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Group theoretic properties of some Schröedinger equations: systematic derivation

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GROUP THEORETIC PROPERTIES OF
SOME SCHROEDINGER EQUATIONS

SYSTEMATIC DERIVATION

A Thesis
Submitted to
the Faculty of the Department of Physics
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Master of Science

by

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This thesis, written and submitted by

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I. Introduction

In this thesis, I study the group theoretic structure of the Schroedinger equations of simple systems by making use of a new systematic method. Group theoretic analyses of Schroedinger equations have been made previously by numerous physicists. The groups found may be classified as

a) geometrical groups

b) dynamical degeneracy groups

c) dynamical groups

The geometrical group arises simply from the spatial symmetry of the system. Although the geometrical groups are very useful, they are not very interesting from the physical viewpoint.

On the other hand, the study of the dynamical degeneracy groups and the dynamical group is very attractive because it reflects the dynamics of the system.

Extensive studies have previously been made by other authors on systems which exhibit nontrivial degeneracy (accidental degeneracy). It turns out that all the states which belong to the same energy level provide a basis for a unitary irreducible representation of some compact group, and the group itself is generated by a set of constants of the motion. These groups are called "dynamical degeneracy groups". Detailed discussion on degeneracy groups will be found in the paper by McIntosh alluded to above.
In 1964, Barut suggested use of larger groups which encompass the states with different energy. This idea was introduced to investigate symmetry breaking in the mass spectrum of elementary particles. This is an extension of the idea of the dynamical symmetry group. In 1965, quite a few articles were published discussing the idea of "dynamical groups". These groups encompass infinitely many energy states in a single unitary irreducible representation. Since 1965, extensive investigations have been made on dynamical groups of simple systems.

Among these, the hydrogen atom has been best studied. The dynamical groups of the nonrelativistic hydrogen atom were proposed to be SO(4,1) and SO(4,2).

The idea of SO(4,1) group was first proposed by Mukunda et. al. and Dothan et. al., and detailed discussions were made by Barut et. al., Böhm, Itzykson, Pratt and Jordan, and Musto.

The idea of SO(4,2) was suggested by Malkin and Man'ko, and detailed studies were made by Barut and Kleinert, and Fronsdal. Definitive discussion of SO(4,2) will be found in the review article by H. Kleinert.

In contrast to the extensive studies of the hydrogen atom, not many papers can be found for other quantum mechanical systems.

The dynamical group of the n-dimensional isotropic and anisotropic oscillator was discussed by Barut. The group is SU(n,1).
The dynamical group of a non-relativistic rigid rotator was suggested to be $SO(4)$ or $SO(3,1)$ by Mukunda et. al. $^5$ and Dothen et. al. But Barut and Bohm $^3$ and Bohm $^8$ derived $SO(3,1)$ for the group from an algebraic discussion. Later, Kyriakopoulos $^7$ derived the larger group $SO(3,2)$ from the analysis of the Schrödinger equation of the rigid rotator. For a relativistic rigid rotator Barut and Bohm $^3$ derived $SO(4,1)$ from an algebraic investigation.

As far as I know, the dynamical groups of the two dimensional hydrogenic atom and symmetric top were not known previous to this thesis. In this thesis, the dynamical groups for these systems will be derived for the nonrelativistic case.

I would like to emphasize a further new result we obtain, that some of the generators of the dynamical groups which will be discussed in Chapter 3 are time-dependent constants of the motion. This type of generator was not known before in quantum mechanics.
II. Theory

2 - 1. General Discussion

The Schrödinger equation is written as

\[(\mathcal{H} - i\partial_t)f = 0\] (1)

or, for a definite energy state

\[(\mathcal{H} - E)f = 0\] (2)

We write these equations in more general form as

\[A \cdot f(x) = 0\] (3)

where \(x = (x_1, x_2, \ldots, x_n)\) denotes the physical variables including time. In this thesis, I restrict myself to the space spanned by solutions \(f(x)\) with discrete spectrum. Therefore, the \(f(x)\) are always square integrable and span a Hilbert space \(\mathcal{H}\) of \(L^2\) functions.

We consider a transformation \(G\) in the vector space \(\{x_1, x_2, \ldots, x_n\}\) and denote by \(T(g)\) the representation of the group \(G\) in the Hilbert space \(\mathcal{H}\).

Under this transformation, equation (3) becomes

\[T(g)A T(g)^{-1} T(g)f(x) = 0\] (4)

This is a trivial relation, but is extremely important if the \(T(g)\) satisfies the relation.
\[ T(g) \cdot T(g^{-1}) = p(x)A \quad (p(x) \neq 0) \quad (5) \]

where \( p(x) \) can be any function of \( x \).

The importance becomes quite clear if one notes that under this assumption the equation (4) can be written as

\[ A \cdot T(g) f = 0 \quad (6) \]

The mathematical meaning is straightforward; the \( T(g) \) produces new solutions of the equation (1). To indicate one physical meaning of the equation (6) we replace \( A \) by \( H - E \);

\[ (H - E) T(g) f(x) = 0. \]

Then one can easily derive the relation

\[ [H, T(g)] = 0 \quad (7) \]

in the Hilbert space \( \mathcal{H} \).

Then, the physical meaning is evident; \( T(g) \) is a time independent constant of the motion. If the group \( G \) involves nontrivial transformations besides the geometrical transformation in the space \( (x_1, x_2, \ldots, x_n) \), the group \( G \) is called a "dynamical degeneracy group", and the representation \( T(g) \) in the Hilbert space \( \mathcal{H} \) is reducible into the direct sum of finite dimensional irreducible representations of \( G \). Each space for this finite representation is spanned by the eigenfunctions \( f_n(x) \) with the same energy eigenvalue \( E \), and the group \( G \) is compact\(^\dagger\).

Another physical interpretation is obtained if we let \( A = H - i\beta \).

\(^\dagger\) For the case of continuous spectra, the group can be noncompact.
For this case, equation (6) becomes

$$(H - \mathcal{A}) T(g) f = 0.$$  

and implies the relation $^+$

$$[H, T(g)] - \frac{i \partial T(g)}{\partial t} = 0$$

in a Hilbert space $\mathcal{H}$.  

The physical meaning of this equation is well-known; $T(g)$ is a time-dependent constant of the motion. The group $G$ which satisfies the relation (8) is called a "dynamical group". For this case the function space for the representation $T(g)$ of the group $G$ is infinite dimensional, and the group $G$ is non-compact. But $T(g)$ is not always irreducible in the space $\mathcal{H}$.  

These two examples show the importance of the group $G$ which satisfies the equation (5) or (6). Before we go into detailed discussion of the method of derivation of such a group $G$, it is important to recognize that the transformation which satisfies the equation (6) covers the transformation which satisfies the equation (5), but not conversely. This can be seen from the following discussion:

From (6) we have

$$T^{(-1)} G \cdot T(g) f = 0.$$  

$^+ 0 = (H - \mathcal{A}) T(g) f = HT(g) f - i(\frac{\partial T(g)}{\partial t}) f - T(g) \mathcal{A} f = \{ [H, T(g)] - \frac{i \partial T(g)}{\partial t} \} f$
From (3) we have, for an arbitrary function \( p(x) \) of \( x \),

\[
p(x) \Lambda f = 0
\]

Subtracting these two, we obtain

\[
(T_{(g)}^{-1}) A T(g) - p(x) A f = 0.
\]

Therefore, the transformation \( T(g) \) which satisfies the relations

(3) and (6) has the property

\[
(T_{(g)}^{-1}) A T(g) - p(x) A = 0
\]

in the Hilbert space \( H \), but the equation (9) does not mean

that the left-hand side is identically zero. On the other hand,

the equation (5) implies that the operator \( T_{(g)}^{-1} A T(g) - p(x) A \)

is identically zero irrespective of the space \( H \).

Therefore, whenever we talk about the relations like (7) and (8) we have

to be careful what we mean by them; in the space \( H \) or identically?

Winternitz, et. al. developed a systematic method to derive the

transformation \( T(g) \) which satisfy the equation \( [H, T(g)] = 0 \). They

formulated their method in such a way that one obtains the transformation

\( T(g) \) which satisfies not the equation \( [H, T(g)] f(x) = 0 \), but the

operator equation \( [H, T(g)] = 0 \). Because of this, a direct application

of their method to other types of equations may lead to smaller trans-

formation groups than really exist. This happens if one seeks operators

\( T(g) \) such that \([H - i \bar{\omega}_L, T(g)] = 0 \). We formulate our method to obtain

the transformation \( T(g) \) which satisfies the relation \( A T(g) f = 0 \).

\[\dagger\] I replaced \( T(g) \) by \( T(g)^{-1} \) in (3) to put equation into the same form as (9).

\[\dagger\]
Now, the problem will be to find the group \( G \) which satisfies the relation (6). As we are interested in continuous transformations, a knowledge of the infinitesimal transformations will be sufficient to obtain the group \( G \). For the infinitesimal transformation
\[
T(g) = e^{\delta aQ} = 1 + \delta aQ
\]
the equation (6) can be rewritten as \( A(1 + \delta aQ)f = 0 \). But from \( Af = 0 \), we obtain
\[
A0f = 0. \tag{10}
\]
In general, the generator \( Q \) will be expressed in a form
\[
Q = q^0 + \sum_{i=1}^{n} q^i \partial x_i + \sum_{i<j} q^{ij} \partial x_i \partial x_j + \cdots
\]
in terms of the physical variables \( x_1, x_2, x_3, \cdots x_n \). Here, the \( q^0, q^i, q^{ij}, \cdots \) are functions of \( x_1, x_2, x_3, \cdots x_n \). In principle, there is no reason that we can put an upper bound to the order of the differential operator in \( Q \). For instance, Runge-Lenz vector
\[
\vec{A} = (-2H)^{-\frac{1}{2}} \left\{ \frac{1}{2} \left( \vec{L} \vec{P} + \vec{P} \vec{L} \right) + \vec{Z}_r \right\}
\]
whose components are well-known to be three of the generators of the dynamical degeneracy group \( SO(4) \) of hydrogen atom, involves infinitely many differential operators after the expansion of the factor \( (-2H)^{-\frac{1}{2}} \). But in the following discussion, for practical reasons, I assume only finite numbers of differential operators in \( Q \).
The program to determine the form of Q which satisfies the equation (10) is the following. First, we assume a certain form for Q. Then, we put the Q into (10), and expand it to obtain an equation

\[ a^0 f + \sum_{i=1}^{n} a^i f_i + \cdots + \sum_{i \leq j \leq k} a^{ij\cdots k} f_{ij\cdots k} = 0 \quad (11) \]

where the subscripts denote the differentiation of \( f(x) \) with respect to corresponding variables. The \( a^0, a^i, \ldots, a^{ij\cdots k} \) involve the \( q^0, q^i, \ldots \), etc., and their derivatives in them, but do not contain any differential operators. Now, we remember that the equations

\[ A f = 0, \partial x_i (Af) = 0, \partial x_i \partial x_j (Af) = 0, \ldots \]

provide a set of relations among \( f, f_i, f_{ij}, \ldots \). Therefore, some of the functions will be expressed in terms of the others, but we can select a set of independent functions. Of course, the choice is not unique, but it does not matter. After determining the independent functions, we eliminate all the dependent functions from equation (11), and regroup the terms in terms of the independent functions. Because of the independence of the functions each coefficient must vanish. This provides a set of simultaneous partial differential equations involving \( q^0, q^i, q^{ij}, \ldots \). Solving these equations, we obtain explicit expressions for the \( q^0, q^i, q^{ij}, \ldots \), as functions of the variables \( x_1, \ldots, x_n \).
Example: Confluent Hypergeometric Functions \( F(-n, a, x) \)

In this example, I demonstrate the procedure step by step using the differential equation of the confluent hypergeometric function. Spectrum generating operators for \( n \) will be obtained. Also it will be shown that for special values of \( a \), the differential equation has higher symmetry.

The confluent hypergeometric functions \( F(-n, a, x) \) are the solutions of the differential equation.

\[
(x \frac{d}{dx})^2 + (a - x) \frac{d}{dx} + n \) \( F(-n, a, x) = 0 \quad \text{(E1)}
\]

Suppose we take the form

\[
(x \frac{d}{dx})^2 + (a - x) \frac{d}{dx} + n)QF(-n, a, x) = 0 \quad \text{(E2)}
\]

for equation (10), and obtain solutions for \( Q \). What will be the effect of \( T(\theta) = e^{\theta Q} \) operating on \( F(-n, a, x) \)? Clearly, the \( Q \) produces solutions with the same eigenvalues \( a \) and \( n \), therefore, \( T(\theta) \) does so too. Thus, the \( Q \) which satisfies the equation (E2) is not interesting. How can we get a \( T(\theta) \) which can generate solutions with different eigenvalues \( a, n \)? The answer is quite clear if we glance at the equation (E2); as long as the \( a \) and \( n \) are sitting in the parenthesis, we never get this type of transformation. Therefore, we have to show these eigenvalues outside the parenthesis. This can be managed if we replace the \( a \) by \( \frac{\partial}{\partial \eta} \) and the \( n \) by \( \frac{\partial}{\partial \xi} \) and the functions.
for the solutions. Then the equation corresponding to (E1) has the form

\[(x^2 + (x + i\alpha - x))\partial_x + i\alpha \partial_t) e^{i\alpha u} e^{-\text{int}} F(-n, a, x) = 0,\]  \tag{E3}

and the equation (10) is written as

\[(x^2 + (x + i\alpha - x))\partial_x + i\alpha \partial_t) e^{i\alpha u} e^{-\text{int}} F(-n, a, x) = 0.\]  \tag{E4}

Now, we have succeeded in shooting them away. The \(a\) and \(n\) are sitting outside of the parenthesis.

In this example, to make the problem simple, we seek the transformation which can generate solutions with different \(n\) but the same \(a\). Then, the equations corresponding to (E1) and (E2) are written as

\[(x^2 + (a - x))\partial_x + i\alpha \partial_t) f = 0\]  \tag{E5}

\[(x^2 + (a - x))\partial_x + i\alpha \partial_t) \tilde{f} = 0\]  \tag{E6}

where \(f = \sum_n c_n e^{-\text{int}} F(-n, a, x)\) with \(c_n\) arbitrary constant.
Now we follow the program. We assume

\[ Q = q_x x^2 + q_t t + q_0. \]  

(E7)

Before expanding (E6) inserting (E7), let us determine the "independent functions" first. The equation (E5) gives rise to the equations

\[ A_f = x f_{xx} + (a - x)f_x + i f_t = 0 \]

\[ \partial_x (A_f) = x f_{xxx} + (a - x)f_{xx} + i f_{xt} + f_{xx} - f_x = 0 \]

\[ \partial_t (A_f) = x f_{xxt} + (a - x)f_{xt} + i f_{tt} = 0. \]

One of the possible choices of the independent functions will be

\[ f, f_x, f_t, f_{xt}, f_{tt} \]

Expanding (E6) and eliminating dependent functions, we obtain

\[ X_1 f_{xt} + X_2 f_t + X_3 f_x + X_4 f = 0 \]  

(E8)

where

\[ X_1 = 2q_x x \]

\[ X_2 = x q_x x^2 + (a - x)q_x t + i q_t + i x^{-1} q_x - 2i q_x \]

\[ X_3 = x q_x x + (a - x)q_x t + 2ax^{-1} q_x + 2q_x \]

\[ X_4 = x q_x x + (a - x)q_x t + 2i q_t \]
Because of the independence of the functions, the conditions

\[ X_1 = X_2 = X_3 = X_4 = 0 \]  \hspace{1cm} (E9)

must be satisfied for the equation (E8) to hold.

The equations (E9) provide a set of determining equations for \( q^x, q^t, q^o \). The solutions are following:

\[ q^x = a_1 e^{it} x - a_2 e^{-it} x + \delta_{a-1,\frac{1}{2}} (a_4 e^{\frac{1}{2}it} x^{\frac{1}{2}} + a_5 e^{-\frac{1}{2}it} x^{\frac{1}{2}}) \]

\[ q^t = a_1 e^{it} + a_2 e^{-it} + a_3 \]

\[ q^o = a_2 e^{-it} (x - a) + \delta_{a-1,\frac{1}{2}} \left( a_4 e^{\frac{1}{2}it} \left( \frac{1}{2}a - \frac{1}{4} \right) x^{\frac{1}{2}} \right. \]

\[ + a_5 e^{-\frac{1}{2}it} \left( \frac{1}{2}a - \frac{1}{4} \right) x^{\frac{1}{2}} - x^{\frac{3}{2}} \right) + a_6 \]

where \( a_i \) are the integration constants and \( \delta_{a-1,\frac{1}{2}} \) denotes the Kronecker delta function. Substituting these into (E7), and regrouping the terms we obtain

\[ Q = \sum_{i=1}^{6} a_i q_i \]

where

\[ Q_1 = e^{it}(i x^3 + \partial_x t) , \quad Q_2 = e^{-it}(-ix^3 + \partial_x t + ix - ia) \]

\[ Q_3 = \partial_t , \quad Q_4 = \delta_{a-1,\frac{1}{2}} e^{\frac{1}{2}it} \left( x^{\frac{1}{2}} \partial_x + \frac{1}{2}a - \frac{1}{4} \right) x^{\frac{1}{2}} \]

\[ Q_5 = \delta_{a,\frac{1}{2}} e^{\frac{1}{2}it} \left( x^{\frac{1}{2}} \partial_x + \frac{1}{2}a - \frac{1}{4} \right) x^{\frac{1}{2}} - x^{\frac{3}{2}} , \quad Q_6 = 1 \].
Because of the arbitrariness of $a_1$, each $Q_1$ will satisfy the equation (E6) independently.

We now investigate these results. First we notice that for the special values of $a$, $a = \frac{1}{2}$ or $\frac{3}{2}$, we have extra generators $Q_4$ and $Q_5$. If we remember that Hermite polynomials are expressed in terms of $F(-n,a,x)$ as

$$H_{2n}(-x) = \frac{(-1)^n}{n!} (2n)! F(-n,\frac{1}{2},x)$$

$$H_{2n+1}(-x) = \frac{(-1)^n}{n!} (2n+1)! 2xF(-n,\frac{3}{2},x),$$

we can see that Hermite polynomials have a special character which the other solutions of the equation (E1) do not have. As the purpose of this section is to illustrate the method of calculation in detail, I do not discuss this property in detail. But it is clear that this higher symmetry is related to the harmonic properties of the harmonic oscillator.

Let's investigate the properties of the generators. The actions of the operators on the eigenfunction is given by

$$Q_1f_n = -\inf_{n-1} f_n, \quad Q_2 f_n = \inf_{n+1} f_n$$

where

$$f_n = e^{-\int} F(-n,a,x) = e^{-\int_{-\infty}^{\infty}} \frac{1}{\Gamma(n+1)} \frac{\Gamma(a)\Gamma(-n+1)}{\Gamma(a+n)} x_n$$
The commutation relations among $Q_1$, $Q_2$ and $Q_o = i \Omega_3 + \frac{1}{2} a$
are given by

\[
\begin{align*}
[Q_2, Q_1] &= 2 \Omega_0 , & [Q_o, Q_2] &= \frac{1}{2} Q_2
\end{align*}
\]

and Casimir Operator

\[
Q^2 = \frac{1}{2} (Q_1 Q_2 + Q_2 Q_1) - (Q_o)^2
\]

has property

\[
Q^2 f_n = -j(j+1)f_n , \quad j = \frac{1}{2} a - 1.
\]

For the case of $a = \frac{1}{2}$ or $\frac{3}{2}$, we have relations

\[
(Q_4)^2 = i \Omega_1 , \quad (Q_5)^2 = i \Omega_2
\]

and "canonical" commutation relation

\[
\left[ \frac{1}{\sqrt{2}} Q_5 , \frac{1}{\sqrt{2}} Q_4 \right] = 1
\]
Example: The degeneracy group of the central potential in two dimensions

In this example I show one possible use of our method; we determine the central potential which admits an accidental degeneracy whose group generators contain differential operators of order no higher than two.

The Schrödinger equation of a particle in the central potential in two dimensions is

\[ -\frac{1}{2} \{ r^2 \partial_\theta^2 \partial_\phi^2 - 2V(r) + 2Ef(r,\phi) \} = 0 \]  

\[(1)\]
or

\[ f_{rr} + r^{-1} f_r + r^{-2} f_\phi \phi - 2V \cdot f + 2Ef = 0 . \]  

\[(1')\]

To determine the independent functions, we differentiate \((1')\) to get

\[ f_{rrr} + r^{-1} f_{rr} + r^{-2} f^r_{\phi \phi} - 2V^r \cdot f + 2Ef_r = r^{-2} f_r + 2r^{-3} f^r_{\phi \phi} + 2V_r f , \]

\[ f_{rr\phi} + r^{-1} f_{r\phi} + r^{-2} f^r_{\phi \phi \phi} - 2V^r \cdot f_{\phi} + 2Ef_{\phi} = 0 , \]  

\[(2)\]

\[ f_{rr\phi \phi} + r^{-1} f_{r\phi \phi} + r^{-2} f^r_{\phi \phi \phi \phi} - 2V^r f_{\phi \phi} + 2Ef_{\phi \phi} = 0 . \]

From \((1)\) and \((2)\), we can choose

\[ f, f_r, f_\phi, f^r_{\phi \phi}, f^r_{r \phi}, f^r_{r \phi \phi}, f^{rr}_{\phi \phi}, f^{rr\phi}_{\phi \phi} \]

for the independent functions \(^\dagger\).

We seek the explicit form of the potential \(V(r)\) for which a 0 operator

\(^\dagger\) Although we can take other combinations for the independent functions, this turns out to be the most convenient one.
of the form

\[ Q = O^\phi \phi + O^r \phi \phi + O^\theta \phi \theta + O^r \theta + O^\phi \theta + O^\theta \]

(3)

which satisfies the equation

\[ (r \phi + r^{-2} \phi \phi - 2V + 2E)Q \cdot f = 0 \]

(4)

Here \( O^\phi \), \( O^r \), \( O^\theta \), and \( O^\theta \) can be functions of \( r \) and \( \phi \).

Expanding (4), and eliminating the dependent functions using relations (1) and (2), we get

\[ f \cdot r \phi \ x_1 + f \cdot r \phi \ x_2 + f \cdot x_3 + f \cdot x_4 + f \cdot x_5 + f \cdot x_6 + f \cdot x_7 = 0 \]

where

\[
\begin{align*}
x_1 &= r^\phi - r^{-2} O^\phi - O^\phi, \\
x_2 &= O^\phi + r^{-2} O^\phi, \\
x_3 &= r^\phi + r^{-2} O^\phi - r^{-2} O^\phi - 2r^{-1} O^\phi - 2O^\phi + 2O^\phi + 2O^\phi, \\
x_4 &= O^\phi + r^{-2} O^\phi - 2O^\phi + 2O^\phi + 2O^\phi + 2O^\phi, \\
x_5 &= O^\phi + r^{-2} O^\phi + 2O^\phi + 4(V-E)O^\phi - 1O^\phi + 2rO^\phi + 4(V-E)O^\phi, \\
x_6 &= r^\phi - r^{-2} O^\phi + r^{-2} O^\phi + 2O^\phi + 2O^\phi \\
x_7 &= O^\phi + r^{-2} O^\phi + 4(V-E)O^\phi + 2rO^\phi \\
\end{align*}
\]

From the linear independence of the functions chosen, it follows that we have seven determining equations \( x_i = 0 \) (i=1,2, \ldots, 7).
Now we solve these equations.

From $x_1 = 0$ and $x_2 = 0$, we obtain

\[
Q^r \phi = \sum_{p} \left( a_p e^{-i(p-1)\phi} + b_p e^{-i(p-1)\phi} \right) r^p \quad (5)
\]

\[
Q^\phi = \sum_{p} \left( -a_p e^{-i(p-1)\phi} + b_p e^{-i(p-1)\phi} \right) r^{p-1} + k \quad (6)
\]

where $a_p$, $b_p$, and $k$ are constants to be determined later.

Solving $x_3 = 0$, $x_4 = 0$ by using (5) and (6), we get

\[
Q^r = \sum_{q} \left( c_q e^{-i(q-1)\phi} + d_q e^{-i(q-1)\phi} \right) r^q \quad (7)
\]

\[
0^\phi = \sum_{q} \left( -c_q e^{-i(q-1)\phi} + d_q e^{-i(q-1)\phi} \right) r^{q-1} + \ell \quad (8)
\]

where $c_q$, $d_q$ and $\ell$ are also constants to be determined.

Substituting (7) into $x_6 = 0$, we have $Q^O_r = 0$, that is,

\[
0^O = 0^O(\phi) = \text{function of } \phi \text{ only} \quad . \quad (9)
\]

Using (5), (6) and (8), we obtain from $x_5 = 0$ the relation

\[
Q^O_\phi = -\sum_{p \neq 1} \left( 2p(V-E) r^r + V_r \right) r^{p+1} \left( a_p e^{-i(p-1)\phi} + b_p e^{-i(p-1)\phi} \right) - (2V-E+rV_r)^2(a_1, b_1) \quad (10)
\]

Using the relations (7), (9) and $x_7 = 0$, we also have

\[
0^O_\phi = -\sum_{q \neq 1} \left( 2q(V-E) r^r + V_r \right) r^{q+1} 2i(q-1) e^{-i(q-1)\phi} \left( -c_q e^{-i(q-1)\phi} + d_q e^{-i(q-1)\phi} \right)

- (2V-E+rV_r)^2(c_1 + d_1) \phi + u(r) \quad , \quad (11)
\]

where $u(r)$ is a function of $r$ to be determined.

Comparing (10) and (11), we obtain the relations

\[
\begin{align*}
    a_p &= -2i(p-1)^{-1} c_p \quad (p \neq 1), & b_p &= 2i(p-1)^{-1} d_p \quad (p \neq 1), \\
    c_1 + d_1 &= 0, \quad - (a_1 + b_1)(2V - 2E + rY_r)^2 = u(r) \quad . \end{align*} \quad (12)
\]
Integrating the equation (10), we get

\[ Q^0 = \sum_{p \neq 1} \frac{1}{(p-1)^{1-1}} (2p(V-E) + iv_r)r^{p+1}(a_p e^{i(p-1)} - b_p e^{-i(p-1)}) \]

\[- (a_1 + b_1)(2V-2E+rV_r)r^2 \phi + h \]  \hspace{1cm} (13)

where h can be a function of \( r \) but not of \( \phi \).

From (9), \( Q^0 \), however, must be a function of \( \phi \) only.

Therefore, we have conditions

\[ \xi_p \{2p(V-E) + iv_r\} r^{p+1} = \text{constant} \quad (\xi_p = a \text{ or } b \text{ for } p \neq 1, \xi_1 = a_1 + b_1) \]  \hspace{1cm} (14)

\[ h = \text{constant} \quad . \]  \hspace{1cm} (15)

Differentiating the first condition by \( r \), we get

\[ \xi_p \{V_r + (3p + 2) r^{-1} V_r + 2p(p-1) r^{-2} V - 2p(p+1) r^{-2} V_r\} = 0 \]  \hspace{1cm} (16)

For nonzero \( \xi_p \), it is clear that \( p \) must be zero or \(-1\) for the potential not to be energy dependent.

For \( p = 0 \), we have a solution

\[ V = -2r^{-1} + V_0 \]  \( (Z \text{ and } V_0 \text{ are arbitrary constants}), \)

and all \( a_p \) and \( b_p \) except \( a_0 \) and \( b_0 \) must vanish. Using (3), (5), (6), (7), (8), (13), (15) and (9), one finds that the allowed form for 0 is

\[ Q = a_0 e^{-i\phi} (a_r r^{-1} + \phi + \frac{1}{2} i a_r^{-1} \phi + \frac{1}{2} i a_r^{-1} \phi + \frac{1}{2} i a_r^{-1} \phi) + b_0 e^{i\phi} (a_r r^{-1} + \phi + \frac{1}{2} i a_r^{-1} \phi + \frac{1}{2} i a_r^{-1} \phi + \frac{1}{2} i a_r^{-1} \phi) + k \phi + l \phi + h \cdot \]

As \( a_0, b_0, k, l, h \) are arbitrary, we have obtained a five parameter generator.
For $p = -1$, the potential $V$ obtained by solving the equation (16) has the form

$$V = \frac{1}{2}kr^2 + V_0$$

where $k$ and $V_0$ are arbitrary. For this case the $Q$ is found to be

$$Q = a_{-1}e^{-2i\phi}(r^{-1} \partial_r \partial_\phi - ir^{-2} \partial_\phi^2 \partial_r - i(r^{-1} \partial_r^{2} - r^{-2} \partial_\phi^2) - i(E-V_0))$$

$$+ b_{-1}e^{2i\phi}(r^{-1} \partial_r \partial_\phi + ir^{-2} \partial_\phi^2 \partial_r + i(r^{-1} \partial_r^{2} - r^{-2} \partial_\phi^2) + i(E-V_0))$$

$$+ k \cdot \partial_\phi^2 \partial_r + \partial_\phi + h.$$ 

For $p = 0$ and $-1$ to exist at the same time, the potential must be constant, $V=V_0$, and $Q$ has the form

$$Q = a_0e^{-i\phi}(r \partial_r \partial_\phi - ir^{-1} \partial_\phi^2 \partial_r - \frac{1}{2}i \partial_r^2 - kr^{-1} \partial_\phi^2)$$

$$+ b_0e^{i\phi}(r \partial_r \partial_\phi + ir^{-1} \partial_\phi^2 \partial_r - \frac{1}{2}i \partial_r^2 - \frac{1}{2}kr^{-1} \partial_\phi^2)$$

$$+ a_{-1}e^{-2i\phi}(r^{-1} \partial_r \partial_\phi - ir^{-2} \partial_\phi^2 \partial_r - ir^{-1} \partial_r^{2} - r^{-2} \partial_\phi^2) - i(E-V_0))$$

$$+ b_{-1}e^{2i\phi}(r^{-1} \partial_r \partial_\phi + ir^{-2} \partial_\phi^2 \partial_r + ir^{-1} \partial_r^{2} - r^{-2} \partial_\phi^2) + i(E-V_0))$$

$$+ k \cdot \partial_\phi^2 \partial_r + \partial_\phi + h.$$ 

It is clear from this result that if we restrict ourselves to the invariants which contain only zeroth, first and second order differential operators, the potentials which admit the accidental degeneracy are only those of the Kepler problem, harmonic oscillator and free particle.
2-2 Linearization of the Spectrum

In the example of confluent hypergeometric functions, we found a spectrum generating algebra by applying our method to the differential equation in which \( n \) was replaced by \( i \partial_t \). But this type of simple replacement is not almighty.

Let us take the differential equation of spherical harmonics:

\[
\{- (1 - x^2) \frac{\partial^2}{\partial x^2} + 2x \frac{\partial}{\partial x} - (1 - x^2)^{-1} \frac{\partial^2}{\partial \phi^2} - \ell(t + 1)\} y_{\ell m} = 0, \quad x = \cos \theta
\]  

To obtain operators which can change the eigenvalue \( \ell(t+1) \) by applying our method, we have to remove \( \ell(t+1) \) to outside the brackets. This can be done if we use \( e^{-it\phi} y_{\ell m} \) for the solution. The corresponding equation will be

\[
\{- (1 - x^2) \frac{\partial^2}{\partial x^2} + 2x \frac{\partial}{\partial x} - (1 - x^2)^{-1} \frac{\partial^2}{\partial \phi^2} - i \partial_t (i\partial_t + 1) \} e^{-it\phi} y_{\ell m} = 0
\]  

But one might ask "why not

\[
\{- (1 - x^2) \frac{\partial^2}{\partial x^2} + 2x \frac{\partial}{\partial x} - (1 - x^2)^{-1} \frac{\partial^2}{\partial \phi^2} - i \partial_t \ell(t + 1) \} e^{-it\phi} y_{\ell m} = 0
\]  

Both seem all right, because there is no \( \ell(t+1) \) in brackets". Both might be right, but within the framework of our method equation (3) is useless. Because if we apply our method to equation (3) assuming a
finite number of differential operators in \( Q \), we will find that the resulting \( Q \) cannot produce eigenstates with different values of \( \lambda \). On the other hand, assuming the simplest form

\[
q^x \partial_x + q^\phi \partial_\phi + q^t \partial_t + q^o
\]

for \( Q \), application of our method to equation (2) gives rise to a full spectrum generating algebra (both \( \mathbb{Z} \) and \( m \) shift operators). Although I cannot give a definite answer to the question "Why you do not need infinitely many differential operators in \( Q \) for equation (2) but you do for equation (3) ?", it seems more reasonable to replace \( \mathbb{Z} \) by \( \mathfrak{A}_t \) instead of \( \mathbb{Z}(\ell + 1) \) by \( \mathfrak{A}_t \) because we are interested in \( \ell \) shift operators.

From this and other examples, it seems to be essential to substitute \( \mathfrak{A}_t \) for not the eigenvalue itself but for the "quantum number". Here, I mean by "quantum number" the quantity which specifies the eigenvalues and takes a series of evenly spaced values. For the above case, the quantum number is \( \ell \).

So far I have been discussing the problem from a purely mathematical viewpoint, and could introduce any "dummy variables" to remove eigenvalues from a differential equation. But it is evident that we should not introduce this kind of physically meaningless variable into a physical problem. For instance, we can interpret equation (1) as the Schrödinger equation of a rigid rotator. Although we can derive a spectrum generating algebra for this problem from equation (2), this algebra is physically meaningless.
unless we can give a definite meaning to the variable $t$. On the other hand, although our method does not work, equation (3) has a definite physical meaning as a time-dependent Schrödinger equation of a rigid rotator.

Then, the natural question will be whether there exists any transformations which can bring the equation (3) into (1). If there exists such a transformation, let's call it $D$, our problem is solved; get a spectrum generating algebra for equation (2), then perform the inverse transformation $D^{-1}$ on the generators obtained. The transformed generators will comprise a spectrum generating algebra (or dynamical algebra) of the time-dependent Schrödinger equation (3).

In the following examples, it will be shown that there exists this type of transformation. We call this technique the "linearization of the spectrum" because after the transformation by $D$, the operator $i\hbar\frac{\partial}{\partial t}$ has a linear spectrum. The technique will be understood best from examples.
An Example of Time-Dilation. The Gegenbauer Equation

As an example illustrating the use of time dilations as an aid in finding spectrum generating invariants, we consider Gegenbauer’s differential equation

\[ \{(1-x^2)\partial_x^2 - (2v + 1)x\partial_x + n(n + 1)}g_n(x) = 0 \]  

which arises by separating the equation

\[ \{(1-x^2)\partial_x^2 - (2v + 1)x\partial_x + \frac{1}{2} \partial_t^2 \} \gamma e^{-i\pi(n + 2v)t} g_n(x) = 0 . \]  

To get a linear spectrum for \( \partial_x \), we transform equation (1) with the dilator

\[ D = \exp \{ \frac{1}{2} \log \frac{n}{n(n + 1)} \} \]  

where

\[ \tilde{n} = -v + \left( v^2 - (1-x^2)\partial_x^2 + (2v + 1)x\partial_x \right)^{1/2} . \]

The transformed equation is

\[ \{(1-x^2)\partial_x^2 - (2v + 1)x\partial_x + \left( \tilde{n}^2 + 2v \right) \partial_t \} \gamma e^{-i\pi \tilde{n}t} g_n(x) = 0 , \]

Within the basis set \{e^{-i\pi \tilde{n}t} g_n(x)\}, we have the operator identity

\[ \partial_t = \tilde{n} . \]

Using this identity we get

\[ \{(1-x^2)\partial_x^2 - (2v + 1)x\partial_x + \partial_t \left( \frac{1}{2} \partial_t + 2v \right) \} f(x,t) = 0 \]  

(4)
where
\[ f(x,t) = \sum_{n=0}^\infty e^{-\int \frac{1}{2}g_n(x)} . \]

From (4'), we have
\[ (1-x^2)f_{xx} - (2\nu + 1)xf_x - f_{tt} + 2ivf_t = 0 , \quad (4') \]
and differentiating this by \( x \) and \( t \), we get
\[ (1-x^2)f_{xxx} - (2\nu + 1)xf_{xx} - f_{xtt} + 2ivf_{xt} = 2xf_{xx} + (2\nu + 1)f_x \quad (5) \]
and
\[ (1-x^2)f_{xxt} - (2\nu + 1)xf_{xt} - f_{ttt} + 2ivf_{tt} = 0 . \quad (6) \]

From (4'), (5) and (6), we can choose
\[ f, f_x, f_{t}, f_{xt}, f_{tt} \]
for the independent functions.

We seek an operator \( Q \) which satisfies the equation
\[ \{(1-x^2)\partial_x \partial_x - (2\nu + 1)x\partial_x - \partial_t \partial_t + 2iv\partial_t\}Qf(x,t) = 0 \quad (8) \]
of the form
\[ Q = q^x \partial_x + q^t \partial_t + q^0 . \quad (9) \]

Substituting \( Q \) into (8), we get
\[ \{(1-x^2)\partial_x \partial_x - (2\nu + 1)x\partial_x - \partial_t \partial_t + 2iv\partial_t\}(q^xf_x + q^tf_t + q^0f) = 0 \quad (10) . \]
Expanding it, using (4), (5), (6) and collecting terms multiplying the functions (7), we obtain the determining equations

\[
q_x^x + x(1 - x^2)^{-1} q_x^t - q_t^t = 0, \quad (1 - x^2) q_x^t - q_t^x = 0,
\]

\[
(1 - x^2)^2 q_{xx}^x - (2v + 1) x q_x^t - q_{tt}^x + 2iv q_t^x - 2 a_0^t - 4iv x (1 - x^2)^{-1} q_x^x - 4iv q_x^t = 0,
\]

\[
(1 - x^2)^2 q_{xx}^x + (2v + 1) x q_x^x - q_{tt}^x + 2iv q_t^x + (2v + 1)(1 + x^2)(1 - x^2)^{-1} q_x^x + 2(1 - x^2) q_x^o = 0,
\]

\[
(1 - x^2) q_{xx}^o - (2v + 1) x q_x^o - q_{tt}^o + 2iv q_t^o = 0. \quad (11 \text{ a-c})
\]

Solving these simultaneous equations, we obtain, for \( v \neq 0, 1, \)

\[
q_x^x = a (1 - x^2) e^{it} + b (1 - x^2) e^{-it}
\]

\[
q_x^t = a x e^{it} - b x e^{-it} + c \quad (12 \text{ a-c})
\]

\[
q_x^o = -2 b v x e^{-it} + d
\]

where \( a, b, c \) and \( d \) are integration constants.

\[\text{for} \ v = 1 \text{ or } 0, 1, \]

\[\text{we have the following solutions for} \ q_x^x, q_x^t, q_x^o : \]

\[
v = 0 : q_x^x = \sum_{k=-\infty}^{\infty} u_k e^{ikt}, \quad q_x^t = \sum_{k=-\infty}^{\infty} \frac{1}{ik} \left( u_k^x + x(1 - x^2)^{-1} u_k^y \right) e^{ikt} + a, \quad q_x^o = b,
\]

\[
v = 1 : q_x^x = \sum_{k=-\infty}^{\infty} \frac{1}{ik} \left( u_k^x + x(1 - x^2)^{-1} u_k^y \right) e^{ikt} + a,
\]

\[
q_x^o = -\sum_{k=-\infty}^{\infty} \frac{1}{ik} \left( u_k^x + (k + 1)x(1 - x^2)^{-1} u_k^y \right) e^{ikt} - x(1 - x^2)^{-1} u_k^o + b
\]

where \( a \) and \( b \) are arbitrary and \( u_k^y \) are the solutions of the equation

\[
\left( (1 - x^2)^2 x x + x(1 + x^2)(1 - x^2)^{-1} + b \right) u_k = 0.
\]
Substituting these into ( 9 ), we get

\[ Q = a \cdot e^{it\{(1-x^2)\hat{\partial}_x + ix\hat{\partial}_t\}} + b \cdot e^{-it\{(1-x^2)\hat{\partial}_x - ix\hat{\partial}_t - 2vx\}} + c \cdot \hat{\partial}_t + d . \]  

(13)

As \( a, b, c \) and \( d \) are arbitrary, we have four independent operators which satisfy the equation ( 8 ):

\[ Q_+ = e^{it\{(1-x^2)\hat{\partial}_x + ix\hat{\partial}_t\}}, \quad Q_+ = e^{-it\{(1-x^2)\hat{\partial}_x - ix\hat{\partial}_t - 2vx\}} \]  

(14)

\[ Q_1 = \hat{\partial}_t, \quad Q_2 = 1. \]

On putting \( Q_0 = iQ_1 + vQ_2 \), we obtain the commutation relations

\[ [Q_+, Q_-] = 2Q_0, \quad [Q_+, Q_-] = \pm Q_+ . \]  

(15)

These show that \( Q_+ \) and \( Q_- \) shift the eigenvalue, \( u + v \), by unit amount:

\[ Q_\pm e^{-int} g_n(x) = C_\pm(n,v) e^{-i(n \pm 1)t} g_{n \pm 1}(x) . \]  

(16)

The Casimir operator, \( \frac{1}{2}(Q_+Q_- + Q_-Q_+) + Q_0^2 \) has the eigenvalue \( v(v - 1) \). This eigenvalue vanishes when \( v = 0 \) or 1. In these cases, as indicated in the footnote, we have infinitely many invariants and, therefore, the equation ( 1 ') has higher symmetry.
An Example of Time and space Dilation

One Dimensional Kepler problem

To illustrate the combined use of the time and space dilations as an aid in finding invariants, we consider the one dimensional Kepler problem.

(i) Linearization of Spectrum

The time dependent Schrödinger equation for the one dimensional Kepler problem is given by

\[
\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} + \frac{1}{r} - \frac{1}{2} \hbar^2 \right) \sum_n C_n e^{-iE_n t} \psi_n (2\sqrt{-2E_n} r), \quad r \geq 0
\]  

(17)

where

\[
E_n = - \frac{1}{2} \hbar^2 n^2
\]

To linearize the spectrum of \( i\hbar t \) we perform the time dilation on the equation (17);

\[
D_t \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} + \frac{1}{r} - \frac{1}{2} \hbar^2 \right) D_t^{-1} \sum_n C_n e^{-iE_n t} \psi_n (2\sqrt{-2E_n} r) = 0
\]

where

\[
D_t = \exp \left\{ i\hbar t \ln (2\sqrt{-2E}) \right\}
\]

(18)
with \( H = -\frac{1}{2} \partial_t \partial_r - 2r^{-1} \). The transformed equation is then

\[
\{-\frac{1}{2} \partial_t \partial_r - 2r^{-1} - (2z)^{-1} (\frac{1}{2} r^{-1} + \frac{1}{2} r^{-3}) \} \sum_n c_n \int_n e^{i \psi_n (2^{1-2z} r)} = 0 \quad (19)
\]

In this basis set we have an operator identity

\[
(-2H)^{1/2} = iz(\partial_t)^{-1}
\]

which allows one to rewrite (19) as

\[
\{-\frac{1}{2} \partial_t \partial_r - 2r^{-1} - i z^2 (\partial_t)^{-2} \} \sum_n c_n \int_n e^{i \psi_n (2^{1-2z} r)} = 0 \quad (19)
\]

Now to eliminate the negative power of \( \partial_t \) we perform the space dilation

\[
D_r = \exp \left( r \partial_r \log (ai \partial_t) \right)
\]

where \( a \) is an arbitrary constant.

The transformed equation will be

\[
-\frac{1}{2} ((ai \partial_t)^{-2} r \partial_r + 2z (ai \partial_t)^{-1} r^{-1} + z^2 (\partial_t)^{-2} \} \sum_n c_n \int_n e^{i \psi_n (-2z r)} = 0 \quad (20)
\]

and can be rewritten as

\[
(\partial_r \partial_r + 2z r^{-1} \partial_t - z^2 a^{-2} \} \sum_n c_n \int_n e^{i \psi_n (-2z r)} = 0 \quad (20)
\]

By choosing \(-2z^{-1}\) for \( a \), we put the equation in the standard form

\[
(\partial_r \partial_r - i r^{-1} \partial_t - \frac{1}{2}) f(r, t) = 0 \quad (21)
\]

Although we can eliminate negative powers by multiplying through \((\partial_t)^2\), we get a fourth order differential equation.
where

\[ f(r, t) = \sum_{n} C_n e^{i n \theta} \psi_n(r). \]

To determine the independent functions, we differentiate (21) with respect to \( r \) and \( t \) to get

\[ \begin{align*}
    f_{rrr} - 1r^{-1} f_{rt} - \frac{1}{2} f_r &= -ir^{-2} f_t \quad (22.) \\
    f_{rrt} - 1r^{-1} f_{tt} - \frac{1}{2} f_t &= 0 \quad (23.)
\end{align*} \]

From (21), (22), and (23), we can choose \( f, f_r, f_t, f_{rt} \) and \( f_{tt} \) for the independent functions.

We seek an operator \( \phi \) of the form

\[ \phi = q^r \theta^r + q^t \theta^t + q^0 \quad (24.) \]

which satisfies the equation

\[ \left( \theta^r \theta^r - 1r^{-1} \theta^t - \frac{1}{2} \phi \right) \phi \psi(r, t) = 0. \quad (25.) \]

Using (21), (22), (23), and (24), (25) becomes

\[ \begin{align*}
    0 &= (\theta^r \theta^r - 1r^{-1} \theta^t - \frac{1}{2} \phi)(q^r f_r + q^t f_t + q^0 f) \\
    &= (q^r \theta^r \theta^r - 1r^{-1} \theta^t - \frac{1}{2} \phi) f + (q^t \theta^r \theta^t - 1r^{-1} \theta^t - 1r^{-2} \theta^r + 2ir^{-1} \theta^r) f_t \\
    & \quad + (q^r \theta^r - 1r^{-1} \theta^r - 2 \theta^r) f_r + 2 \theta^r f_{rt} \\
    &= (q^r \theta^r \theta^r - 1r^{-1} \theta^t - \frac{1}{2} \phi) f + (q^t \theta^r \theta^t - 1r^{-1} \theta^t - 1r^{-2} \theta^r + 2ir^{-1} \theta^r) f_t \\
    & \quad + (q^r \theta^r - 1r^{-1} \theta^r - 2 \theta^r) f_r + 2 \theta^r f_{rt}. \quad (26.)
\end{align*} \]

Using the linear independence of \( f, f_r, f_t \) and \( f_{rt} \), we have
\[ q^{0}_{rr} - ir^{-1}q^{0}_{t} + \frac{1}{2}q^{r}_{r} = 0 \]
\[ q^{t}_{rr} - ir^{-1}q^{t}_{t} - ir^{-2}q^{r} + 2ir^{-1}q^{r}_{r} = 0 \]  
\[ q^{r}_{rr} - ir^{-1}q^{r}_{t} + 2q^{0}_{r} = 0 \]
\[ q^{t}_{r} = 0 \]

Solving these equations, we have

\[ q^{r} = iare^{it} - ibr_{e^{-it}} \]
\[ q^{t} = ae^{it} + be^{-it} + c \]
\[ q^{0} = -\frac{1}{3}iare^{it} - \frac{1}{3}ibr_{e^{-it}} + \frac{1}{3}d \]

where \( a, b, c, \) and \( d \) are arbitrary constants. Putting these solutions into (24), and collecting terms with the same constant coefficient, we get

\[ 0 = aie^{it}(r\partial_{r} - i\partial_{t} - \frac{1}{3}r) \]
\[ -bie^{-it}(r\partial_{r} + i\partial_{t} + \frac{1}{3}r) + c\partial_{t} + \frac{1}{3}d \]

(29)

As \( a, b, c, \) and \( d \) are arbitrary, the operators

\[ \partial_{1} = iie^{it}(r\partial_{r} - i\partial_{t} - \frac{1}{3}r) \]
\[ \partial_{2} = iie^{-it}(r\partial_{r} + i\partial_{t} + \frac{1}{3}r) \]
\[ \partial_{3} = \partial_{t} \]
\[ \partial_{4} = l \]

(30)

will satisfy the condition (25) independently.
These invariants satisfy the commutation relations

\[
\begin{align*}
[Q_1, Q_2] & = -2iQ_3, \\
[Q_3, Q_1] & = iQ_2, \\
[Q_3, Q_2] & = -iQ_1.
\end{align*}
\] (31)

From these it is clear that \( Q_1 \) and \( Q_2 \) shift the eigenvalue \( n \) by one unit;

\[
Q_1 \cdot e^{i n \theta} |n\rangle = C_1 e^{i (n+1) \theta} |n+1\rangle.
\] (32)

To obtain the shift operators for the eigenfunctions of the original equation (17), we must perform the inverse transformation:

\[
D^{-1} = D^{-1} D^{-1} = \exp\{i \theta \log((2z)^{-1}(-2i\alpha)^{-3})\exp[i \theta \log(2z(-i\alpha)^{-1})].
\] (33)

Then the corresponding operators \( \tilde{Q}_1, \tilde{Q}_2 \) and \( \tilde{Q}_3 \) will be

\[
\begin{align*}
\tilde{Q}_1 & = D^{-1} Q_1 D, \\
\tilde{Q}_2 & = D^{-1} Q_2 D, \\
\tilde{Q}_3 & = D^{-1} Q_3 D.
\end{align*}
\] (34)

In this and higher dimensional Kepler problems the energy shift operators are best left in the form (34) as very complicated expressions are obtained on explicitly carrying out the indicated transformations.

It is important to notice that the sets \( Q_1 \) and \( Q_3 \) still satisfy the same commutation relations.

Now I investigate the group theoretic structure of the Hilbert space \( \mathcal{H} \) spanned by the eigenstates with discrete spectrum. To make a definite group theoretic discussion we have to establish the Hermiticity of the generators of the group with respect to a particular scalar product.
To this end first I determine the effects of the action of the 
\( \hat{Q}_1 \) and \( \hat{Q}_2 \) on the eigenstates \( ^{29} \)

\[
f_n = e^{-iE_n t} \psi_n(2\sqrt{-2E_n^n r}) = -(2n)^{-\frac{1}{2}} e^{-iE_n t} e^{-\frac{x}{n}} 2x \frac{f}{f(n|n|2)}.
\]

(35)

The calculations are straightforward. The results are

\[
\hat{Q}_1 f_n = \frac{i}{2} \frac{n + 1}{n} f_{n+1}
\]

\[
\hat{Q}_2 f_n = \inf f_n.
\]

(36)

Because of the factors \( \frac{n + 1}{n} \) in the coefficients of (36) it is not possible to construct skew adjoint operators for the natural scalar product

\[
(f, g) = \int_0^\infty f^* g \, dr.
\]

(37)

by taking linear combination of the \( \hat{Q}_1 \) and \( \hat{Q}_2 \). This factor can be removed, if we introduce new operators

\[
\bar{Q}_1 = (\hat{Q}_3)^{-1} \hat{Q}_1 \hat{Q}_3
\]

(38)

It is clear that the set \( \{\bar{Q}_1\} \) still satisfy the same commutations as (31). The \( \bar{Q}_1, \bar{Q}_2 \) and \( \bar{Q}_3 = \hat{Q}_3 \) satisfy the relations
\[
\bar{Q}_{\frac{1}{2}} f_n = i(n(n+1))^{\frac{1}{2}} f_{n+1}
\]  

Then the operators

\[
J_1 = -\frac{1}{2}(\bar{Q}_1 + \bar{Q}_2), \quad J_2 = -\frac{1}{2}i(\bar{Q}_1 - \bar{Q}_2), \quad J_3 = \bar{Q}_3
\]

are skew adjoint under the scalar product (37), and satisfy the commutation relations of \(SO(2,1)\);

\[
[J_1, J_2] = -J_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2
\]

Therefore, the space \(\mathcal{H}\) is the representation space of a unitary irreducible representation of \(SO(2,1)\).
III. Group Theoretic Properties of the Schrödinger Equations.

In this chapter, we apply the method discussed in the preceding chapter to well-known Schrödinger equations to derive their dynamical groups.

In Section 3-1, we obtain the dynamical algebra of an isotropic harmonic oscillator (I.H.O) in two dimensions. In Section 3-2, we show that a two dimensional anisotropic harmonic oscillator (A.H.O) has the same dynamical group as the two dimensional I.H.O. The generators of the groups of the I.H.O. and A.H.O are connected continuously, and it will be shown that some of the time independent constants of the motion of I.H.O. become time dependent ones of A.H.O.

In section 3-3, we derive the dynamical group of the Kepler problem in two dimensions. We will find that there exist square integrable states with either integer or half odd integer angular momentum quantum number, and that under certain transformation of the variables all the generators of the group can be transformed into those of the two dimensional harmonic oscillator.

In Section 3-4 the generators of the well-known SO(4,2) dynamical group of three dimensional Kepler problem will be derived.

In section 3-5, we apply our method to obtain the spectrum generating algebra of the hydrogenic radial equations in two and three dimensions.

In Section 3-6, we obtain SO(3,2) for the dynamical group of the rigid rotator.

In Section 3-7, it will be shown that the dynamical group of the symmetric top is SU(2,2), and that some of the time independent constants of the motion of the spherical top become time-dependent ones of the symmetric top.
In Section 3-9, using the results in Section 3-6, we will derive the operators which shift the eigenvalue \( \lambda \) of the hydrogenic and harmonic oscillator eigenfunction in three dimensions.

In the last section, we will derive the spectrum generating algebra of the radial equation with Morse potential for \( \ell = 0 \).

I would like emphasize the fact that all the generators \( \Omega \) obtained in this chapter are time independent or time dependent constants of the motion because they satisfy the relation

\[
[ H - i \partial_\ell, \Omega ] = 0
\]
3.1. The two dimensional harmonic oscillator.

The Schroedinger equation is

\[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + kr^2 + 2\alpha r \right) \psi(x, y, r, \phi) = 0 \]  

(1)

We assume a Q operator of the form

\[ Q = Q^0_{\phi} \phi^0_{\phi} + Q^r \phi^r_{\phi} + Q^0 \phi^0_{\phi} + Q^r \phi^r_{\phi} + Q^0_{t} t + Q^r_{t} t \]  

(2)

and choose the independent functions to be;

\[ f, f^r_{\phi}, f^0_{\phi}, f^r_{t}, f^0_{t} \]

The determining equations derived from

\[ (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + kr^2 + 2\alpha r) Q^0_{\phi} = 0 \]

are:

\[ Q^t_{r} = 0 \]

\[ Q^0_{\phi} + Q^r_{\phi} = 0 \]

\[ Q^t_{\phi} - 2ir^2Q^0_{\phi} = 0 \]

\[ Q^t_{rr} + r^{-1}Q^t_{r} + r^{-2}Q^t_{\phi} + 210^r_{t} - 410^r_{r} = 0 \]

\[ Q^r_{rr} - r^{-1}Q^r_{r} + r^{-2}Q^r_{\phi} + 210^r_{t} - 2r^{-2}Q^r_{\phi} + 2r^{-2}Q^r_{t} + 20^r_{r} = 0 \]

\[ Q^t_{rr} + r^{-1}Q^t_{r} + r^{-2}Q^t_{\phi} + 210^t_{t} - 2r^{-2}Q^t_{\phi} + 2r^{-2}Q^t_{r} = 0 \]

\[ Q^r_{rr} - r^{-1}Q^r_{r} + r^{-2}Q^r_{\phi} + 210^r_{t} - 2r^{-2}Q^r_{\phi} = 0 \]

\[ Q^0_{rr} + r^{-1}Q^0_{r} + r^{-2}Q^0_{\phi} + 210^0_{t} + 2kr^2Q^0_{r} + 2kr^0_{r} = 0 \]

\[ Q^0_{rr} + r^{-1}Q^0_{r} + r^{-2}Q^0_{\phi} + 210^0_{t} + 2kr^2Q^0_{r} + 2kr^0_{r} = 0 \]
Their general solution give

\[ Q = \sum_{i=1}^{16} a_i q_i \]

where \( a_i \) are the integration constants and \( q_i \) are defined by

\[ q_1 = e^{\frac{i2\pi}{2} (\pm 1) r^2 - \frac{1}{3} r^3 + r^{-1} a_0 + r^{-2} a_1 + a_2 + r^{-3} a_3 \cdots} \]

\[ q_2 = \frac{1}{4} (a_0 q_{14} - a_0 q_{13}) \]

\[ q_3 = \frac{1}{4} (a_1 q_{12} + a_2 q_{13} - a_3 q_{14} + a_4 q_{15}) \]

\[ q_4 = e^{\frac{i2\pi}{2} (r + 1) r^2 + a_3 + i r^2 \pm i r} \]

\[ q_5 = \frac{1}{4} i (a_1 q_{12} - a_2 q_{13} + a_3 q_{14} - a_4 q_{15}) \]

\[ q_6 = e^{\frac{i2\pi}{2} (r - 1) r^2 + a_3 + i r^2 \pm i r} \]

\[ q_7 = \frac{1}{4} i (a_1 q_{12} + a_2 q_{13} - a_3 q_{14} + a_4 q_{15}) \]

\[ q_8 = e^{\frac{i2\pi}{2} (r - 1) r^2 + a_3 + i r^2 \pm i r} \]

\[ q_9 = \frac{1}{4} i (a_1 q_{12} - a_2 q_{13} - a_3 q_{14} + a_4 q_{15}) \]

\[ q_{10} = \frac{1}{4} i (a_1 q_{12} + a_2 q_{13} + a_3 q_{14} - a_4 q_{15}) \]

\[ q_{11} = e^{\frac{i2\pi}{2} (r - 1) r^2 + a_3 + i r^2 \pm i r} \]

\[ q_{12} = e^{\frac{i2\pi}{2} (r - 1) r^2 + a_3 + i r^2 \pm i r} \]

\[ q_{13} = e^{\frac{i2\pi}{2} (r - 1) r^2 + a_3 + i r^2 \pm i r} \]

\[ q_{14} = e^{\frac{i2\pi}{2} (r - 1) r^2 + a_3 + i r^2 \pm i r} \]

\[ q_{15} = e^{\frac{i2\pi}{2} (r - 1) r^2 + a_3 + i r^2 \pm i r} \]

\[ q_{16} = e^{\frac{i2\pi}{2} (r - 1) r^2 + a_3 + i r^2 \pm i r} \]

\( q_i \) \((i = 1, 2, \cdots, 10)\) are expressed in terms of \( q_{11}, q_{12}, q_{13} \) and \( q_{14} \) as listed. The \( q_1 \) and \( q_2 \) have the same forms as obtained in section three when \( a_3 \) is replaced by \( E_n \)
3.2 Two-dimensional Anisotropic Oscillator

The dynamical group of an anisotropic oscillator (N dimension) is known to be SU(N,1). But the exact relation to the dynamical group of an isotropic oscillator has not been clear. We give an explicit relation between the isotropic and anisotropic harmonic oscillator in two dimensions. The generalization to the N-dimensional case is straightforward.

The time-dependent Schrödinger equation of the two dimensional harmonic oscillator is given by

\[ (-\frac{1}{2} \partial_{x_1}^2 - \frac{1}{2} \partial_{x_2}^2 + \frac{1}{2} m_1 \omega_1 x_1^2 + \frac{1}{2} m_2 \omega_2 x_2^2 - i \partial_t) \psi_{n_1 n_2} = 0 \]  

where

\[ \psi_{n_1 n_2} = N e^{-iE(n_1 n_2)t} e^{-\frac{1}{2}m_1 (\omega_1 x_1^2 + \omega_2 x_2^2)} H_{n_1} (\sqrt{m_1} x_1) H_{n_2} (\sqrt{m_2} x_2) \]  

with

\[ E_{n_1 n_2} = \omega_1 (n_1 + \frac{1}{2}) + \omega_2 (n_2 + \frac{1}{2}) \]  

\[ N = (\frac{1}{\sqrt{2 \pi}} m_1^2 \omega_1 \omega_2)^{\frac{1}{4}} \left( \frac{n_1}{n_2} \right)^{\frac{1}{2}} \left( \frac{n_2}{n_1} \right)^{\frac{1}{2}} \]  

We make a transformation on \( \psi_{n_1 n_2} \) with the operator

\[ D(\omega_1 \omega_2) = (m_1^2 \omega_1 \omega_2)^{-\frac{1}{4}} D_x D_t \]

where

\[ D_x = \exp \left( t \partial_x \partial_{x_1} \ln(m_1 \omega_1)^{-\frac{1}{2}} \right) \exp \left( t \partial_x \partial_{x_2} \ln(m_2 \omega_2)^{-\frac{1}{2}} \right), \quad D_t = \exp \left( t \partial_t \frac{\omega_1}{2} H_1 + \frac{\omega_2}{2} H_2 \right) \]

with \( H = \text{Hamiltonian}, \ H_1 = -\frac{1}{2} \partial_{x_1}^2 + \frac{1}{2} m_1 \omega_1 x_1^2, \ H_2 = -\frac{1}{2} \partial_{x_2}^2 + \frac{1}{2} m_2 \omega_2 x_2^2 \).

Then the \( \psi_{n_1 n_2} \) will reduce to the isotropic harmonic oscillator states \( \phi_{n_1 n_2} \) with \( \omega = m = 1 \);

\[ \phi_{n_1 n_2} = (2^{n_1} n_1! 2^{n_2} n_2! \pi)^{-\frac{1}{4}} e^{-i(n_1 + i n_2 + 1)t} e^{-\frac{1}{2} (x_1^2 + x_2^2)} H_{n_1} (\sqrt{1} x_1) H_{n_2} (\sqrt{1} x_2) \]  

(3)
The dynamical group of the isotropic harmonic oscillator and its
generators can easily be derived. Suppose that the dynamical group
of the isotropic harmonic oscillator with the eigenstates of the form
\[ \psi \] has the Lie algebra \( \{X_i\} \). Then the dynamical group for the
anisotropic harmonic oscillator characterized by the frequencies
\( \omega_1, \omega_2 \) will be generated by the set of operators
\[ D^{-1}(\omega_1, \omega_2)X_1D(\omega_1, \omega_2) \]

Now we investigate the effect of the symmetry breaking \( \omega_1 \neq \omega_2 \)
on the group structure. For the limit \( \omega_1 \to \omega_0, \omega_2 \to \omega_0 \), all the
generators \( D^{-1}(\omega_1, \omega_2)X_1D(\omega_1, \omega_2) \) of the anisotropic harmonic oscillator with
the characteristic frequency \( \omega_1 \) and \( \omega_2 \) will approach the generators
\[ D^{-1}(\omega_0, \omega_0)X_1D(\omega_0, \omega_0) \] of the isotropic harmonic oscillator with \( \omega_1 = \omega_2 = \omega_0 \).
Obviously, the sets \( D^{-1}(\omega_1, \omega_2)X_1D(\omega_1, \omega_2) \) and \( D^{-1}(\omega_0, \omega_0)X_1D(\omega_0, \omega_0) \) satisfy
the same commutation relations as the set \( \{X_i\} \). It is important to
notice that under the transformation \( D \) the Hermiticity of the operators
\( X_1 \) is not subject to any change. Therefore, the dynamical groups of
the isotropic and anisotropic oscillator are the same, and one can
transform one to the other continuously by taking the limit \( \omega_1 \to \omega_0, \omega_2 \to \omega_0 \). From this observation it is clear that the time independent
constants of the motion of the isotropic harmonic oscillator become
the time-dependent constants of the motion of the anisotropic
oscillator under the symmetry breaking \( \omega_0 \to \omega_1, \omega_0 \to \omega_2 \).
3.3 The two-dimensional hydrogen-like atom

The two-dimensional Kepler problem has several interesting features. The Schroedinger equation is

\[- \frac{1}{2} \left( \partial_r^2 + \frac{1}{r^2} \partial_r^2 + r^{-2} \partial_{\phi}^2 + \frac{2n-1}{2r} \right) \psi_{nm} (r, \phi) e^{-iE_n t} = 0,\]

where \( E_n \) is given by \(- \frac{1}{2} \left( n - \frac{1}{2} \right)^2 \).

The transformation operator \( D \) leading to the linear spectrum can be chosen to be

\[ D = D_r \cdot D_t = \exp \left( i \frac{n - \frac{1}{2}}{2} \right) \cdot \exp \left( i \frac{n - \frac{1}{2}}{2} \right) \]

and the transformed equation is then

\[- \frac{1}{2} \left( \partial_r^2 + \frac{1}{r^2} \partial_r^2 + r^{-2} \partial_{\phi}^2 - i r^{-1} \partial_t - \frac{1}{4} \right) f(r, \phi, t) = 0\]

where

\[ f(r, \phi, t) = \sum_{nm} c_{nm} e^{i(n - \frac{1}{2})t} \psi_{nm} (\frac{n}{2\pi} \phi) . \]

We choose as independent functions the set

\[ f, f_r, f_\phi, f_t, f_{r\phi}, f_{rt}, f_{\phi t}, f_{r\phi t}, f_{r\phi \phi}, f_{\phi \phi \phi} , \]

and let the \( Q \) operator be

\[ Q = 0^r \partial_r \partial_r + Q^\phi \partial_\phi \partial_\phi + Q^t \partial_t \partial_t + Q^r \partial_r + Q^\phi \partial_\phi + Q^t \partial_t + Q^0 . \]

Then the determining equations for \( Q \) derived from the equation

\[ \left( \partial_r^2 + \frac{1}{r^2} \partial_r^2 + r^{-2} \partial_{\phi}^2 - i r^{-1} \partial_t - \frac{1}{4} \right) 0 f(r, \phi) = 0 \]
are:

\[ q^t_r = 0, \quad q^{\phi\phi}_r + r^{-2}q^r_\phi = 0, \quad q^{\phi\phi}_\phi - q^r_\phi + r^{-1}q^r_\phi = 0, \]

\[ q^t_\phi + i r Q^r_\phi - \frac{1}{2} q^r_\phi = 0, \quad q^{t\phi}_r + r^{-1}q^t_\phi + r^{-1}q^t_\phi - ir^{-1}q^t_\phi + 2ir^{-1}q^r_\phi - i r^{-2}q^r_r = 0, \]

\[ q^{r\phi}_r - r^{-1}q^r_\phi + r^{-2}q^r_\phi + ir^{-1}q^t_\phi - r^{-2}q^r_\phi + 2r^{-2}q^r_\phi + 2q^\phi_r = 0, \]

\[ q^{\phi\phi}_r + r^{-1}q^{\phi\phi}_r + r^{-2}q^{\phi\phi}_\phi - ir^{-1}q^{\phi\phi}_t + 2r^{-3}q^r_\phi + 2r^{-2}q^r_\phi - 2r^{-2}q^r_\phi = 0, \]

\[ q^r_r - r^{-1}q^r_r + r^{-2}q^r_r - ir^{-1}q^r_\phi + r^{-2}q^r_r + 2q^t_\phi = 0, \]

\[ q^t_\phi + r^{-1}q^t_\phi + r^{-2}q^t_\phi + ir^{-1}q^t_\phi + \frac{1}{2} q^r_\phi = 0, \]

\[ q^{r\phi}_r + r^{-1}q^{r\phi}_r + r^{-2}q^{r\phi}_\phi - ir^{-1}q^{r\phi}_t + 2r^{-2}q^t_\phi + \frac{1}{2} q^t_\phi = 0. \]

Solving these equations, one obtains a 16 parameter generator

\[ q = \sum_{i=1}^{16} a_i Q_i = (4) \]

where \( a_i \) are the integration constants, and the \( Q_i \) are

\[ Q_1 = e^{\frac{i}{2} \phi} (x i r^{-1}\phi_\phi + \phi_\phi + x \phi_\phi + \frac{1}{2} x^2 r + \frac{1}{2} x^{-1} \phi_\phi + \frac{1}{2} x^{-1} \phi_\phi) = \frac{1}{2} Q_{13} Q_{12} - Q_{11} Q_{14}, \]

\[ Q_3 = \phi_\phi = \frac{1}{2} (Q_{13} Q_{12} - Q_{11} Q_{14}), \]

\[ Q_5 = \frac{x}{2} (x i r^{-1} \phi_\phi + \phi_\phi + \frac{1}{2} x^2 r + \frac{1}{2} x^{-1} \phi_\phi) = -1 Q_{11} Q_{13}, \]

\[ Q_6 = \phi_\phi = \frac{1}{2} (Q_{11} Q_{14} + Q_{12} Q_{13}), \]

\[ Q_7 = e^{-\frac{i}{2} ti x} (x i r^{-1} \phi_\phi + \phi_\phi + \frac{1}{2} x^2 r - \frac{1}{2} (x^{-1} - 1) \phi_\phi + \frac{1}{2} x^{-1} \phi_\phi - \frac{1}{2} x^{-1} \phi_\phi) \]

\[ = \frac{1}{4} (Q_{11})^2 \]

\[ Q_{10} = \frac{1}{3} \frac{1}{2} (Q_{11})^2 \]

\[ Q_{11} \]
It is interesting to notice that all the operators listed above can be transformed into those of the harmonic oscillator in section three by the transformation $r \rightarrow r^2$, $\phi \rightarrow 2\phi$, $t \rightarrow 2t$.

This is because under this transformation the equation (1) becomes exactly the same as the equation (1). As a result, we can express the $Q_i$ ($i=1,2, \ldots, 10$) in terms of $Q_{11}$, $Q_{12}$, $Q_{13}$ and $Q_{14}$ as listed above as in the case of the harmonic oscillator.

As the $Q_i$ listed above were obtained by using the transformed equation (3), the corresponding operators $\tilde{Q}_i$ for the original equation (1) are given by

$$\tilde{Q}_i = D^{-1} Q_i D,$$

Now we analyze those operators. The commutation relations of $\tilde{Q}_{11}$, $\tilde{Q}_{12}$, $\tilde{Q}_{13}$ and $\tilde{Q}_{14}$ with $\tilde{Q}_3$ and $\tilde{Q}_6$ shows that these operators raise or lower the eigenvalues of $\tilde{Q}_3$ and $\tilde{Q}_6$ by $\frac{1}{2}$. The commutation relations among $\tilde{Q}_{11}$, $\tilde{Q}_{12}$, $\tilde{Q}_{13}$ and $\tilde{Q}_{14}$ are given by

$$[\tilde{Q}_{11}, \tilde{Q}_{12}] = 0, \quad [\tilde{Q}_{11}, \tilde{Q}_{13}] = 0, \quad [\tilde{Q}_{11}, \tilde{Q}_{14}] = 1,$$

$$[\tilde{Q}_{12}, \tilde{Q}_{13}] = -1, \quad [\tilde{Q}_{12}, \tilde{Q}_{14}] = 0, \quad [\tilde{Q}_{13}, \tilde{Q}_{14}] = 0.$$

From this and the fact that $\tilde{Q}_i$ ($i=1,2, \ldots, 10$) can be expressed in terms of $\tilde{Q}_{11}$, $\tilde{Q}_{12}$, $\tilde{Q}_{13}$ and $\tilde{Q}_{14}$, one can see that the set $\{\tilde{Q}_i\}$ ($i=1,2, \ldots, 10$) forms a closed Lie algebra.

† To derive the expression for the $Q_i$ one must use an operator identity $e^{i{\mathcal{A}}_{3\phi} (t \rightarrow t^2)} = e^{i{\mathcal{A}}_{3\phi} (t \rightarrow t^2)} e^{i{\mathcal{A}}_{3\phi} (t \rightarrow t^2)} + 1/4$ which holds in the space $\{e^{-\int \psi_{n p} (\frac{1}{2} r^2, \phi)}\}$. 

\[Q_{11} = e^{i\frac{1}{2} t} e^{i\frac{1}{2} \phi} (r^2 - r^2 \phi + \frac{1}{2} r^2) \quad Q_{12} = e^{i\frac{1}{2} \phi} \phi \quad Q_{16} = 1.\]
This set contains two subalgebra \( \{ \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3 \} \) and \( \{ \tilde{Q}_4, \tilde{Q}_5, \tilde{Q}_6 \} \), for which the commutation relations are given by

\[
\begin{align*}
[\tilde{Q}_1, \tilde{Q}_2] &= -\frac{1}{2}\tilde{Q}_3 , \\
[\tilde{Q}_1, \tilde{Q}_3] &= i\tilde{Q}_1 , \\
[\tilde{Q}_2, \tilde{Q}_3] &= -i\tilde{Q}_2 , \\
[\tilde{Q}_4, \tilde{Q}_5] &= -2i\tilde{Q}_6 , \\
[\tilde{Q}_6, \tilde{Q}_4] &= i\tilde{Q}_4 , \\
[\tilde{Q}_6, \tilde{Q}_5] &= -i\tilde{Q}_5 , \\
[\tilde{Q}_1, \tilde{Q}_4] &= i\tilde{Q}_7 , \\
[\tilde{Q}_1, \tilde{Q}_5] &= -\tilde{Q}_8 , \\
[\tilde{Q}_2, \tilde{Q}_4] &= i\tilde{Q}_9 , \\
[\tilde{Q}_2, \tilde{Q}_5] &= -\tilde{Q}_{10} , \\
[\tilde{Q}_1, \tilde{Q}_6] &= 0 \quad (i=1,2,3) , \\
[\tilde{Q}_4, \tilde{Q}_3] &= 0 \quad (i=4,5,6).
\end{align*}
\]

These imply that \( \tilde{Q}_1 \) and \( \tilde{Q}_2 \) shift the eigenvalue of \( \tilde{Q}_3 \), that is, \( j^m \), by unit, and also \( \tilde{Q}_4 \) and \( \tilde{Q}_5 \) shift the eigenvalue of \( \tilde{Q}_6 \), in, by unit.

As \( \tilde{Q}_1, \tilde{Q}_2 \) and \( \tilde{Q}_3 \) commute with \( \tilde{Q}_6 \) which is a labeling operator of the energy, they comprise the Lie algebra of the degeneracy group, and the operators \(-i(\tilde{Q}_1 - \tilde{Q}_2)\)

and \(-i(\tilde{Q}_1 + \tilde{Q}_2)\) are identified as \( A_x \) and \( A_y \) where \( \tilde{A} \) is the two dimensional analogue of the Runge-Lenz vector defined by

\[
\tilde{\mathbf{A}} = (A_x, A_y, 0) = (-2\mathbf{H})^{-\frac{1}{2}}\left( \frac{1}{2} (\mathbf{L} \times \mathbf{P}^2 - \mathbf{P} \times \mathbf{L}) + \frac{\mathbf{L}^2}{\mu} \right).
\]

Here we have defined the cross product by, for example,

\[
\mathbf{i} \times \mathbf{P} = \begin{vmatrix}
1 & j & k \\
0 & 0 & L_z \\
P_x & P_y & 0
\end{vmatrix}.
\]

The raising and lowering operators of \( \tilde{Q}_4 \) and \( \tilde{Q}_5 \) satisfy the equations

\[
\begin{align*}
\tilde{Q}_4 e^{-iE_nt} \psi_{nm} &= i \left( \frac{n+\frac{1}{2}}{n-\frac{1}{2}} \right)^{\frac{3}{2}} (n+m)(n-m)^{\frac{1}{2}} e^{-iE_{n+1}t} \psi_{n+1m}, \\
\tilde{Q}_5 e^{-iE_nt} \psi_{nm} &= i \left( \frac{n-\frac{3}{2}}{n-\frac{1}{2}} \right)^{\frac{3}{2}} (n-1+m)(n-1-m)^{\frac{1}{2}} e^{-iE_{n-1}t} \psi_{n-1m}, \\
\tilde{Q}_6 e^{-iE_nt} \psi_{nm} &= i ne^{-iE_{nt}} \psi_{nm}.
\end{align*}
\]

† See appendix I for the analogous calculation in three dimensions.
where

\[
\psi_{nm} = \frac{\alpha}{(2\pi)^{3}} \left( \frac{-(n\!+\!m\!-\!1)!}{2\pi(2n\!-\!1)^3(n\!-\!m\!-\!1)!} \right)^{\frac{1}{2}} F\left(-n\!+\!m\!+\!1 \mid 2m\!+\!1 \mid \alpha r \right) \times e^{-\alpha r} (\alpha r)^m e^{im\phi} 
\]

\[
\alpha = 2\sqrt{-2F_n}, \quad F_n = -\frac{1}{2} \pi^2 (n-\frac{1}{2})^{-2}.
\]

Now to obtain the skew-adjoint operators under the scalar product

\[
(f, g) = \int_{\mathbb{R}} \int_{\mathbb{R}} f^* g \, r \, dr \, d\phi
\]

we define the new operators \( \tilde{Q}_1 \) by

\[
\tilde{Q}_1 = (\tilde{Q}_6)^{-\frac{3}{2}} \tilde{Q}_1 (\tilde{Q}_6)^{\frac{3}{2}}.
\]

The \( \tilde{Q}_4 \) and \( \tilde{Q}_5 \) then satisfy the relations

\[
\tilde{Q}_4 e^{-iF_n t} \psi_{nm} = i(n + m)(n - m) \frac{1}{2} e^{-F_{n+1} t} \psi_{n+1m}
\]

\[
\tilde{Q}_5 e^{-iF_n t} \psi_{nm} = i(n-1 + m)(n-1 - m) \frac{1}{2} e^{-F_{n-1} t} \psi_{n-1m}
\]

and the operators

\[
J_{23} = -i(\tilde{Q}_1 + \tilde{Q}_2), \quad J_{31} = -(\tilde{Q}_1 - \tilde{Q}_2), \quad J_{12} = \tilde{Q}_3
\]

\[
J_{53} = -\frac{1}{2}(\tilde{Q}_4 + \tilde{Q}_5), \quad J_{34} = -\frac{1}{2}(\tilde{Q}_4 - \tilde{Q}_5), \quad J_{45} = \tilde{Q}_6
\]

\[
J_{24} = -\frac{1}{2}(\tilde{Q}_7 + \tilde{Q}_8 + \tilde{Q}_9 + \tilde{Q}_{10}), \quad J_{25} = \frac{1}{2}(\tilde{Q}_7 - \tilde{Q}_8 + \tilde{Q}_9 - \tilde{Q}_{10})
\]

\[
J_{14} = \frac{1}{2}(\tilde{Q}_7 + \tilde{Q}_8 - \tilde{Q}_9 - \tilde{Q}_{10}), \quad J_{15} = \frac{1}{2}(\tilde{Q}_7 - \tilde{Q}_8 - \tilde{Q}_9 + \tilde{Q}_{10})
\]

are skew adjoint.
Calculating the commutation relations directly, we find that $J_{ij}$'s generate an $O(3,2)$ algebra:

$$\left[ J_{ab}, J_{cd} \right] = -\epsilon_{bc} J_{ad} + \epsilon_{ac} J_{bd} - \epsilon_{ad} J_{bc} + \epsilon_{bd} J_{ac}$$

with

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = -\epsilon_{44} = -\epsilon_{55} = -1, (J_{ab})^\dagger = -J_{ab}$$

Therefore the set $\{e^{-iE_{nt}} \psi_{nm}\}$ where $n$ and $m$ are both integers or both half odd integers form the basis for a UIR of $O(3,2)$, and $\psi$ or $n$ specify the UIR of an $O(1,2)$ or $O(3)$ subgroup of $O(3,2)$.

If we allow both integral and half integral $n$ and $m$, then the set $\{e^{-iE_{nt}} \psi_{nm}\}$ will comprise the basis of a UIR of the group generated by the set of generators $\{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}, Q_{11}, Q_{12}, Q_{13}, Q_{14}, Q_{15}\}$. For this case the scalar product must be defined by

$$(f, g) = \int \int f^* g r^2 dr d\phi$$
3.4. Hydrogenlike Atom

We shall determine the invariants of the time-dependent Schrödinger equation of the hydrogenlike atom:

\[
-\frac{i}{\hbar} \left( \frac{\partial}{\partial t} + \frac{1}{2} \partial^2_r + 2z \partial_r + i2z \right) \sum_{n,m} C_{n,m} e^{-iE_n t} \varphi_{n/m}(r,\theta,\phi) = 0
\]

where the \( C_{n,m} \) are arbitrary constants, and

\[
L^2 = -y^2 \partial_x \partial_x + 2x \partial_x - y^{-2} \partial_\phi \partial_\phi \quad \text{with} \quad x = \cos \theta, \quad y = \sin \theta.
\]

The transformation operator \( D \) leading to the linear spectrum can be chosen to be

\[
D = D_r \cdot D_t = \exp \{ r \partial_r \cdot \log \left( \frac{L}{2 \hbar} \right) \} \cdot \exp \{ t \partial_t \cdot \log \left( \frac{2 \hbar}{z} \right) \}
\]

where

\[
H = -\frac{1}{2} \partial_r \partial_r - r^{-1} \partial_r + \frac{1}{z} r^{-2} L^2 - 2z^{-1}.
\]

The transformed equation is then

\[
\left( \frac{\partial}{\partial t} + \frac{1}{2} \partial^2_r + 2z \partial_r + i2z \right) f(r,\theta,\phi) = 0
\]

with \( f(r,\theta,\phi) = \sum_{n,m} C_{n,m} e^{int} \varphi_{n/m}(\frac{2\pi}{2z},\theta,\phi) \).

We choose as independent functions, the set

\[
\begin{align*}
\varphi, & \quad \varphi_r, \quad \varphi_x, \quad \varphi_\phi, \quad \varphi_t, \quad \varphi_{rx}, \quad \varphi_{r\phi}, \quad \varphi_{rt}, \quad \varphi_{xx}, \quad \varphi_{x\phi}, \quad \varphi_{xt}, \quad \varphi_{\phi t}, \quad \varphi_{rx}, \quad \varphi_{r\phi}, \quad \varphi_{xx}, \quad \varphi_{x\phi}, \quad \varphi_{xt}, \quad \varphi_{\phi t},
\end{align*}
\]

where \( x = \cos \theta \), and let the \( Q \) operator be

\[
Q = Q_{rx} \varphi + Q_{r\phi} \varphi_\phi + Q_{xx} \varphi_x + Q_{x\phi} \varphi_\phi + Q_{xt} \varphi_t + Q_{\phi t} \varphi_\phi + Q_{rx} \varphi_r + Q_{r\phi} \varphi_r + Q_{xx} \varphi_x + Q_{x\phi} \varphi_x + Q_{xt} \varphi_t + Q_{\phi t} \varphi_t + 0.
\]

(4)
The determining equations derived from the equation

\[
(a + 2r^{-2} - r^{-2}L + r^{-1}L + \frac{1}{4})Qf = 0
\]

are then found to be

\[
A \cdot Q^0 + \frac{1}{2} Q^r = 0 , \quad A \cdot Q^r + 2r^{-2}Q^r - 4r^{-1}Q^r + 2Q^r = 0 ,
\]

\[
A \cdot Q^x + 2r^{-2}Q^x + 4r^{-2}xQ^r - 4r^{-3}xQ^r + (4r^{-2} + \frac{1}{2})Q^r - 4r^{-3}Q^r + 2r^{-2}y^2 Q^r = 0 ,
\]

\[
A \cdot Q^\phi + 2r^{-2}y^2 Q^\phi + \frac{1}{2} Q^r = 0 , \quad A \cdot Q^t + 2r^{-1}Q^r - 4r^{-2}Q^r = 0 ,
\]

\[
A \cdot Q^{rx} - 4r^{-1}Q^{rx} + 4r^{-2}Q^{rx} + 2r^{-2}yQ^r + 2Q^{rx} = 0 ,
\]

\[
A \cdot Q^{r\phi} - 4r^{-1}Q^{r\phi} + 2r^{-2}Q^{r\phi} + 2r^{-2}Q^r + 2Q^{r\phi} = 0 ,
\]

\[
A \cdot Q^{xx} + 6r^{-2}Q^{xx} + 8r^{-2}xQ^{rx} - 8r^{-3}xQ^{rx} - 2r^{-2}yQ^r + 2r^{-3}yQ^r
\]

\[+ 2r^{-2}yQ^r + 2r^{-2}xQ^r = 0 ,
\]

\[
A \cdot Q^{x\phi} + 2r^{-2}Q^{x\phi} + 4r^{-2}xQ^{r\phi} - 4r^{-3}xQ^{r\phi} + 2r^{-2}y^2 Q^{r\phi} = 0 ,
\]

\[
A \cdot Q^{x\phi} - 2r^{-2}y^2 Q^{x\phi} + 2r^{-3}yQ^r - 4r^{-2}xy Q^{rx} + 4r^{-3}xy Q^{rx} = 0 ,
\]

\[+ 2r^{-2}y(1 + 3x^2)Q^{xx} + 2r^{-2}Q^{x\phi} - 2r^{-2}xy Q^{x\phi} = 0 ,
\]

\[
2r^{-1}Q^{rx} - 2r^{-2}Q^{rx} + 2r^{-2}yQ^r = 0 , \quad 2r^{-1}Q^{r\phi} - 4r^{-2}Q^{r\phi} + 2r^{-2}yQ^{r\phi} = 0 ,
\]

\[
r^{-2}y^2 Q^{rx} + r^{-2}xQ^{rx} + Q^{xx} = 0 , \quad y^2 Q^{rx} + r^{-2}Q^{rx} + y^2 Q^{rx} = 0 ,
\]

\[
Q^{x\phi} + r^{-2}Q^{x\phi} - r^{-2}xy Q^{rx} = 0 , \quad y^2 Q^{rx} + r^{-1}yQ^{rx} + y^2 Q^{rx} + 2Q^{xx} = 0 ,
\]
\[
- y^2 \partial^r \phi + r^{-1} y^2 \partial^r \phi + y^2 \partial^x \phi + x \partial^x \phi + y^{-2} \partial^{xx} \phi = 0,
\]
\[
- y^{-2} \partial^r x + r^{-1} y^{-2} \partial^r x + y^{-2} \partial^x \phi + y^2 \partial^\phi \phi - 2xy^{-4} \partial^{xx} \phi = 0,
\]
\[
- y^{-2} \partial^r \phi + r^{-1} y^{-2} \partial^r \phi + y^{-2} \partial^\phi \phi - xy^{-4} \partial^x \phi = 0, \quad \partial^r r = 0,
\]

where

\[
\Lambda = \partial^r r + 2r^{-1} \partial^r + r^{-2} \partial^2 x + 2r^{-2} \partial^x x + r^{-2} y^2 \partial^2 \phi - r^{-1} \partial^2 \phi r - r^{-1} \partial^\phi \phi t.
\]

Solving these equations, one obtains a 22 parameter generator

\[
\mathcal{O} = \{ A_k \}_{k=1}^{a_0 \phi \phi,}
\]

where the \( a_0 \) are integration constants and the \( \mathcal{O}_k \) are

\[
\begin{align*}
\mathcal{O}_1 &= e^{i\phi} (y \partial_x + ixy^{-1} \partial_x), \quad \mathcal{O}_3 = \partial_x, \\
\mathcal{O}_4 &= 2ie^{i\phi} (xy \partial_x + r^{-1} y^2 \partial^2 x + xy \partial_x \partial_x + r^{-1} y^{-1} \partial_x \partial_x + r^{-1} \partial^2 x \partial_x + r^{-1} \partial^2 \partial_x \\
&\quad + r^{-1} \partial_x \partial_x - 2r^{-1} xy \partial_x - \frac{i}{2} y \partial_x t), \\
\mathcal{O}_5 &= 2i(-y^2 \partial_x + r^{-1} y^2 \partial x + xy \partial \phi - \partial_x - 2r^{-1} x^2 \partial_x - \frac{i}{2} x \partial t), \\
\mathcal{O}_6 &= e^{i\phi} (\partial_x + \partial_t - \frac{i}{2} \partial_x + 1), \quad \mathcal{O}_8 = \partial_t, \\
\mathcal{O}_{10} &= 2ie^{i\phi} \left\{ xy \partial_x + r^{-1} y^2 \partial^2 x + \frac{i}{2} y \partial_x \partial_x + \frac{i}{2} y^{-1} \partial_x \partial_x + \frac{i}{2} \partial_x \partial_x - \frac{i}{2} \partial_x \partial_x \right\}, \\
\mathcal{O}_{12} &= 2ie^{i\phi} \left\{ (1 \pm \frac{1}{2} r) y \partial_x - \frac{1}{2} (4r^{-1} + 1) xy \partial_x + \frac{i}{2} y \partial_x \partial_x + \frac{i}{2} \partial_x \partial_x \right\}, \\
\mathcal{O}_{14} &= 2ie^{i\phi} \left\{ (-y^2 \partial_x + r^{-1} xy \partial_x + r^{-1} y^2 \partial_x \partial_x + xy \partial \phi - \partial_x + 1 \pm \frac{1}{2} r) x \partial x \\
&\quad + \frac{1}{2} (xy \partial_x + 4r^{-1} x) \partial_x + x \partial t - \frac{1}{2} \partial_x \right\}.
\end{align*}
\]
As these are obtained from the transformed equation (3) the corresponding operators, $\tilde{O}_i$, for the original equation (1) are given by

$$\tilde{O}_i = D^{-1} O_i D.$$ 

This transformation can be carried through easily for $O_i (i=1,2,\ldots,6)$ as demonstrated in the appendix I.

Now we analyze those operators. First $\tilde{O}_1$, $\tilde{O}_2$, $\tilde{O}_3$ are expressed in terms of the operators $\tilde{O}_1$, $\tilde{O}_2$ and $\tilde{O}_3$ as listed and $\tilde{O}_2$ is a unit operator. $\tilde{O}_4$ ($i=1,2,\cdots,6$) commute with $\tilde{O}_9$ or the Hamiltonian, and are identified as

$$\tilde{O}_1 = -L_+ , \quad \tilde{O}_2 = L_- , \quad \tilde{O}_3 = iL_z ,$$

$$\tilde{O}_4 = i(A_x + iA_y) , \quad \tilde{O}_5 = i(A_x - iA_y) , \quad \tilde{O}_6 = iA_z$$

where $L_+$, $L_-$, and $L_z$ are the usual angular momentum operators and $\vec{A} = (A_x, A_y, A_z)$ is the well-known Runge-Lenz vector defined by

$$\vec{A} = (-2\pi)^{-3/2} \{ 1/2 (\vec{L} \times \vec{P} - \vec{P} \times \vec{L}) + 2\vec{r}/r \}.$$ 

As is well known, $\vec{A}$ and $\vec{L}$ generate an $\mathfrak{so}(4)$ algebra.

\[ \dagger \text{See appendix I} \]
\( \tilde{Q}_7, \tilde{Q}_8 \) and \( \tilde{Q}_9 \) comprise a closed Lie algebra:

\[
[\tilde{Q}_7, \tilde{Q}_8] = -2i\tilde{Q}_9, \quad [\tilde{Q}_9, \tilde{Q}_7] = i\tilde{Q}_8, \quad [\tilde{Q}_8, \tilde{Q}_9] = -i\tilde{Q}_7.
\]

From these it is clear that \( \tilde{Q}_7 \) and \( \tilde{Q}_8 \) shift the eigenvalue of \( \tilde{Q}_9 \), that is \( n \), by unit amount. They satisfy the relations

\[
\tilde{Q}_7 e^{-iE_n t} \psi_{n^*} = (1 + \frac{1}{n})^2 \frac{E_n}{n+1} \tilde{Q}_9 e^{-iE_n t} \psi_{n^*},
\]

\[
\tilde{Q}_8 e^{-iE_n t} \psi_{n^*} = (1 - \frac{1}{n})^2 \frac{E_n}{n-1} \tilde{Q}_9 e^{-iE_n t} \psi_{n^*},
\]

\[
\tilde{Q}_9 e^{-iE_n t} \psi_{n^*} = n e^{-iE_n t} \psi_{n^*}.
\]

where \( \psi_{n^*} \) is a normalized hydrogenic wave function and \( E_n = -\frac{Z^2}{2n^2} \).

Because of the factor \( \frac{1}{n} \) in the above coefficients, no linear combinations of the operators \( \tilde{Q}_7 \) and \( \tilde{Q}_8 \) are skew adjoint under the usual scalar product

\[
(f, g) = \frac{2\pi}{a} \int_0^\infty \int_0^\pi \int_0^{2\pi} f(r^2 \sin\theta drd\theta d\phi \cdot g(r^2 \sin\theta drd\theta d\phi).
\]

To remove these factors we define the new operators \( \tilde{Q}_1 \) by

\[
\tilde{Q}_1 = (\tilde{Q}_9)^{-2} \cdot \tilde{Q}_1 \cdot (\tilde{Q}_9)^2.
\]

† By using a dummy variable, \( t \), Armstrong obtained the identical generators in his treatment of the radial function. Our result shows that his \( t \) is in fact the physical time.

†† It is clear that the new set \( \{ \tilde{Q}_1 \} \) satisfy the same commutation relations as the set \( \{ \tilde{Q}_j \} \), and as \( \tilde{Q}_1 \) \( (i=1,2,\ldots,6,9) \) commute with \( \tilde{Q}_9 \) we have \( \tilde{Q}_1 = \tilde{Q}_4 \) for \( i=1,2,\ldots,6,9 \).
Using the new operators, we have

\[
\begin{align*}
\tilde{Q}_7 e^{-i\mathbf{P}_n \cdot \mathbf{v}_{n|m}} &= \begin{pmatrix} n+1 \end{pmatrix} \begin{pmatrix} n \end{pmatrix}^{\frac{1}{2}}, e^{-i\mathbf{P}_{n+1} \cdot \mathbf{v}_{n+1|m}} \\
\tilde{Q}_8 e^{-i\mathbf{P}_n \cdot \mathbf{v}_{n|m}} &= \begin{pmatrix} n-1 \end{pmatrix} \begin{pmatrix} n+1 \end{pmatrix}^{\frac{1}{2}}, e^{-i\mathbf{P}_{n-1} \cdot \mathbf{v}_{n-1|m}}
\end{align*}
\]

A straightforward analysis then shows

\[
M_1 = -\frac{1}{2}(\tilde{Q}_7 + \tilde{Q}_8), \quad M_2 = -\frac{i}{2}(\tilde{Q}_7 - \tilde{Q}_8), \quad M_3 = \tilde{Q}_9
\]

are skew adjoint under the above scalar product.

They satisfy the relations of the well known \(O(1,2)\) algebra

\[
[M_1, M_2] = M_3, \quad [M_2, M_3] = M_1, \quad [M_3, M_1] = M_2
\]

The remaining operators can be identified as

\[
\begin{align*}
\tilde{Q}_{10} &= -i[\tilde{Q}_4, \tilde{Q}_7], & \tilde{Q}_{11} &= i[\tilde{Q}_4, \tilde{Q}_8], & \tilde{Q}_{12} &= -i[\tilde{Q}_5, \tilde{Q}_7] \\
\tilde{Q}_{13} &= i[\tilde{Q}_5, \tilde{Q}_8], & \tilde{Q}_{14} &= -i[\tilde{Q}_6, \tilde{Q}_7], & \tilde{Q}_{15} &= i[\tilde{Q}_6, \tilde{Q}_8]
\end{align*}
\]

by the direct calculation. It is clear then that a particular complex extension of the set \(\{\tilde{Q}_i\}\) (i=1,2, ..., 15) comprises the well known \(O(4,2)\) algebra of the dynamical group of the hydrogenlike atom, and that the hydrogenic wave functions form the basis set of a UIR of \(O(4,2)\).
3.5 n-shift operators for hydrogenic radial functions

To illustrate the way in which one can proceed in determining the groups of equations which arise on separation of variables we consider the radial equations for the two and three dimensional Kepler problem.

The radial equations of two and three dimensional hydrogenlike atoms are

\[
(\partial_{r} \partial_{r} + \frac{p}{r} \partial_{r} - q r^{-2} + 2 z r^{-1} + \frac{2}{i} \lambda t) r_{n} e^{-i E_{n} \psi r_{n}} (r) = 0
\]

where

\[
p = 1, q = \hbar^{2}, F_{n} = -\frac{\hbar^{2}}{2} n^{2} - 2 \text{ for 2-dimensions}
\]

or

\[
p = 2, q = \hbar (\varepsilon + 1), F_{n} = -\frac{\hbar^{2}}{2} n^{-2} \text{ for 3-dimensions.}
\]

The appropriate compound dilation operator \( D \) is

\[
D = D_{r} D_{t} = \exp \left[ r \partial_{r} \log \left( -\frac{\hbar^{2} z^{-1} \partial_{r}}{2} \right) \right] \exp \left[ t \partial_{t} \log \left( \frac{\hbar^{2}}{2} \varepsilon - E_{n} \right) \right]
\]

where

\[
H = -\frac{\hbar^{2}}{2} \left( \partial_{r} ^{2} + \frac{p}{r} \partial_{r} - q r^{-2} + 2 z r^{-1} \right).
\]

After the transformation by \( D \), the radial equation becomes

\[
(\partial_{r} \partial_{r} + \frac{p}{r} \partial_{r} - q r^{-2} - i \varepsilon r^{-1} \partial_{t} - \frac{1}{\varepsilon}) f(r, t) = 0
\]

where

\[
f(r, t) = \sum_{n} C_{n} e^{-\frac{r}{2 E_{n}}} (\frac{r}{2 E_{n}})^{\frac{\varepsilon}{2}} R_{n} \left( \frac{r}{2 E_{n}} \right).
\]
We choose the independent functions to be
\[ f, f_t, f_r, f_{rt} \]
and let the \( Q \) operator be of the form
\[ Q = Q_r^r \partial_r + Q_r^t \partial_t + Q^o. \]

The determining equations then are
\[ Q_r^t = 0, \quad Q_r^{rr} - pr^{-1}Q_r^r - ir^{-1}Q_r^t + pr^{-2}Q_r^r + 2Q_r^o = 0 \]
\[ Q_r^{tt} + pr^{-1}Q_r^t - ir^{-1}Q_r^o - ir^{-2}Q_r^r + 2ir^{-1}Q_r^r = 0 \]
\[ Q_r^{oo} + pr^{-1}Q_r^o - ir^{-1}Q_r^o - 2qr^{-3}Q_r^o + 2(qr^{-2} + \lambda)Q_r^o = 0. \]

The solution is
\[ Q = \sum_{i=1}^{4} a^i Q_i \]
where \( a^i \)'s are integration constants, and
\[ Q_1 = \frac{1}{2} e^{\frac{i (\ell t)}{e}} (r \partial_r + i\partial_t - \frac{1}{4}r + \frac{3}{2}p) \]
\[ Q_3 = \partial_r, \quad Q_4 = 1. \]

The \( Q_1 \) and the \( Q_2 \) are the same as the n-shift operators obtained in sections 3 and 4.
3-6. The Rigid Rotator

The Schrödinger equation for the rigid rotator is

\[ \left( L^2 - i \alpha \frac{\partial}{\partial t} \right) \sum_{\ell m} e^{-i \ell (\ell+1) \frac{\pi}{2}} Y_{\ell m}(\theta, \phi) C_{\ell m} = 0 \]  (1)

where \( C_{\ell m} \) are arbitrary constants and

\[ L^2 = -y^2 \frac{\partial^2}{\partial x^2} + 2x \frac{\partial}{\partial x} - y^2 \frac{\partial^2}{\partial \phi^2} \]

with

\[ x = \cos \theta , \ y = \sin \theta . \]

The transformation operator leading to a linear spectrum is seen to be

\[ D = \exp \{ t \alpha \frac{\partial}{\partial t} \log \left( \tilde{\ell} + 1 \right) \} \]  (2)

with

\[ \tilde{\ell} = \frac{1}{2} \left\{ -1 + (1 + 4L^2)^{\frac{1}{2}} \right\} \]

Applying it to (1) yields the transformed equation

\[ \left( L^2 - i (1 \alpha + 1) \frac{\partial}{\partial t} \right) f(\theta, \phi, t) = 0 \]  (3)
where

\[ f(0, \phi, t) = \sum_{l, m} c_{lm} e^{-i\phi t} Y_{lm}(0, \phi) \, . \]

To find the spectrum generating algebra of equation (3), and hence of equation (1), we choose as independent functions

\[ f, f_x, f_{\phi}, f_{\phi x}, f_{\phi t}, f_{tt}, f_{xt}, f_{\phi \phi}, f_{\phi x \phi}, f_{\phi t t}, f_{xx t}, f_{\phi \phi t} \]

and let the Q operator be of the form

\[ Q = Q_{\phi\phi} \partial_{\phi} \partial_{\phi} + Q_{x\phi} \partial_{x} \partial_{\phi} + Q_{\phi x} \partial_{\phi} \partial_{x} + Q_{\phi} \partial_{\phi} + Q_{t} \partial_{t} + Q^{o} \, . \quad (4) \]

Then the determining equations derived by expanding the equation

\[ \{ L^2 - i(\partial_t + 1) \partial_t \} Q f = 0 \]

and by making use of the linear independence of the above functions are

\[
\begin{align*}
Q_{t}^{\phi\phi} &= 0, & Q_{t}^{x\phi} &= 0, & Q_{x}^{x\phi} + xy^{-2}Q_{x}^{x\phi} &= 0, & Q_{\phi}^{x\phi} - y^{-4}Q_{\phi}^{x\phi} &= 0, \\
Q_{x}^{\phi\phi} - xy^{-2}Q_{x}^{\phi\phi} &= 0, & Q_{t}^{t} - y^{-2}Q_{t}^{t} &= 0, & Q_{t}^{x} - xy^{-2}Q_{x}^{x} &= 0, \\
Q_{t}^{x\phi} - y^{-2}Q_{t}^{x\phi} &= 0, & \Lambda Q_{x}^{x\phi} - 2Q_{x}^{x\phi} - 2y^{-2}Q_{\phi}^{x\phi} - 2y^{-2}Q_{x}^{x\phi} &= 0, \\
\Lambda Q_{x}^{\phi\phi} + 2y^{-2}Q_{x}^{x\phi} - 4xy^{-6}Q_{x}^{x\phi} - 2y^{-2}Q_{\phi}^{\phi\phi} &= 0, \end{align*}
\]
\[ A Q - 2Q - 4xQ_x - 4x^2y^2Q_x - 2y^2Q_x = 0, \]
\[ A Q^\phi - 2y^{-2}Q^\phi_\phi = 0, \quad A 0^t_t + 2Q^t_t + 2iQ^t_x + 2ixy^{-2}Q^t_x = 0, \quad A Q^0 = 0 \]

with

\[ A = -y^2_x - 2x\partial_x - y^{-2}\partial_\phi \partial_\phi + 3\partial_t - 13t \cdot \]

The Q operator obtained by solving these equations contains 14 parameters:

\[ Q = (4) = \frac{14}{\sqrt{i}} \sum_{i=1}^{14} a^i Q_i \]

where \( a^i \) are integration constants and \( Q_i \) are given by

\[ Q_1 = e^{\frac{i}{2}t} (y\partial_x - ixy^{-1}\partial_\phi), \quad Q_3 = \partial_\phi \]
\[ Q_4 = e^{-it} (iy^2\partial_x + x\partial_t - ix), \quad Q_5 = e^{it} (-iy^2\partial_x + x\partial_t), \quad Q_6 = \partial_t \]
\[ Q_7 = e^{it} e^{-i\phi} (ixy\partial_x + y^{-1}\partial_\phi + y\partial_t), \]
\[ Q_8 = e^{-it} e^{-i\phi} (-ixy\partial_x + y^{-1}\partial_\phi + y\partial_t - iy), \]
\[ Q_9 = e^{-it} e^{-i\phi} (-ixy\partial_x + y^{-1}\partial_\phi + y\partial_t - iy), \]
\[ Q_{11} = e^{i\phi} (-ixy^{-1}\partial_\phi + y\partial_x \partial_\phi), \quad Q_{13} = \partial_\phi \partial_\phi, \quad Q_{14} = 1 \cdot \]
As these operators are derived by using the transformed equation (3),
the corresponding operators $\hat{Q}_1$ for the original equation (1) have the form

$$\hat{Q}_1 = D^{-1} Q_1 D.$$ 

It is clear that the set \{ $\hat{Q}_1$ \} still satisfy the same commutation relations
as the set \{ $Q_1$ \}. 

We investigate the properties of these operators. First we note
that $Q_{11}$, $Q_{12}$, $Q_{13}$ are expressed in terms of the angular momentum operators
$Q_1$, $Q_2$, $Q_3$ as

$$Q_{11} = Q_1^0 Q_3, \quad Q_{12} = Q_2^0 Q_3, \quad Q_{13} = Q_3^0 Q_3.$$ 

The remaining operators satisfy the following commutation relations:

$$[\hat{Q}_1, \hat{Q}_2] = 2i\hat{Q}_3, \quad [\hat{Q}_3, \hat{Q}_1] = i\hat{Q}_1, \quad [\hat{Q}_3, \hat{Q}_2] = -i\hat{Q}_2,$$

$$[\hat{Q}_4, \hat{Q}_5] = -2i\hat{Q}_0, \quad [\hat{Q}_0, \hat{Q}_4] = i\hat{Q}_4, \quad [\hat{Q}_0, \hat{Q}_5] = -i\hat{Q}_5,$$

$$[\hat{Q}_1, \hat{Q}_5] = \hat{Q}_7, \quad [\hat{Q}_2, \hat{Q}_5] = \hat{Q}_8, \quad [\hat{Q}_1, \hat{Q}_4] = \hat{Q}_9, \quad [\hat{Q}_2, \hat{Q}_4] = \hat{Q}_{10},$$

$$[\hat{Q}_3, \hat{Q}_1] = 0 \quad (i = 0,4,5), \quad [\hat{Q}_0, \hat{Q}_1] = 0 \quad (i = 1,2,3).$$

† As $Q_1$, $Q_2$, $Q_3$ commute with the operator $D$, we have the identities

$\hat{Q}_1 = Q_i$ for $i = 1,2,3$. 

where \( \bar{Q}_0 = -\bar{Q}_6 + \frac{1}{2} \bar{Q}_{14} \).

From these it is clear that \( \bar{Q}_4 \) and \( \bar{Q}_5 \) shift the eigenvalue, \( i(\ell + \frac{1}{2}) \), of \( \bar{Q}_0 \) by unit amount. They are found to satisfy the relations

\[
\begin{align*}
\bar{Q}_4 e^{-i(\ell + 1)t} \chi_{\ell m} &= -i \left( \frac{2(\ell+1)(2\ell+3)}{(\ell+1-m)(2\ell+1+m)} \right)^{1/2} e^{-i(\ell + 1)(\ell + 2)t} \chi_{\ell+1m} \\
\bar{Q}_5 e^{-i(\ell + 1)t} \chi_{\ell m} &= -i \left( \frac{(2\ell+1)(2\ell-1)}{\ell+m} \right)^{1/2} e^{-i(\ell - 1)\ell} \chi_{\ell-1m} .
\end{align*}
\]

Because of the presence of the factors \( (2\ell + 1)(2\ell + 3)^{-1} \) and \( (2\ell + 1)(2\ell - 1)^{-1} \) in these coefficients, no linear combinations of the operators \( \bar{Q}_4 \) and \( \bar{Q}_5 \) are skew-adjoint under the ordinary scalar product

\[
(f, g) = \int_0^{2\pi} \int_0^\pi f^* g \sin \theta \, d\theta \, d\phi .
\]

To construct operators with the proper adjointness we define the new operators

\[
\bar{Q}_4 = (\bar{Q}_0)^{1/2} \bar{Q}_4 (\bar{Q}_0)^{-1/2} .
\]

Then the above equations become

\[
\begin{align*}
\bar{Q}_4 e^{-i(\ell + 1)t} \chi_{\ell m} &= -i \left( \frac{2(\ell+1)(2\ell+3)}{(\ell+1-m)(2\ell+1+m)} \right)^{1/2} e^{-i(\ell + 1)(\ell + 2)t} \chi_{\ell+1m} \\
\bar{Q}_5 e^{-i(\ell + 1)t} \chi_{\ell m} &= -i \left( \frac{(2\ell+1)(2\ell-1)}{\ell+m} \right)^{1/2} e^{-i(\ell - 1)\ell} \chi_{\ell-1m} .
\end{align*}
\]

† As \( \bar{Q}_0, \bar{Q}_1, \bar{Q}_2, \bar{Q}_3 \) commute with \( \bar{Q}_0 \), we have the identities

\[
\bar{Q}_i = \bar{Q}_i \text{ for } i = 1, 2, 3, 0 .
\]
and the following operators are skew-adjoint under the scalar product defined above:

\[ J_{23} = -\frac{1}{2} \sigma_1^2 (\bar{Q}_1 - \bar{Q}_2), \quad J_{31} = -\frac{1}{2} (\bar{Q}_1 + \bar{Q}_2), \quad J_{12} = \bar{Q}_3 \]

\[ J_{34} = -\frac{1}{2} \sigma_4 (\bar{Q}_4 - \bar{Q}_5), \quad J_{45} = \bar{Q}_0, \quad J_{53} = -\frac{1}{2} (\bar{Q}_4 + \bar{Q}_5), \]

\[ J_{24} = \frac{1}{4} (\bar{Q}_7 - \bar{Q}_8 - \bar{Q}_9 + \bar{Q}_{10}), \quad J_{25} = -\frac{1}{4} (\bar{Q}_7 - \bar{Q}_8 + \bar{Q}_9 - \bar{Q}_{10}), \]

\[ J_{14} = \frac{1}{4} (\bar{Q}_7 + \bar{Q}_8 - \bar{Q}_9 - \bar{Q}_{10}), \quad J_{15} = \frac{1}{4} (\bar{Q}_7 + \bar{Q}_8 + \bar{Q}_9 + \bar{Q}_{10}). \]

These operators satisfy the O(3,2) algebra

\[ \left[ J_{ab}, J_{cd} \right] = g_{bc} J_{ad} + g_{ac} J_{bd} - g_{ad} J_{bc} + g_{bd} J_{ac} \]

where \( g_{11} = g_{22} = g_{33} = -g_{44} = -g_{55} = -1 \) and \( (J_{ab})^* = -J_{ab} \).

Therefore the set \( \{ e^{-i\xi (l+1)t} Y_{lm}(0,\phi) \} \) provide a basis for a unitary irreducible representation of O(3,2). Although the dynamical group O(3,2) of the rigid rotator is known, the time dependent Schrödinger equation was not used to derive it. It was shown in paper II that all of our generators are the time-independent or time-dependent constants of the motion.
3-7. SYMMETRIC TOP

The time-dependent Schrödinger equation for the symmetric top is given by 22

\[ \left[ -\frac{1}{2I_1} \left( (1-x^2)\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} + \frac{x^2}{1-x^2} \right) + \frac{I_1}{I_3} \right] \psi = 0, \]

where \( x = \cos \beta \), and \( \alpha, \beta, \gamma \) are the Euler angles which determine the direction of the principal axes with respect to the space fixed coordinate axes. The solutions of this equation are expressed in terms of Wigner's D-functions as

\[ \psi(\alpha,\beta,\gamma,t) = \sum_{jmn} C_{jmn} \frac{\sqrt{2j+1}}{4\pi} D_{jm}^j(\alpha,\beta,\gamma) e^{-iE_j t} = \sum_{jmn} C_{jmn} \psi_{jm}^j(\alpha,\beta,\gamma,t). \]

\[ E_j = \frac{1}{2I_1} j(j+1) + \frac{I_1 - I_3}{2I_1 I_3} n^2 \]

where \( j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots \), \(|m|<j, |n|<j\). To obtain a linear spectrum, we perform a time dilation using the operator defined by

\[ D = \exp(t\partial_t \ln \frac{\sqrt{j+\frac{1}{2}}}{\hbar}) \]

\( \dagger \) We use the same D-function as defined in Edmonds 22 p58, but the eigenstates \( \psi_{jm}^j \) are orthonormal under the SU(2) scalar product given in equation (10). Extensive group theoretical discussion of D-functions will be found in Vilenkin's book. 23
where $H$ is the Hamiltonian, and the operator $J + \frac{1}{2}$ is defined by

$$J + \frac{1}{2} = \frac{1}{2} \left[ 1 + 8I_1 \left( H + \frac{1}{2} \frac{I_1 - I_3}{I_1 I_3} q^2 \right) \right]^{1/2}$$

Under this transformation the equation (1) is transformed into

$$\left( (1-x^2) \frac{d^2}{dx} - 2x \frac{d}{dx} + (1-x^2)^{-1} \frac{d^2}{dy} + 2x(1-x^2)^{-1} \frac{d}{dy} + (1-x^2)^2 - \frac{1}{4} \right) f = A \cdot f = 0$$

where

$$f = \sum_{j_{mn}} C_{j_{mn}} \sqrt{2j + 1} D^j_{mn}(\alpha, \beta, \gamma) e^{-i(j+\frac{1}{2})t}$$

It is clear that the operator $1 \cdot \mathbf{t}$ (not the operator $D \cdot \mathbf{t} \cdot D^{-1}$) has the linear spectrum $j + \frac{1}{2}$ for the transformed eigenstates $D \cdot \mathbf{m} \cdot \mathbf{n}$.

Now we determine the operator $Q$ of the form

$$Q = Q^x \frac{d}{dx} + Q^\alpha \frac{d}{d\alpha} + Q^\beta \frac{d}{d\beta} + Q^\gamma \frac{d}{d\gamma} + Q^\mathbf{t} \cdot \mathbf{t} + Q^0$$

which satisfies the equation

$$A \cdot Q \cdot f = 0$$

where the operator $A$ is defined in the equation (3).
Expanding (5), and choosing the functions

\[ f_{tt}, f_{xt}, f_{at}, f_{\gamma t}, f_{xa}, f_{xy}, f_{aa}, \]

\[ f_{\gamma\gamma}, f_{a\gamma}, f_x, f_a, f_\gamma, f_t, f, \]

for independent functions, we obtain fourteen determining equations:

\[
\begin{align*}
Q_t^t + Q_x^x + x(1-x^2)^{-1} Q_x^x &= 0, & Q_x^x + (1-x^2)Q_t^t &= 0, \\
(1-x^2)Q_{\alpha}^\alpha - Q_{\alpha}^t + xQ_{\gamma}^t &= 0, & (1-x^2)Q_{\gamma}^\gamma - Q_{\gamma}^t + xQ_{\alpha}^t &= 0, \\
Q_{\alpha}^x - xQ_{\alpha}^x + (1-x^2)^2 Q_{x}^x &= 0, & Q_{\gamma}^x - xQ_{\gamma}^x + (1-x^2)^2 Q_{x}^x &= 0, \\
Q_{\alpha}^\alpha - xQ_{\alpha}^\alpha - Q_{\alpha}^x - 2x(1-x^2)^{-1} Q_x^x &= 0, & Q_{\gamma}^\gamma - xQ_{\gamma}^\gamma - Q_{\gamma}^x - 2x(1-x^2)^{-1} Q_x^x &= 0, \\
Q_{\gamma}^\gamma - xQ_{\gamma}^\gamma + Q_{\gamma}^\gamma - Q_{\gamma}^\gamma + 2xQ_{x}^x + (1+5x^2)(1-x^2)^{-1} Q_x^x &= 0, & (A + \frac{1}{4})Q_x^x + 4xQ_x^x + 2(1-x^2)^{-1}(1+x^2)Q_x^x + 2(1-x^2)Q_{\alpha}^o &= 0, \\
(A + \frac{1}{4})Q_{\alpha}^\alpha + 2(1-x^2)^{-1} Q_{\alpha}^0 - 2x(1-x^2)^{-1} Q_{\gamma}^0 &= 0, & (A + \frac{1}{4})Q_{\gamma}^\gamma - 2Q_{\gamma}^0 &= 0, \\
(A + \frac{1}{4})Q_{\gamma}^\gamma + 2(1-x^2)^{-1} Q_{\gamma}^0 - 2x(1-x^2)^{-1} Q_{\gamma}^0 &= 0, & (A + \frac{1}{4})Q_t^t - 2Q_{t}^0 &= 0, \\
(A + \frac{1}{4})Q_{\alpha}^0 + \frac{1}{2} Q_{x}^x + \frac{1}{2}x(1-x^2)^{-1} Q_x^x &= 0.
\end{align*}
\]
Solving these equations, we obtain the solution for $Q$ in the form

$$Q = \sum_{i=1}^{16} a_i^4 Q_i = (4)$$

where the $a_i^4$ are the integration constants, and $Q_i$ are defined by

$$Q_1 = (Q_5)^* = e^{i\alpha} \left\{ (1 - x^2)^{1/2} \partial_x - i(1 - x^2)^{-1/2} \partial_\alpha + i(1 - x^2)^{-1/2} \partial_\gamma \right\}$$

$$Q_2 = (Q_6)^* = e^{i\alpha} \left\{ (1 - x^2)^{1/2} \partial_x - i(1 - x^2)^{-1/2} \partial_\alpha + i(1 - x^2)^{-1/2} \partial_\gamma \right\}$$

$$Q_3 = (Q_7)^* = e^{i\alpha} \left\{ (1 - x^2)^{1/2} \partial_x - i(1 - x^2)^{-1/2} \partial_\alpha + i(1 - x^2)^{-1/2} \partial_\gamma \right\}$$

$$Q_4 = (Q_8)^* = e^{i\alpha} \left\{ (1 - x^2)^{1/2} \partial_x - i(1 - x^2)^{-1/2} \partial_\alpha + i(1 - x^2)^{-1/2} \partial_\gamma \right\}$$

As the $a_i^4$ are arbitrary constants, each $Q_i$ satisfies the equation

$$\text{(5)}$$

independently. To obtain the corresponding operators of the equation $(1)$, we just perform the inverse transformation $D^{-1}$ on each $Q_i$. Then the operators $\tilde{Q}_i = D^{-1} Q_i D$ satisfy

$$\left( H - i\Omega \right) \tilde{Q}_i \psi(\alpha, \beta, \gamma, t) = 0 \quad \text{(6)}$$

where $H - i\Omega$ and $\psi$ are given in the equation $(1)$.

† It is important to notice that the commutation relations among $Q_i$ are not changed under this transformation.
The equation (6) implies the relation

\[
[H - i\alpha_\tau, \hat{Q}_1] = 0 \quad (i = 1, 2, 3, \ldots, 16).
\]

Therefore the \( \hat{Q}_1 \) are the invariants which we are looking for.

The action of these operators on the normalized eigenfunctions \( \psi_{mn}^j \) defined in (2) are given by

\[
\begin{align*}
\hat{O}_{1/2} \psi_{mn}^j &= \sqrt{2} \left( \frac{2j+1}{2j} \right)^{1/2} \{(j + \frac{1}{2}m) \}^{1/2} \psi_{m, n + \frac{1}{2}}^{j} \quad (7) \\
\hat{O}_5 \psi_{mn}^j &= \sqrt{2} \left( \frac{2j+1}{2j} \right)^{1/2} \{(j + \frac{1}{2}m + 1) \}^{1/2} \psi_{m, n + \frac{1}{2}}^{j + 1/2} \quad (8) \\
\hat{O}_{10} \psi_{mn}^j &= \{(j + \frac{1}{2}m) \}^{1/2} \psi_{m, n + \frac{1}{2}}^{j} \quad (9) \\
\hat{O}_{11} \psi_{mn}^j &= \{(j + \frac{1}{2}m + 1) \}^{1/2} \psi_{m, n + \frac{1}{2}}^{j + 1/2} \quad (10) \\
\hat{O}_{13} \psi_{mn}^j &= \im \psi_{mn}^j, \quad \hat{O}_{14} \psi_{mn}^j = \im \psi_{mn}^j, \quad \hat{O}_{15} \psi_{mn}^j = -i (j + \frac{1}{2}) \psi_{mn}^j.
\end{align*}
\]

From these results it is clear that the following operators shift the eigenvalue \( j \) alone by one unit;

\[
\begin{align*}
\hat{Q}_- &= \hat{O}_{1/2}^\dagger = -\hat{O}_{1/2}, \quad \hat{Q}_+ = \hat{O}_{1/2} \quad (9) \\
\hat{Q}_+ &= \hat{O}_{1/2}^\dagger = -\hat{O}_{1/2}, \quad \hat{Q}_- = \hat{O}_{1/2} \quad (10)
\end{align*}
\]
As these give rise to two term recursion relations among the functions $D_{mn}^{j}$, they will be useful for practical purposes.

To elucidate the group theoretical properties of the differential equation (1), we introduce the new operators $\overline{Q}_{i}$ defined by

$$\overline{Q}_{i} = (Q_{15})^{-\frac{1}{2}} Q_{i} (Q_{15})^{\frac{1}{2}}.$$  

This is necessary to obtain skew-adjoint operators.†

Then the operators $C_{j}^{i}$ defined by

$$\begin{align*}
C_{2}^{1} & = -\overline{Q}_{9}, & C_{2}^{2} & = \overline{Q}_{10}, & C_{3}^{1} & = \frac{1}{\sqrt{2}} \overline{Q}_{9}, & C_{3}^{3} & = \frac{1}{\sqrt{2}} \overline{Q}_{4}, \\
C_{3}^{4} & = \frac{1}{\sqrt{2}} \overline{Q}_{6}, & C_{4}^{1} & = \frac{1}{\sqrt{2}} \overline{Q}_{2}, & C_{3}^{2} & = \frac{1}{\sqrt{2}} \overline{Q}_{5}, & C_{2}^{3} & = \frac{1}{\sqrt{2}} \overline{Q}_{1}, \\
C_{4}^{2} & = -\frac{1}{\sqrt{2}} \overline{Q}_{7}, & C_{4}^{4} & = -\frac{1}{\sqrt{2}} \overline{Q}_{3}, & C_{3}^{3} & = -\overline{Q}_{11}, & C_{3}^{4} & = \overline{Q}_{12}, \\
C_{1}^{1} & = 1(\overline{Q}_{13} - \overline{Q}_{15}), & C_{2}^{2} & = 1(-\overline{Q}_{13} - \overline{Q}_{15}), \\
C_{3}^{3} & = 1(\overline{Q}_{14} + \overline{Q}_{15}), & C_{4}^{4} & = 1(- \overline{Q}_{14} + \overline{Q}_{15})
\end{align*}$$

† The factors $(Q_{15})^{-\frac{1}{2}}$ and $(Q_{15})^{\frac{1}{2}}$ in this expression remove the factors

$$(\frac{2}{2(11)+1})^{\frac{1}{2}}$$

of the coefficients in equations (7) and (8). However, it is clear that the set $\{\overline{Q}_{i}\}$ still satisfy the same commutations as the sets $\{\overline{Q}_{i}\}$ and $\{Q_{i}\}$. 
satisfy the SL(4,R) algebra:

\[ [c^k_j, c^l_k] = \delta^l_k c^k_j - \delta^k_l c^l_j \]

and following linear combinations of the \( c^i_j \)

\[
\begin{align*}
X^k_k &= 1 c^k_k; \quad k = 1, 2, 3, 4 \\
X^\ell_k &= 1 (c^\ell_k + c^\ell_k), \quad X^\ell_k = c^\ell_k - c^\ell_k; \quad k < \ell \text{ with } k, \ell = 1, 2 \\
X^k_\ell &= 1 (c^k_\ell + c^k_\ell), \quad X^k_\ell = c^k_\ell - c^\ell_k; \quad k < \ell \text{ with } k, \ell = 3, 4 \\
X^k_\ell &= 1 (c^k_\ell - c^\ell_k), \quad X^k_\ell = -c^\ell_k - c^k_\ell; \quad k = 1, 2, \ell = 3, 4
\end{align*}
\]

satisfy the commutation relations of \( SU(2,2) \), and are skew-adjoint under the \( SU(2) \) scalar product

\[
(f, g) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f^* f \sin \theta \, d\alpha \, d\beta \, d\gamma 
\]  

(10)

Thus it is clear that the set \( \{ y^j_m (\alpha, \beta, \gamma, \epsilon) | j = 0, \frac{1}{2}, 1, \ldots, -j < m, n < j \} \) which are the matrix elements of the regular representation of \( SU(2) \) provide a basis for a unitary irreducible representation of \( SU(2,2) \).

\[ \text{† A general discussion of the group } U(p,q) \text{ in relation to the group } GL(p+q,R) \text{ will be found in the paper by}
\]

R.L. Anderson et. al., \textit{24}
The SU(2,2) group contains a variety of subgroups. Here we investigate the
SU(2)XSU(2) subgroup generated by the operators $\tilde{Q}_i$ (i=9,10, . . . , 14). For
the case of the spherical top ($I_1 = I_2 = I_3$), we have the identities $\tilde{Q}_i = Q_i$
(i = 9, 10, . . . , 14) because the $Q_i$ commute with both the dilation operators
D and the operator $Q_{15}$. One can easily check that they also commute with the
Hamiltonian. Therefore, they comprise the well-known SU(2)XSU(2) degeneracy
group of the spherical top. On the other hand, for the case of a symmetric
top ($I_1 = I_2 \neq I_3$), we have the identities $\tilde{Q}_i = Q_{11}$ only for i = 9, 10, 13, 14,
and therefore $Q_i$ commute with the Hamiltonian, so that degeneracy group will be
SU(2)XU(1). The $\tilde{Q}_{11}$ and $\tilde{Q}_{12}$ are no longer time-independent constants of the
motion. Here we have obtained an important result: Under the symmetry breaking
$I_1 = I_2 = I_3 + I_1 = I_2 \neq I_3$ two of the time-independent constants of the motion,
$Q_{11}$ and $Q_{12}$, of the spherical top turn into the time-dependent ones. This is an
example of a more general phenomenon which will be discussed in detail in a
future communication.

So far, we have found that the eigenstates $\{\psi_{j m n}^{j} (\alpha, \beta, \gamma, t)\}, j = 0, \frac{1}{2}, 1, . . . ,
-j \leq m \leq j$, which comprise the regular representation of SU(2), also form the
basis for a unitary irreducible representation of SU(2,2). However, it is clear
that physically the integer and half-odd integer states cannot be mixed. If the
mixing is allowed, the probability amplitude $|\psi|^2$ is no longer invariant under
a rotation of 360°. Therefore, if there exists some observable which causes
the mixing of integer and half odd integer states, one type of state has to be
eliminated. If we restrict ourselves to the integer or half odd-integer states,
then SU(2,2) is no longer the dynamical group in the ordinary sense. In this
case the dynamical group may be generated by the operators
$\{\tilde{O}_+, \tilde{O}_-, \tilde{O}_9, \tilde{O}_{10}, \tilde{O}_{11},
\tilde{O}_{12}, \tilde{O}_{13}, \tilde{O}_{14}, \tilde{O}_{15}\}$ where the operators $\tilde{O}_+$ and $\tilde{O}_-$ are defined by
\[ \bar{Q}_\pm = (\bar{Q}_{15})^{\frac{1}{2}} Q_\pm (\bar{Q}_{15})^{-\frac{1}{2}} \]

where the \( Q_\pm \) are given by (9). However, these operators do not close under a finite number of commutation operations, that is, they generate an infinite dimensional Lie Algebra.
3.8 Solid Harmonics \( r^\ell Y_{\ell m}(\theta, \phi) \)

So far we have used time dependent basis functions and it was essential to obtain a dynamical group which involves only the physical variables. This, however, is not a necessary condition for obtaining a spectrum generating algebra of the special functions. We can use a different type of basis functions by introducing a purely artificial variable for this purpose. We illustrate this alternative by using spherical harmonics.

The spherical harmonics are the solutions of the differential equation

\[
(y^2 \frac{\partial}{\partial x} \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} + \ell (\ell + 1)) \sum_m C_{\ell m} Y_{\ell m}(\theta, \phi) = 0,
\]

where \( x = \cos \theta \), \( y = \sin \theta \). This equation can be rewritten as

\[
(y^2 \frac{\partial}{\partial x} \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} + r^2 r (r \frac{\partial}{\partial r} \frac{\partial}{\partial r} + 1)) \sum_{\ell, m} C_{\ell m} Y_{\ell m}(\theta, \phi) r^\ell = 0
\]

by using the solid harmonics \( r^\ell Y_{\ell m}(\theta, \phi) \) as the basis functions. It is clear that this is the Laplace equation in polar coordinates. To simplify the problem we replace \( \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} \) by \( -m^2 \) to get

\[
(y^2 \frac{\partial}{\partial x} \frac{\partial}{\partial x} - y^2 m^2 + r^2 r (r \frac{\partial}{\partial r} \frac{\partial}{\partial r} + 1)) f(r, \theta, \phi) = 0
\]

with

\[
f(r, \theta, \phi) = \sum_{\ell} C_{\ell} r^\ell Y_{\ell m}(\theta, \phi)
\]
We seek a $Q$ operator of the form

$$Q = Q^x_x + Q^r_r + Q^0_0.$$  \hspace{1cm} (1)

By choosing $f, f_x, f_r, f_{xr}$ and $f_{rr}$ for the independent functions, we get the determining equations:

\begin{align*}
    r^2 Q^x_r + y^2 Q^r_x &= 0, \\
    Q^x_x + xy y^2 Q^x_r - Q^r_r + r^{-1} Q^r_r &= 0, \\
    y^2 Q_{xx} - 2xy Q^r_x + r^2 Q^r_{rr} + 2r Q^r_r - 2Q^r_x - 4r y^2 Q^x_r - 4xy y^2 Q^r_r + 2r^2 Q^r_{rr} &= 0, \\
    y^2 Q_{xx} + 2xy Q^r_x + r^2 Q^r_{rr} + 2r Q^r_r + 2(1 + x^2) y^2 Q^x_r + 2y^2 Q^r_r &= 0, \\
    y^2 Q_{xx} - 2xy Q^r_x + r^2 Q^r_{rr} + 2r Q^r_r + 2m^2 y^2 Q^x_r + 4m^2 xy y^2 Q^r_r &= 0.
\end{align*}

The solutions of these equations contain four free parameters and the $Q$ is determined as

$$Q = \sum_{i=1}^{4} a_i Q_1 = (1)$$

with

$$Q_1 = -y^2 r^3 x + r^2 x^3 r + rx, \hspace{0.5cm} Q_2 = -y^2 r^{-1} x - x^3 r,$$

$$Q_3 = r^3 r, \hspace{0.5cm} Q_4 = 1.$$

It is convenient to define a new operator $Q_0 = Q_3 + \frac{1}{2} Q_4$.

The $Q_0$, $Q_1$, and $Q_2$ then satisfy the relations

$$[Q_0, Q_1] = Q_1, \hspace{0.5cm} [Q_0, Q_2] = -Q_2, \hspace{0.5cm} [Q_1, Q_2] = 2Q_0.$$  \hspace{1cm} (2)

These imply that $Q_1$ and $Q_2$ shift the eigenvalue of $Q_0$, that is $\{ Q_0 \}$, by one.

The action on an eigenfunction is
If we define new operators $\overline{Q}_1$ by

$$\overline{Q}_1 y_{\ell m} = -i(\ell+1+m)(\ell+1-m)^{1/2} y_{\ell+1m},$$

$$\overline{Q}_2 y_{\ell m} = -i(\ell+m)(\ell-m)^{1/2} y_{\ell-1m},$$

$$\overline{Q}_0 y_{\ell m} = i(\ell+1)y_{\ell m},$$

and the linear combinations

$$J_{12} = \overline{Q}_0, \quad J_{23} = -\frac{i}{2}(\overline{Q}_1 - \overline{Q}_2), \quad J_{31} = -\frac{i}{2}(\overline{Q}_1 + \overline{Q}_2),$$

are skew adjoint under the scalar product

$$\langle y_{\ell m} , y_{\ell' m'} \rangle = \int_0^{2\pi} \int_0^{2\pi} (y_{\ell m}^* y_{\ell' m'}) \sin \theta \, d\phi \, d\theta \quad (2)$$

The $J_{12}$, $J_{23}$ and $J_{31}$ satisfy the commutation relations of an $O(2,1)$ algebra

$$[J_{23}, J_{31}] = J_{12}, \quad [J_{31}, J_{12}] = -J_{23}, \quad [J_{12}, J_{23}] = -J_{31}.$$  

The Casimir operator of this algebra has an eigenvalue $m^2 - \frac{1}{4}$.

Therefore the set $\{y_{\ell m}(r,\theta,\phi)\}$ with fixed $m$ form a basis of a UIR of $O(2,1)$, and $m$ specifies the representation.
Now to obtain the full spectrum generating algebra of the solid harmonics we add the Cartesian components $L_x^1$, $L_x^2$, $L_x^3$ of the angular momentum operator which allow the n-shift operators, where $x^1 = rsin\theta cos\phi$

$x^2 = rsin\theta sin\phi$. $x^3 = rcos\theta$. By direct calculation one finds a set

$[\bar{0}_1, \bar{0}_2, \bar{0}_0, iL_x^1, iL_x^2, iL_x^3, [\bar{0}_1, iL_x^1], [\bar{0}_1, iL_x^2], [\bar{0}_2, iL_x^1], [\bar{0}_2, iL_x^2]$ form a closed Lie algebra. It is straightforward to show that they comprise an $O(3,2)$ algebra under the scalar product defined by (2). We therefore conclude that the set $\{v^m_{\lambda}(r,0,\phi)\}$ comprise the basis of a UIR of an $O(3,2)$.

It is helpful to express these generators in terms of Cartesian coordinates $(x^1, x^2, x^3)$ to understand the geometrical meaning of the generators. They are expressed as

$$Q_1 = 2x^3(x^1\partial_1 + x^2\partial_2 + x^3\partial_3) - \vec{\partial} \cdot \vec{\partial} + x^3$$

$$Q_2 = -\partial_3, \quad Q_0 = x^1\partial_1 + x^2\partial_2 + x^3\partial_3 + \frac{1}{2},$$

$$iL_x^1 = x^1\delta_{jk} - x^k\partial_j \quad (i,j,k = 1,2,3\ \text{cyclically}),$$

$$[Q_1 , iL_x^1] = -2x^2(x^1\partial_1 + x^2\partial_2 + x^3\partial_3) + \vec{\partial} \cdot \vec{\partial} - x^2,$$

$$[Q_1 , iL_x^2] = 2x^1(x^1\partial_1 + x^2\partial_2 + x^3\partial_3) - \vec{\partial} \cdot \vec{\partial} + x^1,$$

$$[Q_2 , iL_x^1] = \partial_2, \quad [Q_2 , iL_x^2] = -\partial_1 .$$

Here we have used the original $Q_1$ instead of skew-Hermitian generator $\bar{Q}_1$. It is clear that this ten member algebra contains an Euclidean group algebra in three dimensions.

In light of the discussion at the beginning of this paragraph, we see that in effect we have shown that Laplace equation in 3-dimensions has the symmetry $O(3,2)$. Further, we can treat the n-dimensional Laplace equation corresponding to an indefinite metric with the techniques discussed here.
We emphasize that from the point of view of the theory of special functions both this determination and the one in Section 3-6 are essentially equivalent in that both yields a spectrum generating algebra for the spherical harmonics, but if one is seeking operators that map solutions of the Schroedinger equation onto solutions of the same equation, we have to take the first determination.
3-9. \( \varepsilon \)-shift operators for the hydrogenic and harmonic oscillator wave functions

It seems that no one has obtained \( \varepsilon \)-shift operators which satisfy the relations

\[
\frac{\partial}{\partial t} \psi_{n \ell m} = \omega(n \ell m) \psi_{n \ell +1 m} \tag{1}
\]

either for the hydrogenic or the harmonic oscillator wave function \( \psi_{n \ell m} \). In this section we derive such operators.

As in the preceding problems, first we transform the eigenfunctions into that form for which the operator \( i \partial_t \) has a linear spectrum. The time-dependent eigenfunctions for a coulombic or a harmonic oscillator central potential are written as

\[
\psi_{n \ell m} = e^{-i F n \ell} R_n(r) Y_{\ell m}(\theta, \phi) \tag{2}
\]

The linearization of the spectrum of \( i \partial_t \) with respect to the eigenvalue \( \varepsilon \) will be achieved by the operator

\[
D = \exp( t \partial_t \ln \frac{\varepsilon + \lambda}{N} ) \tag{3}
\]

where the \( N \) is the Hamiltonian and the operator \( \lambda \) is defined by

\[
\frac{\varepsilon}{2} + \frac{1}{2} = \frac{1}{2} \sqrt{1 + 4L^2} \tag{4}
\]

with \( L^2 = L_x^2 + L_y^2 + L_z^2 \).
The transformed wave functions have the form
\[ \phi_{n\ell m} = D \psi_{n\ell m} = e^{-i\left(\ell + \frac{1}{2}\right)t} R_{n\ell}^{\theta}(r) Y_{\ell m}(\theta, \phi) \] (5)
for which the operator \( i\partial_t \) has a linear spectrum.
Now we use the results obtained in the preceding section.

In Section 3.5, we derived operators
\[ Q_{4,5} = e^{-i t \left(\ell + \frac{1}{2}\right)} \left( -i \sin \theta \partial_\theta + \cos \theta \partial_\phi - \frac{1}{2} \cos \theta \right) \] (6)
which satisfy the relations
\[ Q_{4,5} e^{-i(\ell + \frac{1}{2})t} Y_{\ell m} = p_+(\ell, m) e^{-i(\ell + \frac{1}{2} \pm 1)t} Y_{\ell \pm 1 m} \] (7)
where \( p_\pm(\ell, m) = -i \left( \frac{b}{b \pm \frac{1}{2}} \right)^{1/2} \left( (b + c)(b \pm c \pm 1) \right)^{1/2} \)
with \( b = \ell + \frac{1}{2}, \ a = m - \frac{1}{2} \). Suppose that we have derived the operators \( Q_1 \) which have the properties
\[ Q_1 e^{-i(\ell + \frac{1}{2})t} R_{n\ell}(r) = q_+(n, \ell) e^{-i(\ell + \frac{1}{2} \pm 1)t} R_{n\ell \pm 1}(r) \].

\( \dagger \) Because the \( Q_4 \) and \( Q_5 \) in section 3.5 act on the basis \( \{ e^{-i\ell t} Y_{\ell m} \} \), whereas the \( Q_{4,5} \) act on \( \{ e^{-i(\ell + \frac{1}{2})t} Y_{\ell m} \} \), the \( Q_4 \) and \( Q_5 \) have the different forms than \( Q_4 \) and \( Q_5 \). The latter are obtained from the former simply by replacing the operator \( i\partial_t \) by \( i\partial_t - \frac{1}{2} \).
This is because the \( i\partial_t \) and the \( i\partial_t - \frac{1}{2} \) have eigenvalue \( \ell \) for the basis \( \{ e^{-i\ell t} Y_{\ell m} \} \) and \( \{ e^{-i(\ell + \frac{1}{2})t} Y_{\ell m} \} \) respectively.
Then the operators
\[ \Omega_\pm = \frac{Q}{2} \pm \frac{it}{2}, \]  
will have the properties
\[ \Omega_\pm e^{-i(\lambda + \frac{1}{2})t} R_{n\lambda} Y_{\lambda} \pm \Omega_\pm e^{-i(\lambda + \frac{1}{2} \pm 1)t} R_{n\lambda \pm 1} Y_{\lambda \pm 1}. \]  

To obtain the \( \lambda \)-shift operators \( \Omega_{\lambda} \) for the functions \( e^{-i(\lambda + \frac{1}{2})t} R_{n\lambda} \), we first notice that the function

\[ \sum_{\lambda} C_{\lambda} e^{-i(\lambda + \frac{1}{2})t} R_{n\lambda} \quad (C_{\lambda} = \text{arbitrary constants}) \]

is a solution of the differential equation
\[ \left\{ -\frac{1}{2} \frac{\partial^2}{\partial r^2} - \frac{1}{2} \frac{\partial}{\partial r} + \frac{1}{2} \frac{\partial^2}{\partial t^2} - \frac{1}{4} + V(r) - E \right\} f(r,t) = 0 \]  
where \( V(r) \) is the potential and the \( R_{n\lambda} \) is the energy. Now we use our systematic method to determine the invariants of this equation. Assuming the form
\[ Q = Q^r \partial_r \partial_t + Q^t \partial_r \partial_t + Q^r \partial_r + Q^t \partial_t + Q^0 \]

\[ \dag \] This equation is easily obtained if we notice that in the transformed space \( \{ e^{-i(\lambda + \frac{1}{2})t} R_{n\lambda} Y_{\lambda}\} \), \( L^2 \) can be replaced by \( -\frac{\partial^2}{\partial t^2} - \frac{1}{4} \).
for the Q operator, and choosing

\[ f, f_r, f_t, f_{tt}, f_{rt}, f_{ttt} \]

for independent functions, we obtain the following determining equations:

\[
\begin{align*}
q_{rt}^r - q_{tt}^r - r q_{tt}^t &= 0, \\
2 r q_{tt}^r + q_{tt}^t &= 0,
\end{align*}
\]

\[
\begin{align*}
q_{rr}^t + 2 r^{-1} q_{tt}^r + r^{-2} q_{tt}^t + 2 r^{-2} q_{tt}^r - 2 r^{-2} q_{tt}^t + 2 r^{-3} q_{tt}^r &= 0, \\
q_{rr}^r - 2 r^{-1} q_{tt}^r + r^{-2} q_{tt}^t + 2 r^{-2} q_{tt}^r + 2 r^{-2} q_{tt}^t + 2 q_{tt}^c &= 0, \\
q_{rr}^c + 2 r^{-1} q_{tt}^r + r^{-2} q_{tt}^t + 2 r^{-2} q_{tt}^r - q_{tt}^r + 2 g q_{tt}^r &= 0, \\
q_{rr}^o + 2 r^{-1} q_{tt}^r + r^{-2} q_{tt}^t - q_{tt}^r - 2 g q_{tt}^r &= 0
\end{align*}
\]

where \( g = \frac{1}{4} r^{-2} - 2V(r) + 2E_n \).

We investigate two cases separately.

(1) Coulombic potential: \( V(r) = -\frac{Z}{r} \)

For the case of the Coulombic potential, the determining equations
give rise to the solution

\[ Q = (11) = \sum_{i=1}^{5} a_i Q_i \]

where the \( a_i \) are integration constants and the \( Q_i \) are

\[ Q_1 = e^{\frac{-i\lambda t}{2}} (\lambda^2 t + i r - \frac{1}{2} t + \frac{1}{2} r - \frac{1}{4} + iz) \]

\[ Q_3 = a_3 t, \quad Q_4 = a_4 t, \quad Q_5 = 1. \]

The operators \( Q_+ = \frac{1}{\sqrt{2\pi n}} Q_1 \) have the expected properties, that is

\[ Q_+ e^{-i(\xi + \frac{1}{2}) t} R_n^\xi = +i((n - \xi - 1)(n + \xi + 1))^\frac{1}{2} e^{-i(\xi + \frac{1}{2}) t} R_{n+1}^\xi \]

\[ Q_+ e^{-i(\xi + \frac{1}{2}) t} R_n^\xi = -i((n + \xi)(n - \xi))^\frac{1}{2} e^{-i(\xi + \frac{1}{2} - 1) t} R_{n-1}^\xi , \]

or, by defining \( a = n - \frac{1}{2} , \ b = \xi + \frac{1}{2} , \ f_{ab} = \frac{1}{\sqrt{4\pi}} e^{-i(\xi + \frac{1}{2}) t} R_n^\xi \)

\[ Q_+ f_{ab} = +i((a - \xi)(a + \xi + 1))^\frac{1}{2} f_{ab+1} \]

where the \( R_n^\xi \) is the normalized radial function

\[ R_n^\xi = \frac{2}{n^2(2\xi + 1)!} \left( \frac{(n + \xi)!}{(n - \xi - 1)!} \right)^\frac{1}{2} \frac{2r^\xi}{n} e^{-\frac{r}{n}} F(-n + \xi + 1, 2\xi + 2, \frac{2r}{n}) . \]
Before we construct the $\xi$-shift operators for the total eigenfunctions $\psi_{n\xi m}$ defined by (2), we investigate the group generated by the operators $Q_+$, $Q_-$, and $Q_3$. From (12) it is clear that the ladder terminates at $b = \pm a$, not only for the half odd integer $a$ but also for the integer $a$. We also note that the operator $Q_-$ creates the eigenfunctions $f_{ab}$ with negative values of $b$, and, therefore, the ladders involving the actual radial functions also contain the physically non-existing radial functions with negative values of $\xi$.

The functions $f_{ab}$ with integer or half odd integer $a,b$ satisfy the orthonormality relation

$$
(f_{ab}, f_{cd}) = \delta_{ac} \delta_{bd} \quad (a, c = 0, \frac{1}{2}, 1, \ldots)
$$

for the scalar product

$$
(f, g) = \int_{-2\pi}^{2\pi} \int_0^\infty f(r) g(r) r^2 dr dt.
$$

Hence, in the space spanned by the basis $f_{ab}$, the operators

$$
J_1 = \frac{i}{2} (Q_+ + Q_-), \quad J_2 = \frac{1}{2} (Q_+ - Q_-), \quad J_3 = \mathbb{I}
$$

which satisfy the commutation relations of an SU(2) algebra

$$
$$
are skew-adjoint under the scalar product defined above. Of course, the Casimir operator $J^2 = J^2_1 + J^2_2 + J^2_3$ has the eigenvalues $-a(a+1)$. Therefore, it is obvious that the set $\{f_{ab}\}$ with integer and half odd integer $a$ provides a basis for a $2a+1$ dimensional unitary irreducible representation of $SU(2)$.

Finally, we may obtain the shift operators of $\ell$ for the total wave functions $\Psi_{n\ell m} = (2\ell)$. The operators $\Omega^\pm_{\ell}$ of (8) shift the $\ell$ of the transformed wave function $\phi_{n\ell m} = (5)$. Using the results (9), (7), (12), we obtain the following relations;

$$\Omega^\pm_{\ell} \phi_{n\ell m} = \mp \sqrt{2\ell+1} \left( \frac{2\ell + 1}{2\ell + 1 + 2} \right)^{\frac{1}{2}} \{ (a+b)(a+b+1)(b+c)(b+c+1) \}^{\frac{1}{2}} \phi_{n\ell m}$$

(14)

where $a = n - \frac{1}{2}$, $b = \ell + \frac{1}{2}$, $c = m + \frac{1}{2}$. Also we have

$$\Omega_0 \phi_{n\ell m} = i\ell \phi_{n\ell m} = (\ell + \frac{1}{2}) \phi_{n\ell m}.$$  

(15)

Just by inspecting the coefficients of (14), it is obvious that the ladder terminates at $a = h$ and at $b = \frac{1}{2} + |c + \frac{1}{2}|$, that is, at $\ell = n - 1$, $m = |m|$. Therefore, the operators $\Omega^\pm_{\ell}$ do not produce "ghost" states. On the other hand, $\Omega_+$, $\Omega_-$ and $\Omega_0$ do not form a closed Lie algebra. To finish the discussion we obtain the $\ell$ shift operators $\Delta_{\ell}$ and the labeling operator $\Delta_0$ for the actual wave functions $\Psi_{n\ell m} = (2\ell)$.  

These will be obtained simply by acting with the inverse operator of the $D$ defined by (3) on the $\Omega_+$, $\Omega_-$ and $\Omega_0$, that is,

$$
\tilde{\Omega}_+ = D^{-1} \Omega_+ D, \quad \tilde{\Omega}_0 = D^{-1} \Omega_0 D = \tilde{\nu} + \frac{1}{2} \quad \dagger.
$$

However, no linear combinations of $\tilde{\Omega}_+$ and $\tilde{\Omega}_-$ give self-adjoint operators. To obtain self-adjoint operators we perform the transformation

$$
\tilde{\Omega}_+ = i (\tilde{\Omega}_0)^{-\frac{1}{2}} \tilde{\Omega}_+ (\tilde{\Omega}_0)^{-\frac{1}{2}}.
$$

The $\tilde{\Omega}_+$ satisfy the relations

$$
\tilde{\Omega}_+ \Psi_{\nu m} = \pm i \sqrt{2 \nu + 1} \{ (a^+ b)(a^+ b + 1)(b^+ c)(b^+ c + 1) \}^{\frac{1}{2}} \Psi_{\nu m}\quad \dagger\dagger
$$

and the operators

$$
\tilde{\Omega}_1 = \tilde{\Omega}_+ + \tilde{\Omega}_-, \quad \tilde{\Omega}_2 = i (\tilde{\Omega}_+ - \tilde{\Omega}_-)
$$

are self-adjoint under the scalar product

$$
(f, g) = \int_0^\infty \int_0^\Pi \int_0^{2\pi} f^* g r^2 \sin \theta \, d\phi d\theta dr. \quad (17)
$$

$\dagger$ This is because we have $D^{-1} \Omega_0 D = \tilde{\nu} + \frac{1}{2}$, which holds in the space spanned by the functions $\Psi_{\nu m}$.

$\dagger\dagger$ One can easily eliminate the factor $\sqrt{2 \nu + 1}$ in the coefficients introducing the operators $(-2\nu)^{-\frac{1}{2}} \tilde{\Omega}_\pm$.
Because \( \tilde{\Omega}_1 \) and \( \tilde{\Omega}_2 \) do not mix different energy states, they are time-independent constants of the motion.

The operators \( Q_1 \) and \( Q_2 \) of (12) were obtained by D. Herrick and O. Sinanoglu \(^{27}\) by introducing a dummy variable. But we would like to emphasize that because our \( t \) is a physical variable the operators \( \tilde{\Omega}_1 \) and \( \tilde{\Omega}_2 \) which are constructed from the \( Q_1 \) and the \( Q_2 \) have a definite physical meaning; while physical meaning can not be attached to operators involving dummy variables. It is particularly interesting to find the relationship between the set \( \tilde{\Omega}_+, \tilde{\Omega}_-, \tilde{\Omega}_0 \) and the generators of the well-known \( SO(4) \) degeneracy group of the hydrogenic atom. This can be done by comparing the matrix elements of the \( \tilde{\Omega}_+, \tilde{\Omega}_-, \tilde{\Omega}_0 \) to those of the \( SO(4) \) generators.

The calculations of the relation are given in Appendix III. The results are following:

\[
(-2H)^{-1/2} (\tilde{\Omega}_+ + \tilde{\Omega}_-) = 2A \tilde{\Omega}_0
\]

\[
(-2H)_{L_z}^{1/2} \tilde{\Omega}_- = A_+ L^- (\tilde{\Omega}_o - L_z - \frac{1}{2}) + A_- L^+ (\tilde{\Omega}_o + L_z - \frac{1}{2}),
\]

\[
2 (-2H)_{L_z}^{-1/2} \tilde{\Omega}_+ = A_+ L^- (\tilde{\Omega}_o + L_z + \frac{1}{2}) + A_- L^+ (\tilde{\Omega}_o - L_z + \frac{1}{2}),
\]

where

\[
A = (-2H)^{-1/2}(\frac{1}{2}(1 \cdot 1 - \hat{\imath} \times \hat{\imath}) + \hat{\imath} \cdot \hat{\imath}) \quad \hat{\imath} \times \hat{\imath},
\]

\[
A_+ = \hat{\imath} \times A_0, \quad L_+ = \hat{\imath} \times L_0.
\]
(2) Harmonic Oscillator Potential \( V = \frac{1}{2}kr^2 \)

The determining equations for the potential \( V = \frac{1}{2}kr^2 \) provide the following solution for \( Q \):

\[
Q = \sum_{i=1}^{5} a^i Q_i
\]

where the \( a^i \) are integration constants and the \( Q_i \) are

\[
Q_1 = e^{\mp 2it} (r^{-\frac{1}{2}} \frac{\partial}{\partial r} + ir^{-\frac{3}{2}} \frac{\partial}{\partial t} + r^{-\frac{1}{2}} \frac{\partial}{\partial r} - \frac{1}{2} r^{-1} \frac{\partial^2}{\partial t^2} + \frac{1}{2} ir^{-2} + iE_n ),
\]

\[
Q_2 = a^2_t, \quad Q_3 = a_t^3, \quad Q_4 = 1.
\]

To eliminate the explicit dependence of these generators upon \( E_n \), we replace \( E_n \) by the operator

\[
- \frac{1}{2} \frac{\partial^2}{\partial r^2} + \frac{1}{2} \frac{\partial^2}{\partial t^2} + \frac{1}{4} \frac{\partial^2}{\partial r^2} + \frac{1}{2} r \frac{\partial}{\partial r} + \frac{1}{2} r \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial t^2} + \frac{3}{2} ir^{-2} + \frac{1}{2} ir^{-2} + iE_n
\]

which has the eigenvalue \( E_n \) for the state \( e^{-\frac{1}{2}(b + \frac{1}{2})t} \) \( r_n^2 \) (see (10)).

Then \( Q_1 \) and \( Q_2 \) become

\[
Q_1 = e^{\mp 2it} \left( \pm \frac{1}{2} r \frac{\partial}{\partial r} + r^{-\frac{1}{2}} \frac{\partial}{\partial t} + \frac{1}{2} ir^{-2} \frac{\partial^2}{\partial t^2} - \frac{1}{2} r^{-1} \frac{\partial^2}{\partial r^2} + \frac{1}{2} ir^{-2} + \frac{1}{2} ir^{-2} + \frac{1}{2} ir^{-2} \right).
\]

(19)
The operators $Q_\pm = \frac{1}{2} Q_1$ have the properties

$$Q_\pm f_{ab} = \mp \iota \iota (a \pm b)(a \mp b + 1)^{1/2} f_{ab} \pm 1$$

(20)

where $a = \frac{1}{2} n + \frac{1}{4}$, $b = \frac{1}{2} \ell + \frac{1}{4}$, and

$$f_{ab} = (2\pi)^{-\frac{1}{2}} e^{-i(\ell + 1)t} r_n \ell, \quad r_n = \frac{\Gamma\left(\frac{1}{2} n + \frac{1}{2} \ell + \frac{3}{2}\right)^{1/2}}{\Gamma(\ell + 1)\left(\frac{1}{2} n - \frac{1}{2} \ell\right)!^{1/2}} e^{-\frac{1}{2} r^2} r^{\frac{1}{2}(\frac{1}{2} n + \frac{1}{2} \ell + \frac{3}{2})}.$$

The ladder terminated only for integer or half odd integer values of $a$, $b$. Therefore, the ladder which involves the actual harmonic oscillator radial functions (indexed by $n = 0, 1, 2, \ldots$) does not have a bottom, and the lowering operator $Q_-$ produces infinitely many non square-integrable radial functions. Therefore, only in the space spanned by the basis $\{f_{ab}\}$ with integer or half odd integer $a$, the operators

$$J_1 = \frac{1}{2} (Q_+ + Q_-), \quad J_2 = \frac{1}{2} (Q_+ - Q_-), \quad J_3 = \frac{1}{2} \ell$$

which satisfy the relations

are skew-adjoint for the scalar product

$$(f_{ab}, f_{cd}) = \int_0^{2\pi} \int_0^\infty f_{ab}^* f_{cd} r^2 dr dt$$

and the set \(\{f_{ab}\}\) provides a basis for the \(2a + 1\) dimensional unitary irreducible representation of \(SU(2)\). For other cases (non-integer or non-half odd integer \(a\)), it is not possible to define the adjointness of the operators properly.

Now, we construct the shift operators and the labeling operator of \(k\) for the total eigenfunction \(\psi_{n\ell m}\) by the same procedure as for the hydrogen atom. They are

$$\overline{\Omega}_\pm = (\Omega_0)^{1/2} i\Omega_\pm (\Omega_0)^{-1/2}, \quad \Omega_0 = \ell + \frac{1}{2},$$

with

$$\Omega_\pm = D^{-1/2} Q_4 e^{\pm i \ell t} Q_4^* D$$

where \(Q_4\), \(Q_4^*\) are given by (19) and (6). They satisfy the relations

$$\overline{\Omega}_\pm \psi_{n\ell m} = \pm i \left( (a - b') (a + b + 1) (b + c) (b + c + 1) \right)^{1/2} \psi_{n\ell \pm 2m}$$

and there exists both a top (\(n = 2\)) and a bottom (\(\ell = |m|\)) to the ladder. The recovery of the bottom of the ladder is due to the fact that the ladder \(\{Y_{\ell m}\}\) produced by the operators \(Q_4\) and \(Q_5\) have bottom at \(\ell = |m|\) for integer \(\ell\).
As the states produced by operating the $\tilde{\Omega}_2$ on the $\psi_{n\ell m}$ are square-integrable, we can define the adjointness of the operator in the space $\{\psi_{n\ell m}\}$.

The operators

$$\tilde{\Omega}_1 = \tilde{\Omega}_+ + \tilde{\Omega}_-, \quad \tilde{\Omega}_2 = i(\tilde{\Omega}_+ - \tilde{\Omega}_-), \quad \tilde{\Omega}_0$$

are self-adjoint under the same scalar product as (17), and, therefore, are physically meaningful constants of the motion. However, they do not form a closed Lie algebra.
3-10. MORSE POTENTIAL

In this section the spectrum generating algebra of a particle in "Morse potential" will be derived for a special value of \( \varepsilon \).

The Morse potential has been playing an important role in the theory of diatomic molecules. This is because Schrödinger equation for this potential is partly soluble. Solutions of this equation are known only for zero angular momentum states.

For the case of \( \varepsilon = 0 \), the Schrödinger equation can be written as

\[
\left[ -\frac{1}{2m} \left( \frac{\partial^2}{\partial^2 r} + 2r^{-1} \frac{\partial}{\partial r} \right) + V(r) - i\varepsilon \frac{\partial}{\partial t} \right] \sum \psi \exp \left( -i\varepsilon t \right) \psi = 0 \tag{1}
\]

where

\[
V(r) = \delta^d \left( e^{-2a(r-r_0)} - 2e^{-a(r-r_0)} \right) \tag{2}
\]

The spectrum for this equation is well-known to be

\[
P_n = -\frac{\alpha^2}{2m}(n + \frac{1}{2} - \kappa)^2, \quad \kappa = a^{-1}\left(2mp_{\varepsilon}\right)^{1/2}
\]

Using this knowledge of the spectrum, we perform time dilation on equation (1) by an operator

\[
D = \exp \left( t\frac{\alpha}{\hbar} \left( \frac{n + \frac{1}{2}}{\hbar} - \kappa \right) \right)
\]

where

\[
\hbar = -\frac{1}{2m} \left( \frac{\partial^2}{\partial^2 r} + 2r^{-1} \frac{\partial}{\partial r} \right) + V(r), \quad n + \frac{1}{2} - \kappa = (-2ma^{-2})^{1/2}
\]
The transformed equation is

\[- \frac{1}{2m} \left( \frac{\partial^2}{\partial r^2} + 2r^{-1} \frac{\partial}{\partial r} \right) + V(r) - \frac{a^2}{2m} \frac{\partial^2}{\partial t^2} \right] f = 0 \tag{5}\]

where

\[f = \sum_n C_n e^{-1(n + \frac{1}{2} - k)t} \psi_n(r) \tag{6}\]

Assuming a form

\[Q = q_r f + q_t^2 + q_r^2 + q_t + q^0 \tag{7}\]

and choosing functions

\[f, f_r, f_t, f_{rr}, f_{rt}, f_{tt}, f_{rrr}, f_{rtr}, f_{ttt}\]

for independent functions, we obtain following determining equations:

\[2q_{rr} + q_{tt} = 0, \quad q_{rt} - q_{tt} = 0,\]

\[q_{rr} + a^2 q_{tt} - 2r^{-1} q_r + 2r^{-2} q_{rr} + 2q_t + 2a^2 q_r = 0,\]

\[q_{rr} + a^2 q_{tt} + 2r^{-1} q_r - 2a^2 q_r + 2a^2 q_t = 0,\]

\[q_{rr} + a^2 q_{tt} - 2r^{-1} q_r + 2r^{-2} q_r + 2q_t = 0,\]

\[q_{rr} + a^2 q_{tt} + 2r^{-1} q_r + 2a^2 q_t + 2mV_r q_{rt} + 4mV_q q_r = 0,\]

\[q_{rr} + a^2 q_{tt} + 2r^{-1} q_r + 2mV_r q_r + 4mV_q = 0.\]
Solving these equations, we obtain

\[ Q = (7) = \sum_{i=1}^{5} a_i Q_i \]

where the \( a_i \) are integration constants, and the \( Q_i \) are defined by

\[
Q_1 = e^{ikt} e^{\frac{1}{2}r^2 \frac{i}{2}} \left( r t + i a \right) e^{\frac{1}{2}r^2 \frac{i}{2}} \left( r t + i a \right) \]

\[
- \frac{1}{2} r^{-1} \frac{2}{a} e^{a(r_0 - r)} \}
\]

\[
Q_3 = \partial_t , \quad Q_4 = \partial_t \partial_t , \quad Q_5 = 1 .
\]

The commutation relations are

\[
[Q_+, Q_-] = 20_0 , \quad [Q_0, Q_+] = \partial_+ Q_+ \]

where

\[
Q_+ = \frac{e^{-ar_0}}{\sqrt{2}} \cdot \frac{1}{2} Q_0 , \quad Q_0 = 10_3 .
\]
3.11 Poschl - Teller Potentials

It is wellknown that the Morse potential is not the only potential which gives rise to the spectrum \((n + c)^2\) where \(n\) is integer. Poschl and Teller showed\(^{(30)}\) that the potentials

\[
V_1 = \frac{a^2}{2m} \left( \frac{p(p + 1)}{\sin^2 a(r - r_0)} + \frac{q(q + 1)}{\cos^2 a(r - r_0)} \right)
\]

\[
V_2 = \frac{a^2}{2m} \left( \frac{p(p + 1)}{\sinh^2 a(r - r_0)} - \frac{q(q + 1)}{\cosh^2 a(r - r_0)} \right)
\]

also exhibit the same type of spectrum\(^{\dagger}\);\(^{(31)}\)

\[
E^1 = \frac{n^2}{2m} (2n - p - q)^2, \quad n = 0, 1, 2, \ldots
\]

\[
E^2 = -\frac{n^2}{2m} (2n - p - q)^2, \quad n = 0, 1, 2, \ldots < \frac{1}{2}(p + q)
\]

The solutions for these potentials are known only for \(s\) states \((\ell = 0)\). In this section the spectrum generating algebras for \(s\) states will be derived.

\(^{\dagger}\) If \(p\) or \(q\) vanishes, the spectrum is different from these given here.\(^{(32)}\)
(1) Spectrum generating algebra for $V_1$

The state wave functions are the solutions of the

Schroedinger equation

$$\left\{ -\frac{1}{2m} \left( \partial_r \partial_r + 2r^{-1} \partial_r \right) + V_1 - i\alpha_t \right\} \sum c_n e^{-\frac{iE_n t}{\hbar}} f_n(r) = 0. \quad (1)$$

By making the time dilation

$$D = \exp(t \partial_t - \frac{2n - p - q}{2\hbar}) \quad (2)$$

where

$$H = -\frac{1}{2m} \left( \partial_r \partial_r + 2r^{-1} \partial_r \right) + V_1$$

$$\alpha = \frac{1}{2} \left\{ (2ma^2 \hbar) \gamma + p + q \right\}$$

we obtain the transformed equation

$$\left\{ -\frac{1}{2m} \left( \partial_r \partial_r + 2r^{-1} \partial_r \right) + V_1 + \frac{2a^2}{m} \alpha^2 \right\} f = 0 \quad (3)$$

where

$$f = \sum c_n e^{-i(n - p - q)t} f_n(r)$$

By assuming the form

$$\phi = \phi_{tt}^t + \phi_{tt}^t + \phi_{tt}^t + \phi_{tt}^t + \phi_{tt}^t$$

(4)
for the Q operators and taking

\[ f, f_r, f_t, f_{rt}, f_{tt}, f_{rtt}, f_{ttt} \]

for the independent functions, the equation

\[ \left( -\frac{1}{2m} (a r \partial_r + 2r^{-1} \partial_r) + v_1 + \frac{2a^2}{m} \partial_t^2 \right) \partial f = 0 \]

yields the determining equations

\[ Q^r_{rt} - Q^t_{tt} = 0, \quad 4a^2 Q^t_t - Q^r_r = 0, \]
\[ Q^r_{rr} - 2r^{-1} Q^r_{rt} - 4a^2 Q^t_{tt} + 2r^{-2} Q^r_{rt} - 8a^2 Q^r_t + 20^r_r = 0, \]
\[ Q^t_{tt} + 2r^{-1} Q^t_{rt} - 4a^2 Q^t_{tt} + 8a^2 Q^r_r - 8a^2 Q^t_t = 0, \]
\[ Q^r_{rr} - 2r^{-1} Q^r_{rr} - 4a^2 Q^t_{tt} + 2r^{-2} Q^r_{rr} + 20^r_r = 0, \]
\[ Q^t_{rr} + 2r^{-1} Q^t_{rr} - 4a^2 Q^t_{tt} - 8a^2 Q^t_t + 2mV_1 Q^t_{rt} + 4mV_1 Q^t_{rtt} = 0, \]
\[ Q^0_{rr} + 2r^{-1} Q^0_{rr} - 4a^2 Q^0_{tt} + 2mV_1 Q^0_{rt} + 4mV_1 Q^0_{rtt} = 0. \]

The solutions of these equations give rise to

\[ Q = \sum_{n=1}^{13} h_0^n n \]
where

\[\frac{0_1}{2} = (4a)^{-1} e^{\frac{\pi i}{4}} \left\{ \begin{array}{l} 2 i s a \tilde{e}_t r_t + 4 a \ c_3 \tilde{e}_t - s_3 r_t \\ - 2 i (r^{-1} s + a c) \tilde{e}_t - r^{-1} s - a p (p + 1) + a q (q + 1) \end{array} \right\},\]

\[0_3 = \delta_{0, q(q+1)}^{-1} e^{\frac{3\pi i}{4}} (s_3 r_t + 2 i a c_3 \tilde{e}_t + r^{-1} s),\]

\[0_5 = \delta_{0, p(p+1)}^{-1} e^{\frac{3\pi i}{4}} (c_3 r_t + 2 i a s_3 \tilde{e}_t + r^{-1} c),\]

\[0_7 = i \tilde{e}_t, \quad 0_9 = 0_3 \cdot 0_7, \quad 0_{10} = 0_5 \cdot 0_7, \quad 0_{12} = 0_7^2, \quad 0_{13} = 1\]

with

\[s = \sin 2a (r - r_o), \quad c = \cos 2a (r - r_o),\]

\[s = \sin a (x - r_o), \quad c = \cos a (r - r_o).\]

The invariants \(\bar{Q}_4\) corresponding to the equation (1) can be obtained by making inverse transformation on \(Q_4\) by \(D^{-1}\). These results show that in case of \(q(q-1) = 0\) or \(p(p-1) = 0\) the equation (1) gives rise to extra invariants \(\bar{Q}_3, \bar{Q}_4\) or \(\bar{Q}_5, \bar{Q}_6\). If neither \(p(p+1)\) nor \(q(q+1)\) vanishes, we have three nontrivial invariants \(\bar{Q}_1, \bar{Q}_2, \bar{Q}_7\). The commutation relations are

\[\left[ \bar{Q}_1, \bar{Q}_2 \right] = - 4 (\bar{Q}_7)^3 + (p(p+1) + q(q+1) - \frac{1}{2}) \bar{Q}_7,\]

\[\left[ \bar{Q}_7, \bar{Q}_{1/2} \right] = \pm \frac{1}{2} \bar{Q}_{1/2}.'
Although the $\bar{Q}_1$, $\bar{Q}_2$, $\bar{Q}_7$ do not form a closed Lie algebra, it is clear that the $\bar{Q}_1$ and $\bar{Q}_2$ shift the quantum number by one.

If $q(q+1)$ or $p(p+1)$ vanishes, we have three non-trivial invariants $\bar{Q}_3$, $\bar{Q}_4$, $\bar{Q}_7$ or $\bar{Q}_5$, $\bar{Q}_6$, $\bar{Q}_7$. For these cases the $\bar{Q}_1$ and $\bar{Q}_2$ are expressed as

$$\bar{Q}_1 = -4 \left( \frac{\bar{Q}_3}{q} \right)^2 \text{ if } q(q+1) = 0,$$

or

$$\bar{Q}_1 = 4 \left( \frac{\bar{Q}_3}{q} \right)^2 \text{ if } p(p+1) = 0.$$

These operators satisfy the commutation relations

$$\left[ \bar{Q}_3, \bar{Q}_4 \right] = 2 \bar{Q}_6, \quad \left[ \bar{Q}_5, \bar{Q}_6 \right] = 2 \bar{Q}_6, \quad \left[ \bar{Q}_o, \bar{Q}_q \right] = \pm \bar{Q}_3$$

or

$$\left[ \bar{Q}_5, \bar{Q}_6 \right] = 2 \bar{Q}_6, \quad \left[ \bar{Q}_o, \bar{Q}_q \right] = \pm \bar{Q}_6$$

where $\bar{Q}_0 = 20_7$. Therefore, the $\bar{Q}_3$, $\bar{Q}_4$ or $\bar{Q}_5$, $\bar{Q}_6$ shift the quantum number by one half unit.

It is interesting to notice that first by performing a transformation

$$r = \frac{Q_3}{6}^{-1}$$

on $Q_6$ (not on $\bar{Q}_6$), and then by making the transformations of the variables

$$x = \tan a(r - r'), \quad \phi = -\frac{1}{2}t$$
we obtain

\[ r_0 e^{r_0^2} = i \delta_{\theta, p(p+1)} e^{i\phi} \left( (1 - x^2)^{1/2} x + ix(1 - x^2)^{-1/2} \phi \right) \]

which are formally equivalent to angular momentum operators \( L_+ \) and \( L_\phi \) if we put \( x = \cos \theta \). The reason becomes quite clear if we perform the same transformations on the equation (3).

The resulting equation has a form

\[ \frac{\partial}{\partial x} \left( (1 - x^2) \phi \frac{\partial}{\partial x} x - 2x \phi + (1 - x^2)^{-1} \phi^2 + q(q+1) \right) \phi = 0 \]

which is formally equivalent to the differential equation of the spherical harmonics. However, we have to remember that the variable \( x \) takes all values from \(-i\infty\) to \(+i\infty\) on the imaginary axis. A similar type of transformation exists also for \( \phi_3 \) and \( \phi_4 \).

(ii) Spectrum generating algebra for \( V_2 \)

For this potential we perform a time dilation

\[ D = \exp(\text{t} \frac{2n - p - q}{2m}) \]

where

\[ n = -\frac{1}{2m} (\delta r^2 + 2r^{-1} \delta r) + V_2 \]

\[ n = \frac{1}{2} \left( (-2\lambda \omega^2) \right)^{1/2} \oint p + q \]

\[ n = \frac{1}{2} \left( (-2\lambda \omega^2) \right) \oint p + q \]
Then, the Schrödinger equation is transformed into

$$\left\{ -\frac{1}{2m}\left( \frac{\partial^2}{\partial r^2} + \frac{2r^{-1}}{r} \partial_r \right) + V_2 - \frac{2\alpha^2}{m} \partial_t^2 \right\} f = 0$$

(7)

where

$$f = \sum c_n e^{\frac{i}{\hbar}(n - \frac{1}{2}p - \frac{1}{2}q)t} \hat{f}_n (r).$$

Now we notice that the replacement of $a$ in the equation (3) by $ia$ yields the equation (7). Therefore, by simply replacing all $a$'s in the operators (5a-h) by $ia$ we can get the invariants of the equation (7). The corresponding invariants for the Schrödinger equation can be obtained by making the inverse transformation $D^{-1}$ of (6) on these invariants.
Appendix I. The inverse transformation of constants of the motion to the original physical space.

We illustrate the process of inverse transformation by using the operators which arise in the treatment of the hydrogenlike atom.

The inverse transformation operator $\mathbf{D}^{-1}$ is given by

$$
\mathbf{D}^{-1} = (\mathbf{D}_t)^{-1}(\mathbf{D}_r)^{-1} = \exp\{t\mathbf{\partial}_r \log (2\mathbf{z})^{-1/2}(2\mathbf{H})^{-1}\}
$$

and the transformed operator $\tilde{\mathbf{Q}}_i$ will be

$$
\tilde{\mathbf{Q}}_i = \mathbf{D}^{-1}\mathbf{Q}_i\mathbf{D} = (\mathbf{D}_t)^{-1}(\mathbf{D}_r)^{-1}\mathbf{Q}_i\mathbf{D}_r\mathbf{D}_t.
$$

As $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$ commute with $\mathbf{D}^{-1}$, $\tilde{\mathbf{Q}}_1$, $\tilde{\mathbf{Q}}_2$ and $\tilde{\mathbf{Q}}_3$ have the same form as $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$.

We next calculate the transformation of $\mathbf{Q}_6$ explicitly. (The transformation of $\mathbf{Q}_4$ and $\mathbf{Q}_5$ can be done in a similar manner.) Performing the transformation $(\mathbf{D}_r)^{-1}$ on $\mathbf{Q}_6$, we get

$$
(\mathbf{D}_r)^{-1}(-2iy^2\partial_x^2 + 2ir^{-1}xy^2\partial_x^2 + 2ir^{-1}xy^2\partial_x^2 + 2ix\partial_r \\
- 4ir^{-1}x^2\partial_x + x^2\partial_t) \mathbf{D}_r = z^{-1}\partial_t (-y^2\partial_x^2 + r^{-1}xy^2\partial_x^2 + r^{-1}x^2\partial_x^2 \partial_t + 2x) + r^{-1}xy^2\partial_x^2 + x\partial_t - 2r^{-1}x^2\partial_x + 2x). \tag{1}
$$

Here we have used the relations

$$
[\partial_t, \mathbf{Q}_6] = 0, \quad (\mathbf{D}_r)^{-1}\partial_r\mathbf{D}_r = -\frac{1}{2}i\mathbf{z}^{-1}\partial_r\mathbf{z}, \quad (\mathbf{D}_r)^{-1}\mathbf{D}_r = -\frac{1}{2}i\mathbf{z}^{-1}\mathbf{D}_r.
$$

Next we perform the time dilation $(\mathbf{D}_t)^{-1}$ on (1) to get
\[
\hat{Q}_6 = 2 (-2\pi)^{-\frac{1}{2}} \partial_t (-y^2 \partial_x \partial_x + r^{-1} xy^2 \partial_x \partial_x + r^{-1} xy^2 \partial_x \partial_x \\
+ x \partial_x - 2r^{-1} x^2 \partial_x + 2x)
\]

after using the relations

\[
[H, (D_t)^{-1} \partial_x r] = 0, \quad (D_t)^{-1} \partial_x r = 2 \pi (-2\pi)^{-\frac{1}{2}} \partial_x r.
\]

As we have an operator identity \( H = i \partial_x \) in the original space \( \{ e^{-i\pi\eta n^2} \}_{nm} \), the last equation can be rewritten as

\[
\hat{Q}_6 = i (-2\pi)^{-\frac{1}{2}} (-y^2 \partial_x \partial_x + r^{-1} xy^2 \partial_x \partial_x + r^{-1} xy^2 \partial_x \partial_x \\
+ x \partial_x - 2r^{-1} x^2 \partial_x + 2x).
\]

This is a \( Z \) component of the well-known Runge-Lenz vector to within the phase factor \( i \).
Appendix II

Sometimes we have to make use of an operator identity implied by the original differential equation to obtain a closed Lie algebra. We illustrate the process by using the two dimensional hydrogenlike atom, taking the commutator $[Q_1, Q_2]$ as an example.

By calculating the commutator explicitly, we get

$$[Q_1, Q_2] = e^{it} \left( -\frac{1}{2} \left( \frac{\partial}{\partial r} + r^{-1} \frac{\partial}{\partial r} + r^{-2} \frac{\partial}{\partial \phi} - ir^{-1} \partial_t - \frac{1}{4} \right) \frac{\partial}{\partial \phi} \right)$$

But as we have an operator identity (see equation 4.3)

$$\frac{\partial}{\partial r} + r^{-1} \frac{\partial}{\partial r} + r^{-2} \frac{\partial}{\partial \phi} - ir^{-1} \partial_t - \frac{1}{4} = 0$$

in the transformed space $\{ e^{i(n-l_2)t} \sum \frac{n}{2} \frac{\partial}{\partial r}, \phi \}$, the above equation reduces to

$$[Q_1, Q_2] = -\frac{i}{2} e^{it} (i r \partial_r + \partial_t - \frac{1}{2} \partial_r + \frac{i}{2}) = Q_4.$$
APPENDIX III

RELATIONS BETWEEN \( A_+ \), \( A_- \), \( A_z \) AND \( \Omega_+ \), \( \Omega_- \), \( \Omega_0 \)

The Runge-Lentz vector is defined by

\[
\vec{\Lambda} = (-2\hbar)^{-1/2} \left( \frac{1}{2} (\vec{L} \times \vec{P} - \vec{P} \times \vec{L}) + z \frac{\vec{r}}{r} \right).
\]

The action of the operators

\[
A_+ = A_x \pm i A_y \quad A_z
\]

on normalized hydrogenic wave function \( |n\ell m> \) is given by

\[
A_\pm |n\ell m> = \frac{1}{2} ((\ell + \frac{1}{2})(\ell - \frac{1}{2}))^{-1/2} \{ (\ell + m)(\ell - m - 1)(n - \ell)(\ell + n) \}^{1/2} |n\ell-1\pm m> \]

\[
A_z |n\ell m> = \frac{1}{2} ((\ell + \frac{1}{2})(\ell - \frac{1}{2}))^{-1/2} \{ (\ell - m)(\ell + m)(n - \ell)(\ell + n) \}^{1/2} |n\ell-1m> \]

Using these relations we obtain

\[
2A_\pm J_\pm |n\ell m> = \pm ((\ell + \frac{1}{2})(\ell - \frac{1}{2}))^{-1/2} ((\ell + m + 1)(\ell - m - n)(\ell + n))^{1/2} |n\ell-1m> \]

\[
A_z |n\ell m> = \frac{1}{2} ((\ell + \frac{1}{2})(\ell - \frac{1}{2}))^{-1/2} ((\ell + m)(\ell - m + 1)(n - \ell - 1)(\ell + 1 + n))^{1/2} |n\ell+1m>
\]

\( ^{\dagger} \) Bohm's results are not for the hydrogenic wave function.
Using the relations

\[ (-2H)_{\pm}^{1/2} \tilde{\Omega} \mid n \pm m \rangle = \pm (\frac{a}{2} + \frac{b}{2} \pm \frac{1}{2})^{1/2} \{(a \pm b + 1)(b \mp c)(b \mp c \pm 1)\}^{1/2} \mid n \pm m \rangle \]

where \( a = n - \frac{1}{2}, \ b = \frac{1}{2}, \ c = m - \frac{1}{2} \),

above equations can be rewritten as

\[ 2A_{\pm} \mid n \pm m \rangle = \frac{\mp (\frac{1}{2} + \frac{1}{2})}{(-2H)^{1/2}} \{(\frac{1}{2} \mp m + 1)\tilde{\Omega}_- + (\frac{1}{2} \mp m)\tilde{\Omega}_+ \} \mid n \pm m \rangle \]

\[ 2A_{0} \mid n \pm m \rangle = \mp (\frac{1}{2} + \frac{1}{2}) (-2H)^{1/2} (\tilde{\Omega}_+ - \tilde{\Omega}_-) \mid n \pm m \rangle \]

or using \( \tilde{\Omega}_+ \mid n \pm m \rangle = (\frac{1}{2} + \frac{1}{2}) \mid n \pm m \rangle \)

\[ 2(-2H)^{1/2} A_{\pm} \tilde{\Omega}_0 \mid n \pm m \rangle = \mp \{(\frac{1}{2} \mp m + 1)\tilde{\Omega}_- + (\frac{1}{2} \mp m)\tilde{\Omega}_+ \} \mid n \pm m \rangle \]

\[ 2(-2H)^{1/2} A_{0} \tilde{\Omega}_0 \mid n \pm m \rangle = \mp (\tilde{\Omega}_+ - \tilde{\Omega}_-) \mid n \pm m \rangle \]

From these we can easily obtain the relations

\[ (-2H)^{1/2} \{(A_{\pm} \tilde{\Omega}_0 \pm J_z \mp \frac{1}{2}) + A_{\pm} \tilde{\Omega}_0 \pm J_z \mp \frac{1}{2} \} = 2iJ \tilde{\Omega}_0 \]

\[ (-2H)^{1/2} A_{0} \tilde{\Omega}_0 \mp \frac{1}{2} (\tilde{\Omega}_+ - \tilde{\Omega}_-) \].
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