

On the Motion of Planets and the Determination of Orbits (an English translation of *De Motu Planetarum et Orbitarum Determinatione*)

Leonhard Euler

All footnotes are comments by the translator, Patrick Headley¹.

1. At this time it is agreed that planets move in ellipses, with the sun located at one of the foci, and that the motion is arranged so that times are proportional to areas described around the sun. Two questions about the motion of the planets arise, one asking for properties of the ellipse, namely, the position of an apsis and the eccentricity, and the other for the motion of the planet itself in its orbit. Both questions are studied here, and, as far as difficulties of calculation permit, I will try to resolve them.

2. In fact I will first take the orbit of the planet to be known, and I will direct my attention to determining the motion of the planet in it. Therefore let ADB be half the orbit of a planet at P whose aphelion is at A and perihelion at B , with the sun at focus S of the ellipse (*see footnote²*). Furthermore, let C be the center of the orbit, and let the semiaxis AC or BC equal a , and the distance from focus S to center C , or eccentricity CS (*see footnote³*), equal b , so that the minor semiaxis CD is $\sqrt{a^2 - b^2}$. We now assume the planet has arrived at P from aphelion A , and then progressed in a brief interval of time to p . From P and p lines are

Fig. 1

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²The figures accompanying this article can be found on page 4 of eulerarchive.maa.org/publications/journals/figures.CASP07.pdf

³In the modern definition, the eccentricity would be the ratio of CS to AC .

drawn to S , as well as perpendiculars to the axis AB . Letting $CQ = r$, then $PQ = \frac{\sqrt{(a^2-b^2)(a^2-r^2)}}{a}$ and $PS = a + \frac{br}{a}$.

3. In this situation, the angle ASP represents the true anomaly (*see footnote*⁴), which will be called z . The mean anomaly, however, is proportional to the time in which the planet moves from A to P , or to the area ASP . Therefore, the area ADB will be to the area ASP as two right angles are to the mean anomaly. We now consider a circle of radius 1, with an arc equal to the true anomaly z . If we wish to find the mean anomaly, equal to the arc x , the ratio of area ADB to two equal right angles, or to twice the area of the semicircle, will be the same as that of $AC \cdot CD$ to 2, or $a\sqrt{a^2-b^2}$ to 2. Therefore $a\sqrt{a^2-b^2} : 2 = \text{Area } ASP : x$, so that $x = \frac{2\text{Area } ASP}{a\sqrt{a^2-b^2}}$.

4. With the true anomaly equal to angle ASP , the differential dz will be equal to PSp . The angle PSp is equal to twice the area PSp divided by the square of PS ; that is to say, $dz = \frac{2\text{Area } PSp}{(a+\frac{br}{a})^2} = \frac{2a^2 PSp}{(a^2+br)^2}$. But, from the equation shown earlier, $dx = \frac{2 PSp}{a\sqrt{a^2-b^2}}$. It remains therefore to express the area PSp in a convenient form, and it will be done from consideration of the entire area. For, indeed, $\text{Area } ASP = \frac{PQ \cdot QS}{2} + \int PQ \cdot Qq = \frac{(b+r)\sqrt{(a^2-b^2)(a^2-r^2)}}{2a} - \int \frac{dr\sqrt{(a^2-b^2)(a^2-r^2)}}{a}$, whose differential is $\frac{-dr(a^2+br)\sqrt{a^2-b^2}}{2a\sqrt{a^2-r^2}}$, which therefore is equal to PSp . From this therefore the differential of the mean anomaly is $dx = \frac{-dr(a^2+br)}{a^2\sqrt{a^2-r^2}}$, and the differential of the true anomaly is $dz = \frac{-a dr\sqrt{a^2-b^2}}{(a^2+br)\sqrt{a^2-r^2}}$.

5. A relation between the mean anomaly and the true anomaly is contained in these two equations. To express this it is necessary to integrate both equations so that in the end an equation between z and x can be derived. Concerning the first equation, it transforms at once to $dx = \frac{-dr}{\sqrt{a^2-r^2}} - \frac{br dr}{a^2\sqrt{a^2-r^2}}$, whose integral is $x = A \cdot \frac{\sqrt{a^2-r^2}}{a} + \frac{b}{a^2}\sqrt{a^2-r^2}$, where A signifies the arc of a circle whose sine is the subsequent quantity, the whole sine (*see footnote*⁵) being 1. Therefore let this

⁴In the modern definitions, the anomalies are measured from the perihelion and not the aphelion.

⁵The whole sine, or *sinus totus*, is a value chosen as a radius, so that Euler's *sine* is what would now be considered the radius multiplied by the sine. The tables that Euler used had varying choices for the whole sine. The logarithms of sine values that Euler uses in this article assume a whole

sine $\frac{\sqrt{a^2-r^2}}{a}$ equal s , so that $x = A.s + \frac{bs}{a}$.

6. The equation of the other differential is $dz = \frac{-a dr \sqrt{a^2-b^2}}{(a^2+br)\sqrt{a^2-r^2}}$, which is completely integrable, but can also be integrated in several ways using series. The chosen method will be examined before proceeding. It yields a series in which the powers of b increase in the numerator, so that for small eccentricities it suffices to use two or three initial terms.

7. Thus, I first convert $\sqrt{a^2-b^2}$ into a series, which will be $a - \frac{1 \cdot b^2}{2a} - \frac{1 \cdot 1 \cdot b^4}{2 \cdot 4 \cdot a^3} - \frac{1 \cdot 1 \cdot 3 \cdot b^6}{2 \cdot 4 \cdot 6 \cdot a^5}$ etc. $= a - \frac{b^2}{2a} - \frac{b^4}{8a^3} - \frac{b^6}{16a^5}$ etc. Next, $\frac{a}{a^2+br} = \frac{1}{a} - \frac{br}{a^3} + \frac{b^2r^2}{a^5} - \frac{b^3r^3}{a^7} +$ etc. Thus, these two series multiplied term-by-term will give $\frac{a\sqrt{a^2-b^2}}{a^2+br} = 1 - \frac{br}{a^2} - \frac{b^2(a^2-2r^2)}{2a^4} + \frac{b^3r(a^2-2r^2)}{2a^6} - \frac{b^4(a^4+4a^2r^2-8r^4)}{8a^8} + \frac{b^5r(a^4+4a^2r^2-8r^4)}{8a^{10}} -$ etc. If now each of the terms of this series is preceded by $\frac{-dr}{\sqrt{a^2-r^2}}$, the differential dz of the true anomaly will be obtained. All of the terms after the first will be completely integrable, so it is found that $z = A \cdot \frac{\sqrt{a^2-r^2}}{a} - \frac{b\sqrt{a^2-r^2}}{a^2} + \frac{b^2r\sqrt{a^2-r^2}}{2a^4} - \frac{b^3(a^2+2r^2)\sqrt{a^2-r^2}}{6a^6} + \frac{b^4r(a^2+2r^2)\sqrt{a^2-r^2}}{8a^8} -$ etc.

8. The arc or angle whose sine is $\frac{\sqrt{a^2-r^2}}{a}$ will be called V . Then $x = V + \frac{b}{a} \sin V$, and z will be determined from V , its sine, and sines of multiples of V as follows: $z = V - \frac{b}{a} \sin V + \frac{b^2}{4a^2} \sin 2V - \frac{b^3}{12a^3} (\sin 3V + 3 \sin V) + \frac{b^4}{32a^4} (\sin 4V + 4 \sin 2V) - \frac{b^5}{80a^5} (\sin 5V + 5 \sin 3V + 10 \sin V) + \frac{b^6}{192a^6} (\sin 6V + 6 \sin 4V + 15 \sin 2V) - \frac{b^7}{448a^7} (\sin 7V + 7 \sin 5V + 21 \sin 3V + 35 \sin V) +$ etc. (see footnote⁶) The rule for this series extends easily, for the denominators consist of the numbers in this series: $1 \cdot 1, 2 \cdot 2, 3 \cdot 4, 4 \cdot 8, 5 \cdot 16, 6 \cdot 32$, etc.

9. Thus, the true anomaly z will be most conveniently determined from the value x of the mean anomaly if, first, the angle V is determined from angle x through the equation $x = V + \frac{b}{a} \sin V$, and this then is substituted into the other equation producing the true anomaly z . It is difficult, however, to see how to determine V from x with this equation, since the equation is transcendental, and

sine of 10^{10} , but the trigonometric values assume a whole sine of 10^7 . As a result, Euler's use of the decimal point (actually a comma in the original text) seems somewhat erratic to the modern reader. For more information on the *sinus totus* see González-Velasco, Enrique A. *Journey through Mathematics*. New York, Springer, 2011.

⁶Euler actually uses the symbol \int for sin in this and subsequent formulas, in spite of using it for integration earlier in the text.

therefore V cannot entirely be represented algebraically using x . It is therefore necessary, in order for V to be determined accurately and with as little work as possible, that it is presented in the following way which seems to me to be most convenient, namely, that $V = x - \frac{b}{a} \sin(x - \frac{b}{a} \sin(x - \frac{b}{a} \sin(x - \dots$. It will be very easy to find the angle V from this, since, after $\log \sin x$ is found, from this $\log \frac{a}{b}$ is subtracted, and then from this the logarithm 6.4637261 is subtracted; this number is required for logarithms of natural numbers so that the corresponding number will give the angle in minutes (*see footnote*⁷). Then this angle is subtracted from the mean anomaly x , and the logarithm of the sine of the resulting angle is found, from which $\log \frac{a}{b}$ and 6.4637261 are subtracted, and the number corresponding to the resulting logarithm will give the number of minutes to be subtracted from x . If the value is to be estimated correctly, the resulting angle is then treated in the same manner until at last it is neither increased nor decreased any further. Then that angle will be the true value of V itself.

10. Angle V having been found in this way, both $\log \frac{a}{b}$ and 6.4637261 are subtracted from the logarithm of its sine. The number corresponding to the remainder will give the angle, expressed in minutes, that should be subtracted from V , and what remains will already be the true anomaly almost exactly. However, it will turn out to be more accurate if the logarithm of the sine of twice angle V (*see footnote*⁸), twice $\log \frac{a}{b}$, $\log 4$, and also 6.4637261 are subtracted from this, for the number corresponding to the remainder will give the minutes that are to be further added or subtracted, depending on whether V is less than or greater to 90° . Moreover, the subsequent terms in the series found for z can be calculated in the same way, continuing as long as angles are found that we do not wish to ignore. Always, however, when working with logarithms, besides the usual logarithmic operations, the logarithm 6.4637261 should be subtracted so that the corresponding number will give minutes. If, in place of minutes, seconds are desired, then in place of that logarithm 4.6855749 should be used (*see footnote*⁹).

11. The distance $PS = a + \frac{br}{a}$ of a planet to the sun is easily known once V is found. Let the ratio of the whole sine to $\cos V$ equal that of b to a fourth quantity;

⁷The logarithms are base 10. Recall that the whole sine will be 10^{10} , and note that 10^{10} minutes = $10^{6.4637261}$ radians.

⁸As $\log \sin 2V$ might be undefined, Euler actually uses $\log |\sin 2V|$, which is why it matters whether V is less than or greater than 90° .

⁹ 10^{10} seconds = $10^{4.6855749}$ radians.

this quantity, when added to or subtracted from the mean distance a , depending on whether $\cos V$ is itself positive or negative, will give the true distance of the planet to the sun. The distance PS can also be found from angle z , the true anomaly itself, since $PS = \frac{(a+b)(a-b)}{a-b \cos z}$.

12. Furthermore, having found angle V , it is possible to find the true anomaly z more simply, for $\cos z = \frac{a \cos V + b}{a + b \cos V}$, and $\sin z = \frac{\sin V \sqrt{a^2 - b^2}}{a + b \cos V}$. These expressions are simpler and shorter than the series found above. Nevertheless, I do not know which is to be preferred if ease of calculation is considered. Indeed, the earlier formula is perhaps more useful for what follows.

13. Our example is the orbit of Mars, in which $a : b = 152369 : 14100$ (*see footnote*¹⁰), and thus $\log \frac{a}{b} = 1.0336775$. Given that the mean anomaly is $2S.$, 20° (*see footnote*¹¹), or 80° , the true anomaly is required. The work is done as follows.

$\log \frac{a}{b}$	=	1.0336775
		4.6855749
		<hr/> 5.7192524
$\log \sin 80^\circ$	=	9.9933515
subtr.		5.7192524
		<hr/> 4.2740991
num.		18797''
this is		5°, 13', 17''
from		80°
leaves		<hr/> 74°, 46', 43''
$\log \sin$ of this angle	=	9.9844906
subtr.		5.7192524
		<hr/> 4.2652382
num.		18418''
this is		5°, 6', 58''
from		80°
leaves		<hr/> 74°, 53', 2''

¹⁰The units for a and b are 0.00001 astronomical units (AU).

¹¹The S stands for *signum* or sign, and is 30° , representing the traditional division of the ecliptic into the 12 zodiacal signs.

log sin of this angle	=	9.9847070
subtr.		5.7192524
		<hr/> 4.2654546
num.		18427''
this is		5°, 7', 7''
from		80°
leaves		<hr/> 74°, 52', 53''
log sin of this angle	=	9.9847035
subtr.		5.7192524
		<hr/> 4.2654511
num.		18427''
this is		5°, 7', 7''
Therefore V	=	74°, 52', 53''
subtr. $\frac{b}{a} \sin V$	=	5°, 7', 7''
approx. value of z	=	<hr/> 69°, 45', 46''
2V	=	149°, 45', 46''
Next, take		30°, 14', 14''
log sin of this angle		9.7020703
subtr. $2 \log \frac{a}{b}$		2.0673550
		<hr/> 7.6347153
subtr.		4.6855749
		<hr/> 2.9491404
subtr. log 4		0.6020600
		<hr/> 2.3470804
num.		222''
this is		3', 42''
add for more accurate value of z		69°, 49', 28''
$3V = -44°, 38', 39''$, sine	=	-7025671
+3 sin V	=	28961904
sin 3V + 3 sin V	=	<hr/> 21936233
its log	=	10.3412220
subtr. $3 \log \frac{a}{b}$	=	3.1010325
		<hr/> 7.2401895
subtr. log 12	=	1.0791812
		<hr/> 6.1610083
subtr.		4.6855749
		<hr/> 1.4754334
num.		37'' subtr.
correct value of z or true anomaly	=	69°, 48', 51''.

14. We check whether we find the same true anomaly with the other method by which z can be found from V , which is $\cos z = \cos V + \frac{b \sin V \sin V}{a + b \cos V} = \cos V + \frac{\sin V \sin V}{\frac{a}{b} + \cos V}$. The work will be as follows:

$$\begin{array}{rcl}
 \cos V & = & 2608181 \\
 \text{whole sine } \frac{a}{b} & = & 108063121 \\
 \frac{a}{b} + \cos V & = & 110671302 \\
 \text{its log} & = & 11.0440348 \\
 2 \log \sin V & = & 19.9694070 \\
 \text{diff.} & = & 8.9253722 \\
 \text{corresponding sine} & & 842116 \\
 \text{to } \cos V & & 2608181 \\
 \cos z & = & 3450297 \\
 \text{Therefore the true anomaly} & = & 69^\circ, 48', 59''
 \end{array}$$

This nearly coincides with what was found before, noting however that the quantity added would be extremely small if the term $\frac{b^4}{32a^4}(\sin 4V + 4 \sin V)$ (*see footnote*¹²) were included; in fact this term does not amount to an entire second. As for the other calculation, the intermediate values that I summed in the last part were not proportional, and in that calculation tables of logarithms were not sufficient (*see footnote*¹³).

14. (*see footnote*¹⁴) The distance from Mars to the sun corresponding to this anomaly is equal to $a + b \cos V$; and, by what we found before, $\frac{a}{b} + \cos V = 110671302$. Thus, the whole sine is to this number as b is to Mars' distance to the sun. Therefore, by logarithms,

$$\begin{array}{rcl}
 \log(\frac{a}{b} + \cos V) & = & 11.0440348 \\
 + \log b & & 4.1492191 \\
 - \log \text{ whole sine} & = & -10.0000000 \\
 \log \text{ dist. to } \odot & = & 5.1932539
 \end{array}$$

¹²This an error in the original text, since the $\sin 2V$ found in Section 8 should appear in place of $\sin V$.

¹³This is a confusing passage, perhaps because Euler has misdiagnosed the problem. The final subtraction in Section 13 should have been $10^{1.4754334} \approx 30$ seconds, accounting for the discrepancy.

¹⁴This is an error in the original text, as this is actually section 15.

16. Finally, it should be noted that, if the true anomaly corresponding to a given mean anomaly is found, and if the mean anomaly is increased by a small amount, the small change in the true anomaly can be found easily. Indeed, if the mean anomaly is increased by angle dx , then the change in the true anomaly will be $dz = \frac{dx}{(1 + \frac{b}{a} \cos V)^2}$. In our case, $\log(1 + \frac{b}{a} \cos V) = 0.0103573$ and $(1 + \frac{b}{a} \cos V)^2 = 1.04885$. Thus, $dz = \frac{dx}{1.04885} = dx - \frac{2dx}{43}$, and, therefore, if the mean anomaly was 81° , the true anomaly will be $70^\circ, 46', 4''$.

17. If, however, someone wishes to compute a table of true anomalies by this method, this goal is more easily achieved if the angles denoted by V , and not the mean anomalies, are assumed known, and from these both the mean and true anomalies are found by calculation, since a table is easily produced in this way. For, given a selected angle V , we have $x = V + \frac{b}{a} \sin V$ and $z = V - \frac{b}{a} \sin V + \frac{b^2}{4a^2} \sin 2V - \frac{b^3}{12a^3} (\sin 3V + \sin 3V)$ (*see footnote*¹⁵). For example, for the orbit of Mars assume $V = 20^\circ$ (*see footnote*¹⁶).

	$\log \sin V$	=	9.5340517
	subtr. as before		5.7192524
			<hr/> 3.8147993
	corresp. num.		6528''
	this is		<hr/> 1°, 48', 48''
Therefore the mean anomaly will be			<hr/> 21°, 48', 48''
and the approx. true anomaly			<hr/> 18°, 11', 12''
Now take $\log \sin 2V$			9.8080675
subtr. $2 \log \frac{a}{b}$			<hr/> 2.0673550
			<hr/> 7.7407125
subtr. $\log 4$			<hr/> 0.6020600
			<hr/> 7.1386525
Moreover, subtr.			<hr/> 4.6855749
			<hr/> 3.4530776
	num. 284''	=	4', 44''
more accurate anomaly	=		18°, 15', 56''

¹⁵This is an error in the original text, as the $3 \sin V$ found in Section 8 should appear in place of the second $\sin 3V$.

¹⁶The fourteenth line in the following calculation is an error in the original text. The following line is correct, however.

Finally, take $\sin 3V$	=	8660254
and $3 \sin V$	=	10260606
		<hr/> 18920860
its log.		10.2769406
subtr. $3 \log \frac{a}{b}$		3.1010325
		<hr/> 7.1759081
subtr. $\log 12$		1.0791812
		<hr/> 6.0967269
subtr.		4.6855749
		<hr/> 1.4111520
num.		26''

Therefore, the true anomaly is $18^\circ, 15', 30''$, corresponding to the mean anomaly $21^\circ, 48', 48''$. However, in tables the true anomaly corresponding to this mean anomaly is $18^\circ, 16', 14''$.

18. Thus, I proceed to the other question I mentioned in the beginning, which involves the shape of the ellipse in which the planet revolves, and the determination of the planet's position. In order to find these, I assume that the period of the planet is known to be T . Next, three arbitrary heliocentric positions FS , GS , and HS of the planet are required, together with the elapsed times between these observations. The heliocentric positions determine the angles FSG and FSH , so let $FSG = f$ and $FSH = g$. Moreover, the period T is to the time between two observations as 360 degrees is to the difference in the mean anomalies between the same observations. Since, therefore, the differences in the mean anomalies are given by the time between observations of the planet, let that between positions F and G equal m , and that between positions F and H equal n . Now the ratio AC to CS is set equal to that of 1 to v , so that $\frac{b}{a} = v$; furthermore, let the mean anomaly at F be x , and the true anomaly, or angle ASF , be z . Fig. 2

19. Assuming this, the mean anomaly at G will be $x + m$, and at H will be $x + n$; the true anomaly at G will be $z + f$, and at H will be $z + g$. Then let $x = P + v \sin P$, $x + m = Q + v \sin Q$, and $x + n = R + v \sin R$; thus, $\cos z = \frac{\cos P + v}{1 + v \cos P}$, and $\cos P = \frac{\cos z - v}{1 - v \cos z}$, and also $\sin P = \frac{\sin z \sqrt{1 - v^2}}{1 - v \cos z}$. In a similar fashion, $\cos Q = \frac{\cos(z+f) - v}{1 - v \cos(z+f)}$ and $\sin Q = \frac{\sin(z+f) \sqrt{1 - v^2}}{1 - v \cos(z+f)}$; also, $\cos R = \frac{\cos(z+g) - v}{1 - v \cos(z+g)}$ and $\sin R = \frac{\sin(z+g) \sqrt{1 - v^2}}{1 - v \cos(z+g)}$. If the values of P , Q , and R are now substituted into three previous equations, we have three equations from which the three unknowns x , z ,

and v ought to be determined. But this method immediately arrives at completely irresolvable equations; we are not able to reach our goal in this way. For this reason, it will instead be advantageous to resolve this question by approximation.

20. To do this most conveniently, it will help to express the true anomalies as series by the method shown previously. Therefore, they will be as follows (*see footnote*¹⁷):

$$z = P - v \sin P + \frac{v^2}{4} \sin 2P - \frac{v^3}{12} (\sin 3P + 3 \sin P) + \text{etc.}$$

$$z + f = Q - v \sin Q + \frac{v^2}{4} \sin 2Q - \frac{v^3}{12} (\sin 3Q + 3 \sin Q) + \text{etc.}$$

$$z + g = R - v \sin R + \frac{v^2}{4} \sin 2R - \frac{v^3}{12} (\sin 3R + 3 \sin R) + \text{etc.}$$

Eliminating z from these equations, we have

$$f = Q - P - v(\sin Q - \sin P) + \frac{v^2}{2} (\sin 2Q - \sin 2P) - \frac{v^3}{12} (\sin 3Q - \sin 3P + 3 \sin Q - 3 \sin P) + \text{etc.}$$

$$g = R - P - v(\sin R - \sin P) + \frac{v^2}{2} (\sin 2R - \sin 2P) - \frac{v^3}{12} (\sin 3R - \sin 3P + 3 \sin R - 3 \sin P) + \text{etc.}$$

But eliminating x from the earlier equations will give

$$\begin{aligned} m &= Q - P + v(\sin Q - \sin P) \text{ and also} \\ n &= R - P + v(\sin R - \sin P) \end{aligned}$$

We let the terms containing v^2 and higher powers disappear; the associated equations will be $\frac{f+m}{2} = Q - P$ and $Q = P + \frac{f+m}{2}$, and also $R = P + \frac{g+n}{2}$. Likewise, we next have equations $v = \frac{m-f}{2 \sin(P + \frac{f+m}{2}) - 2 \sin P} = \frac{n-g}{2 \sin(P + \frac{g+n}{2}) - 2 \sin P}$. From these equations is derived $\tan P = \frac{(m-f) \sin \frac{n+g}{2} - (n-g) \sin \frac{m+f}{2}}{(m-f) \text{versin} \frac{n+g}{2} - (n-g) \text{versin} \frac{m+f}{2}}$, where versin denotes the versed sine (*see footnote*¹⁸). The angle P can now be approximated from this equation; with this known the value of v itself can be approximated at the same time.

¹⁷The $\frac{v^2}{12}$ in the second equation is an error in the original text.

¹⁸Also known as the versine, and defined by $\text{versin } x = 1 - \cos x$. Euler actually uses $\int v$. for the versed sine.

21. Angle P being found in this manner, from this the values of Q and R are found through the equations $Q = P + \frac{f+m}{2}$ and $R = P + \frac{g+n}{2}$, and then the value of v itself through the equation $v = \frac{m-f}{2 \sin Q - 2 \sin P}$. In truth, the values found by this method are not yet accurate, but they are close, and better values are found when the succeeding terms are not neglected. Indeed, $\frac{f+m}{2} = Q - P + \frac{v^2}{8}(\sin 2Q - \sin 2P) - \frac{v^3}{24}(\sin 3Q - \sin 3P + 3 \sin Q - 3 \sin P)$ etc., and therefore $Q = P + \frac{f+m}{2} - \frac{v^2}{8}(\sin 2Q - \sin 2P) + \frac{v^3}{24}(\sin 3Q - \sin 3P + 3 \sin Q - 3 \sin P)$, and, similarly, $R = P + \frac{g+n}{2} - \frac{v^2}{8}(\sin 2R - \sin 2P) + \frac{v^3}{24}(\sin 3R - \sin 3P + 3 \sin R - 3 \sin P)$. The values of P , Q , R , and v found previously may be substituted after the = in these expressions; when this is done, better values will be obtained for Q and R . Suppose for the sake of brevity that $Q = P + M$ and $R = P + N$; then $v = \frac{m-M}{\sin(P+M) - \sin P} = \frac{n-N}{\sin(P+N) - \sin P}$. From these equations is derived $\tan P = \frac{(m-M) \sin N - (n-N) \sin M}{(m-M) \operatorname{versin} N - (n-N) \operatorname{versin} M}$, from which a much better value of P itself is obtained. This being substituted again, much better values for Q and R as well as for v result.

22. If the last values found for P , Q , R , and v are substituted into the equations $M = \frac{f+m}{2} - \frac{v^2}{8}(\sin 2Q - \sin 2P) + \frac{v^3}{24}(\sin 3Q - \sin 3P + 3 \sin Q - 3 \sin P)$ —etc. and $N = \frac{g+n}{2} - \frac{v^2}{8}(\sin 2R - \sin 2P) + \frac{v^3}{24}(\sin 3R - \sin 3P + 3 \sin R - 3 \sin P)$ —etc., then the values obtained for M and N are approximately correct, and consequently those for Q and R are approximately correct, and again for P and v . Therefore, the repetition of these operations, if performed as many times as necessary until the values of P and v are no longer changed, finally produce values that will be certain to be correct. Once these are determined, then z , which is the angle or difference between the position of the first observation F and the aphelion, is found from P and v by the rules stated earlier, and v in fact expresses the ratio of the orbit's eccentricity to the semi-major axis, or mean distance.

23. However, when undertaking such calculations, it is advantageous to work with more than three observations, not only so that the work is better confirmed, but also so that the most suitable observations can be selected. There are those observations in which the differences in the mean and true anomalies are equal, and this is unsuitable, as the first step immediately gives $v = \frac{0}{0}$ (see footnote¹⁹). To avoid this, such observations are chosen in which the differences in the anomalies are as large as possible. Yet this precaution is not even necessary if the eccentric-

¹⁹Perhaps Euler means $\tan P$, which would be $\frac{0}{0}$ if $m = M$ and $n = N$.

ity was known approximately in advance. Indeed, the desired eccentricity can be estimated and substituted for v at the beginning, from which better values for Q and R are immediately discovered, and the work can then be done as before.

24. By way of example for this method we determine an apsis and the eccentricity of Earth's orbit from the three observations given below, which are taken from Comment. Ac. R. Scient. Paris. A. 1720 (*see footnote*²⁰).

Year 1716		Position of Sun
March 20	11h, 57', 44"	0S, 0°, 0', 0"
May 12	11h, 55', 53"	1S, 21°, 44', 35"
July 28	12h, 5', 48"	4S, 5°, 22', 10"

From these observations, the difference in the true anomaly between the first and second observation, which we called f , is $51^\circ, 44', 35''$, and the difference between the first and third, or g , is $125^\circ, 22', 10''$.

25. To find the differences in the mean anomaly, I take the period, or tropical year, to be $T = 365d., 5h., 49', 8''$, or $31556948''$. The difference in time between the first and second observations is $52d., 23h., 58', 9''$, or $4579089''$, and from this is derived the difference between the mean anomalies of these observations, which is $m = 52^\circ, 14', 17''$. The difference in time between the first and third observations is $130d., 0h., 8', 14''$, or $11232494''$. From this the difference between the mean anomalies of these observations is $n = 128^\circ, 8', 23''$.

26. Now to perform the calculations we have

$$\begin{aligned}
 m - f &= 29', 42'' = 1782'' \\
 \text{and } n - g &= 2^\circ, 46', 13'' = 9973'' \\
 \text{and also } \frac{m+f}{2} &= 51^\circ, 59', 26'' \\
 \text{and } \frac{n+g}{2} &= 126^\circ, 45', 16''
 \end{aligned}$$

Then for convenience of calculation we have

²⁰The observations are from de Louville, Construction et Theorie des Tables du Soleil, which can be found at pages 35-84 of the section Memoires de Mathematique et de Physique of *Histoire de l'Académie Royale des Sciences* for the year 1720. The relevant pages are 43-45, which can be viewed beginning at <https://www.biodiversitylibrary.org/item/88038#page/189/mode/1up>. The seconds in the July 28 observation are given incorrectly as 48 instead of 58, but Euler uses the correct number in the calculations that follow.

$$\begin{array}{rcl}
\log \frac{n-g}{m-f} & = & 0.7479181 \\
\text{Then, } \sin \frac{n+g}{2} & = & 8012073 \\
\text{and also } \text{versin} \frac{n+g}{2} & = & 15983867 \\
\text{Then, } \log \sin \frac{m+f}{2} & = & 9.8964761 \\
\text{to which is added } \log \frac{n-g}{m-f} & = & 0.7479181 \\
& & \hline
& & 10.6443942
\end{array}$$

$$\begin{array}{rcl}
= \log \frac{n-g}{m-f} \sin \frac{m+f}{2}, & \text{for which logarithm the corresponding number is} & \\
44095498 & \text{from which subtract} & \\
8012073 & \sin \frac{n+g}{2} & \\
\hline
36083425 & = \frac{n-g}{m-f} \sin \frac{m+f}{2} - \sin \frac{n+g}{2} &
\end{array}$$

which number is the numerator of the fraction equaling $\tan P$. The value of $\log \text{versin} \frac{m+f}{2}$ is

$$\begin{array}{rcl}
9.5845671 & \text{to which } \log \frac{n-g}{m-f} & \\
0.7479181 & \text{is added} & \\
\hline
10.3324852 & = \log \frac{n-g}{m-f} \text{versin} \frac{m+f}{2} &
\end{array}$$

This corresponds in tables to the number

$$\begin{array}{rcl}
21502313 & \text{from which is subtracted} & \\
15983867 & \text{versin} \frac{n+g}{2} & \\
\hline
5518446 & \text{denominator of the fraction} &
\end{array}$$

Therefore, $\tan P = \frac{36083425}{5518446} = 6.5387 \dots$, giving angle $P = 81^\circ, 18', 17''$.

$$\begin{array}{rcl}
\text{Next } \log \frac{m-f}{2} & = & 2.9498777 \\
\text{and } P + \frac{m+f}{2} & = & Q = 133^\circ, 17', 43''
\end{array}$$

whose sine equals $\sin 46^\circ, 42', 17'' = 7278292$.

$$\begin{array}{rcl}
\text{Also } \sin P & = & 98885062, \\
\text{and therefore } \sin(P + Q) - \sin P & = & -2606770,
\end{array}$$

from which value (*see footnote*²¹) v itself will be negative, indicating that the position that the calculation of the aphelion should give is not the aphelion but the perihelion. In fact, v is determined from a table of sines by taking the logarithm of the sine 2906770 (*see footnote*²²),

²¹Note that the value for $\sin P$ in the preceding calculation should be 9885062, and $\sin Q$ appears incorrectly as $\sin(P + Q)$, but the -2606770 is correct.

²²This should read 2606770. The calculations that follow are correct.

which will be	9.4160988
from which is subtracted the number as before	4.6855749
log. of the denom. of the fract. for v	4.7305239
subtr.	2.9498777
Hence, $\log -v$	$= -1.7806462$

therefore $\frac{100}{6035}$, or, the mean distance from the Earth to the sun is to the eccentricity as 6035 is to 100, which ratio, however, is not yet corrected. If from here we wish to find the position of an apsis before applying subsequent corrections, the value produced for z is approximately $82^\circ, 14'$ or $2S, 22^\circ, 14'$, which when subtracted from the position of the first observation gives $9S, 7^\circ, 46'$ for the position of perihelion, and $3S, 7^\circ, 46'$ for the position of aphelion, which already nearly coincides with the accepted value. However, these values will be obtained most accurately if sufficient correction is made first.

27. Moreover, in making this correction, it should be noted that the sidereal year ought to be used in place of the tropical year, since the sun returns to the aphelion from the aphelion in this time. This makes $m = 52^\circ, 14', 10''$ and $n = 128^\circ, 8', 5''$, and thus $m - f = 1775''$ and $n - g = 9955''$. Also, $\frac{m+f}{2} = 51^\circ, 59', 22''$ and $\frac{n+g}{2} = 126^\circ, 45', 8''$. Then $P = 81^\circ, 18'$, $Q = 133^\circ, 18'$, and $R = 208^\circ, 3'$, where the seconds are omitted on purpose, as they contribute nothing to finding M and N . Indeed, $M = \frac{f+m}{2} - \frac{v^2}{8}(\sin 2Q - \sin 2P)$ and $N = \frac{n+g}{2} - \frac{v^2}{8}(\sin 2R - \sin 2P)$. From these are produced $M = 51^\circ, 59', 17''$ and $N = 126^\circ, 45', 4''$; and $m - M = 0^\circ, 14', 53'' = 893''$, and $n - N = 1^\circ, 23', 1'' = 4981''$, from which $\log \frac{n-N}{m-M} = 0.7464650$, and, by the rule given previously, $\tan P = \frac{\left(\frac{n-N}{m-M}\right) \sin M - \sin N}{\left(\frac{n-N}{m-M}\right) \text{versin } M - \text{versin } N}$. From this is found $P = 81^\circ, 23', 0''$, and $P + M = 133^\circ, 22', 17'' = Q$. Also, v again takes a negative value, and $\log(-v) = -1.7815349$, or, the Earth's mean distance to the sun is to the eccentricity as 6047 is to 100. These values being the most accurate, z will be $82^\circ, 19', 16''$, which angle, when subtracted from the vernal equinox, gives $9S, 7^\circ, 40', 44''$ for the position for the Sun's perigee. Thus, the apogee of the Sun is at $\mathfrak{Q}, 7^\circ, 40', 44''$ (see footnote²³). In fact, Street's table (see footnote²⁴) table gives $\mathfrak{Q}, 7^\circ, 52', 32''$ for

²³The symbol stands for the sign Cancer, corresponding to $3S$.

²⁴Thomas Street (or Streete) (1621-1689) was an English astronomer and author of *Astronomia Carolina*, which is available online through Google Books. It would be interesting to know how Euler derived this value for the apogee from Street's work; the value implied by the tables on page 105 of *Astronomia Carolina* would seem to be a little different.

this time. If one wished to make one more correction, I doubt whether the number of seconds would be affected.