2} - \log(\sin Q - \sin P)$, in which expression it should be noted that, for the sines to be compatible with the angles, 4.6855704 *(see footnote*[10]*)* should be subtracted from the logarithm of the sines, in order that logarithms of seconds are obtained. For the same purpose, $\frac{f-m}{2}$ will be 49.5 in seconds, and $\log \frac{f-m}{2} = 1.6946052$. Now, in truth,

$$\begin{aligned}
\sin Q &= 9982658 \\
\text{and } \sin P &= \underline{9839292} \\
\text{and so } \sin Q - \sin P &= 143366 \\
\text{and also } \log(\sin Q - \sin P) &= 8.1564482 \\
\text{from which subtr.} &= \underline{4.6855704} \\
\text{leaving} & 3.4708778 \\
\text{subtr. from } \log \tfrac{f-m}{2} &= \underline{1.6946052} \\
\log -v &= -1.7762726 \\
& \text{or } v = \tfrac{-99}{5915}
\end{aligned}$$

This value, like the rest, needs subsequent correction.

8. To apply this correction, the values of the letters M and N from the following formulas are required:

$$M = \frac{f+m}{2} + \frac{v^2}{8}\sin 2P - \frac{v^2}{8}\sin 2Q$$
$$N = \frac{g+n}{2} + \frac{v^2}{8}\sin 2P - \frac{v^2}{8}\sin 2R$$

where 4.6855704 should be subtracted from the logarithms of the sines. Because each sine is multiplied by $\frac{v^2}{8}$, for which the logarithm is -4.4556252, the logarithm 9.1412056 should be subtracted from the logarithms of the sines, and the number corresponding to the resulting logarithm will give the number of seconds.

[10] The logarithms are base 10. Recall that the whole sine is 10^{10}, and that 10^{10} seconds is approximately $10^{4.6855704}$ radians. Euler used the more accurate value 4.6855749 in the previous article.

However, for the sake of brevity, it is not necessary to calculate angles $2P$, $2Q$, and $2R$ to the second: such accuracy is in fact superfluous. With this premised we have the following:

$$\begin{aligned}\sin 2P &= \sin 159°, 26' = \sin 20°, 34' \\ \sin 2Q &= \sin 173°, 15' = \sin 6°, 45' \\ \sin 2R &= \sin 534°, 1' = \sin 5°, 59'\end{aligned}$$

From this will be

$$\begin{array}{rcr}\log \sin 2P & = & 9.5456745 \\ \text{subtr.} & & 9.1412056 \\ \hline & & 0.4044689\end{array}$$

Therefore $\frac{v^2}{8} \sin 2P = 2\frac{1}{2}''$.
Similarly,

$$\begin{array}{rcr}\log \sin 2Q & = & 9.0701761 \\ \text{subtr.} & & 9.1412056 \\ \hline & & (-1).9289705\end{array}$$

Therefore, $\frac{v^2}{8} \sin 2Q = 0.85''$, and $\frac{v^2}{8} \sin 2R = 0.75''$. From these the result is

$$\begin{aligned}M &= 6°, 54', 42'' \quad \text{and} \\ N &= 187°, 17', 31''.\end{aligned}$$

9. Since the values of M and N differ by so little from $\frac{m+f}{2}$ and $\frac{n+g}{2}$, the correction produced for P and v will be imperceptible. Nevertheless, I will do the calcuations in order to show how the rules I have related are carried out. I have shown that $\tan P = \frac{\sin M - \left(\frac{m-M}{n-N}\right) \sin N}{\text{versin } M - \left(\frac{m-M}{n-N}\right) \text{versin } N}$, assuming the whole sine = 1. To this end I work to identify angle P in the following way *(see footnote*[11]*)*.

$$\begin{array}{rrcrcr} & m - M & = & -50.69' & & \\ & n - N & = & 1°, 54', 3.21'' & = & 6843.21'' \\ \text{thus} & -\left(\frac{m-M}{n-N}\right) & = & 0.0074073 & & \end{array}$$

[11] The value of $m - M$ is incorrectly given as minutes instead of seconds.

Moreover

$$\begin{aligned} \sin M &= 1203003 \\ \sin N &= -1296242 \\ \text{thus } -\left(\tfrac{m-M}{n-N}\right)\sin N &= -9400 \\ \text{thus the numerator} &= 1193603 \end{aligned}$$

Next

$$\begin{aligned} \operatorname{versin} M &= 72670 \\ \operatorname{versin} N &= 19919123 \end{aligned}$$

So it will be that

$$\begin{aligned} -\left(\tfrac{m-M}{n-N}\right)\operatorname{versin} N &= 149306 \quad \text{and thus} \\ \text{denominator} &= 221976 \end{aligned}$$

Therefore, it is found that

$$\begin{aligned} \tan P &= 53771000 \\ \text{and thus} \qquad P &= 79^\circ, 27', 54''. \end{aligned}$$

10. Therefore, assuming this value to be the true value of P, it will be that $Q = P + M = 86^\circ, 22', 36''$ and $R = P + N = 266^\circ, 45', 25''$, and, from these, the true value of v itself will be $\frac{m-M}{\sin Q - \sin P} = \frac{n-N}{\sin R - \sin P}$, which should be worked out as before. The equation $v = \frac{n-N}{\sin R - \sin P}$ now produces $\log -v = -1.7761733$, which is the true value for v, and the ratio of the mean distance to the eccentricity is as 100000 to 1674.

11. From these both anomalies of the solar position at the first observation can now be determined, but it should be noted that the computed anomalies are measured from the perigee on account of the negative value of v. The first observed solar position is $11S, 27^\circ, 21', 27''$, the mean anomaly is $x = P + v \sin P$, and the true anomaly is $z = P - v \sin P + \frac{v^2}{4} \sin 2P +$ etc.; to find these values we have

$$\begin{aligned}
P &= 79^\circ, 27, 54''\\
\log\sin P &= 9.9926192\\
\text{subtr.} & \quad 4.6855704\\
& \quad 5.3070488\\
\log -v &= -1.7761733\\
\log -v\sin P &= 3.5308755\\
\text{thus } -v\sin P &= 3395'' = 56', 35''\\
\log\sin 2P &= 9.5556433\\
\text{subtr.} & \quad 4.6855704\\
& \quad 4.8700729\\
\log\frac{v^2}{4} &= -4.1655562\\
\log\frac{v^2}{4}\sin 2P &= 0.7045167\\
\text{thus} \frac{v^2}{4}\sin 2P &= 5''.
\end{aligned}$$

Therefore, from these

$$\begin{aligned}
x &= 78^\circ, 31', 19'' = 2S, 18^\circ, 31', 19''\\
z &= 80^\circ, 24', 34'' = 2S, 20^\circ, 24', 34''
\end{aligned}$$

12. Therefore,when the Sun was at $11S, 27^\circ, 21', 27''$ on the ecliptic on March 7, 1690 at $12h., 8', 24''$, this was the mean motion plus $1^\circ, 53', 15''$ added through the equation of time. For which reason the mean motion of the Sun at that time was $11S, 25^\circ, 28', 12''$, and on that same March 7, 1690, the mean motion of the Sun was $11S, 25^\circ, 27', 52''$ at approximately noon mean time. Therefore, at the end of 1689, or beginning of 1690, the mean motion of the Sun was $9S, 20^\circ, 24', 42''$, which, if compared with the tables of solar mean motion in Harris' Lexicon *(see footnote*[12]*)*, will be found to be $40''$ greater, and for this reason these tables should be adjusted higher by $40''$ to accord with the Greenwich observatory. Therefore, it will be

[12]The tables to which Euler is referring are in Volume II of *Lexicon Technicum*, by John Harris (c.1666-1719). The second edition from 1723 can be viewed online at Google Books. The relevant tables are on the third page of tables in the Astronomy section.

Year	mean motion of the Sun
1701	$9S, 20^\circ, 44', 30''$
1721	$9S, 20^\circ, 53', 34''$
1741	$9S, 21^\circ, 2', 38''$
1761	$9S, 21^\circ, 11', 42''$
1781	$9S, 21^\circ, 20', 46''$
1801	$9S, 21^\circ, 29', 51''$

In the table for intermediate years nothing should be changed.

13.

From the position of the Sun	$11S, 27^\circ, 21', 27''$
the true anomaly is subtracted.	$2S, 20^\circ, 24', 34''$
The perigee of the solar orbit is produced.	$9S, 6^\circ, 56', 53''$

For which reason the apogee of the solar orbit was

on March 7, 1690 at	$3S, 6^\circ, 56', 53''$
and at the start of 1690 at	$3S, 6^\circ, 56', 44''$
and at the start of 1701 at	$3S, 7^\circ, 5', 54''$

Therefore, $34', 16''$ should always be subtracted from the position of the apogee of the Sun in the cited astronomical tables; the apogee is too far advanced in these tables by more than half a degree.

14. We find the logarithm of the ratio of the eccentricity to the mean distance to be -1.7761733; therefore, the mean distance is to the eccentricity as 100000 is to 1674, or as 5973 is to 100. If, now, by the rule presented in the next article *(see footnote[13])*, 5.6154596 is added to this logarithm, it will produce 3.8392863, which corresponds to $6907''$ for the maximum of the equation *(see footnote[14])*, and therefore the maximum is $1^\circ, 55', 7''$.

15. It remains to determine the mean anomaly to which the maximum of the equation corresponds; it can be done by my rule in the following way.

[13]This is *Solutio Problematum Quorundam Astronomicorum*, which is E039 in the Euler Archive.

[14]Euler is referring to *the equation of the center*, which is the difference between the true anomaly and the mean anomaly.

$$
\begin{array}{rcr}
\text{To} \log \text{ whole sine} & & 10.0000000 \\
\text{is added } \log \frac{v^3}{4} & = & -5.9305799 \\
\hline
 & & 4.0694200 \\
\text{which is the log sine of angle} & & 14''' \\
\text{and therefore} & & q = 14''' \\
\text{Next to} & & 5.3144295 \\
\text{is added } \log v & = & -1.7761733 \\
\hline
 & & 3.5382562
\end{array}
$$

which corresponds *(see footnote*[15]*)* to $3454''$ or $57', 34''$. Therefore, the mean anomaly corresponding to the maximum of the equation will be $90°, 57', 34''$, and the corresponding true anomaly will be $89°, 2', 27''$.

[15]Note that the unit for q is *thirds*, or sixtieths of seconds, denoted by a triple prime. Thus, while q is supposed to be added to $3454''$ to give the correct answer, it does not affect the value when rounded to the second.