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College of the Pacific
Stockton Calif.

A SYLLABUS OF LINE GEOMETRY

BY

ALICE WILLMARTH

A SYLLABUS OF LINE GEOMETRY

A Thesis

Submitted to the Department of Mathematics

College of the Pacific

In partial fulfillment

of the

Requirements for the

Degree of Master of Arts

Aproved...

Head of Department

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CHAPTER I

DUALITY

In the study of advanced geometry, we shall deal with a certain important relation between pairs of figures in space, and also between their properties. There are two distinct parts to analytic geometry, the analytic work and the geometric interpretation. Two systems of geometry, depending upon different elements with the same number of coordinates, will have the same analytic expressions and will differ only in the interpretation of the analysis. In such a case it is often sufficient to know the meaning of the coordinates and the interpretation of a few fundamental relations in each system in order to find for a theorem in one geometry a corresponding theorem in the other. The nature of this relation is explained by the theorem of duality which asserts that a dual, or reciprocal, statement can be derived from a given statement, for example

1. Two lines a and b in a plane intersect and so determine the point ab . Also

2. Two points A and B are joined by and so determine the line AB .

Comparing statement (1) and statement (2), we see that either may be derived from the other by a simple interchange of point with line. The two statements are said to be plane duals, or reciprocal, of each other. The <u>elements</u> of <u>point</u> and <u>line</u> are <u>dual elements in the plane.</u>

This interchange of dual elements may be illustrated by writing one statement over the other, thus:

Two $\begin{bmatrix}\text{lines } a \text{ and } b\\ \text{points } A \text{ and } B\end{bmatrix}$ determine the $\begin{bmatrix}\text{point } ab\\ \text{line } AB\end{bmatrix}$.

In the same way, from any geometrical figure consisting of points and lines in a plane, can be derived the dual, or reciprocal, figure by interchanging point with line. The following are examples:

1. All the $\begin{bmatrix}\text{points on a line}\\ \text{lines through a point}\end{bmatrix}$ constitute a $\begin{bmatrix}\text{point-row}\\ \text{sheaf of rays}\end{bmatrix}$.

2. All the $\begin{bmatrix}\text{points}\\ \text{lines}\end{bmatrix}$ in a plane constitute a field of $\begin{bmatrix}\text{points}\\ \text{lines}\end{bmatrix}$.

3. Three $\begin{bmatrix}\text{points}\\ \text{lines}\end{bmatrix}$ not $\begin{bmatrix}\text{lying on a line}\\ \text{passing through a point}\end{bmatrix}$ are the $\begin{bmatrix}\text{vertices}\\ \text{sides}\end{bmatrix}$ of a triangle.

4. A moving $\begin{bmatrix}\text{point describes}\\ \text{line envelopes}\end{bmatrix}$ a curve.

Similarly, any theorem involving only the relative positions of the dual elements, line and point.

Let us consider the following statements in the duality of space.

1. Three $\begin{bmatrix}\text{planes}\\ \text{points}\end{bmatrix}$ not $\begin{bmatrix}\text{passing through}\\ \text{lying on}\end{bmatrix}$ the same $\begin{bmatrix}\text{line}\\ \text{line}\end{bmatrix}$ determine a $\begin{bmatrix}\text{point}\\ \text{plane}\end{bmatrix}$.

2. Two $\begin{bmatrix}\text{points } A \text{ and } B\\ \text{planes } \alpha \text{ and } \beta\end{bmatrix}$ determine the $\begin{bmatrix}\text{line } AB\\ \text{line } \alpha\beta\end{bmatrix}$

3. A $\begin{bmatrix}\text{point}\\ \text{plane}\end{bmatrix}$ and a $\begin{bmatrix}\text{line}\\ \text{line}\end{bmatrix}$ determine a $\begin{bmatrix}\text{plane}\\ \text{point}\end{bmatrix}$.

4. Two $\begin{bmatrix}\text{lines}\\ \text{lines}\end{bmatrix}$ which have a common $\begin{bmatrix}\text{point}\\ \text{plane}\end{bmatrix}$ determine a $\begin{bmatrix}\text{plane}\\ \text{point}\end{bmatrix}$.

A study of these statements shows that the point and the plane are dual elements in space, while the line is its own dual, or self-dual.

Also,

5. All the $\begin{bmatrix}\text{points on a line}\\ \text{planes through a line}\end{bmatrix}$ constitute a $\begin{bmatrix}\text{point-row}\\ \text{sheaf of planes}\end{bmatrix}$.

6. All the $\begin{bmatrix}\text{points lying in a plane}\\ \text{planes passing through a point}\end{bmatrix}$ constitute

a $\left[\begin{array}{l}\text{field of points} \\ \text{bundle of planes}\end{array}\right]$.

From these examples it will be seen that each primitive form has a space dual, or reciprocal, primitive form.

The following is an example of a theorem and its reciprocal placed in parallel columns.

If four points A, B, C and D are so situated that the lines AB and CD intersect, then all the points lie in one plane and consequently the lines AC and BD and also the lines AD and BC intersect.

If four planes α, β, γ, and δ are so situated that the lines $\alpha\beta$ and $\gamma\delta$ intersect, then all the planes pass through one point and consequently the lines $\alpha\gamma$ and $\beta\delta$, and also the lines $\alpha\delta$ and $\beta\gamma$ intersect.

And now we may state the principle of duality as follows:

Corresponding to any figure in space which is made up of or generated by points, lines and planes there exists a second figure which is made up of or generated by planes, lines, and points, such that to every point, every line, and every plane of the first figure, there corresponds respectively a plane, a line, and a point of the second figure, and such that to every proposition which relates to points, lines and planes of the first figure, but which does not essentially involve ideas of measurement, there corresponds a similar proposition regarding the planes, lines, and points of the second figure, and these two propositions are either both true or both false.

Exercises in duality:

1. What is the plane dual of a triangle? Space-dual?
2. State the dual of the following.
   If the vertices of a triangle are points of a curve, the triangle is said to be inscribed in a curve.
3. If the points A, E, C lie on a line a, and the points D, B, F lie on a second line b, then the lines AB, ED; AF, DC; and EF, BC determine three points which lie on the third line c.

   Write out the reciprocal theorem in the plane and draw the corresponding figure. Also write out the space-dual of the given theorem.

CHAPTER II

DUALITY AS APPLIED TO COORDINATES

The whole of analytical geometry as hitherto studied depends on the possibility of representing the position of a point in a plane by two coordinates, with the dependend possibility of representing the position of a point in ordinary space by three coordinates. These coordinates in the case of plane geometry were regarded initially as the distances from two selected lines to the point. But other systems were occasionally used; for example, polar coordinates, where the distance from a fixed pole to the point, and the direction of the line joining the fixed pole to the point, were the two determining quantities. The fundamental idea of coordinates from elementary plane analytic geometry is therefore that they are any two quantities that serve to determine the position of a point in a plane. Here there are implied certain limitations; for there are geometrical elements other than a point, whose position we may wish to specify, and we have no assurance that the number of coordinates required is necessarily two. We generalize, therefore, by dropping these limitations and saying: Coordinates are quantities that determine the position of a geometrical element. The nature of these depend on (1) the space assumed, (2) the problem considered, (3) the element selected.

Let us here consider the possibility that the primary element may not be the point. The point presents itself to us as __the element__ probably because all our drawing is done with the point. But the straight line is essentially as simple. It is possible that we might have learned to do all our drawing with a straight-edge instead of a point. We should then regard the point as a secondary element, uniquely determined by two straight lines. This secondary element, the point, would suggest to us an infinity of straight lines passing through it, just as with our present ideas the secondary element, the straight line, uniquely determined by two points, suggests to us an infinity of points lying on it. We shall constantly have occasion to notice in detail the correspondence between the two geometrical theories; that is, the two in which the field being restricted to a plane, the primary elements are respectively the point and the straight line, the secondary elements, the straight line and the point. The element then need not be a point, it may be some other geometrical entity.

Henceforth, we shall confine ourselves to geometry in a plane, using for element sometimes the point and sometimes the line. In Cartesian geometry, we obtain the two coordinates of a point by means of two fixed lines of reference. We here begin by assuming three non-concurrent fixed lines of reference, a, b and c (Figure 1). Let the distances from these lines to a point P be denoted by α, β, and γ. Using the ordinary method of signs, each line has a positive and a negative side which may be arbitrarily assigned. The ratio of any two $\alpha:\beta$, determines P as lying on a line through the intersection of ab and the ratio of $\alpha:\gamma$ determines P as lying on a line through the intersection of ac. P is therefore determined by two ratios $\alpha:\beta, \alpha:\gamma$ by which a third $\beta\gamma$ is implied. If, then, we select three lines a, b and c, not concurrent; the position of P is determined by any two of the ratios $\alpha:\beta:\gamma$ which are the coordinates of the point. It is essential that the three lines a, b, and c be not concurrent, for otherwise the two lines $P(ab)$ and $P(ac)$ whose intersection has to give P would intersect only at abc, the intersection of the lines of reference. Therefore, the point P could not be determined. (Figure 2)

Now, regarding the line as element, let us assign its position in a corresponding way. The position of the point was referred to fixed fundamental non-concurrent lines; we refer the position of the line to fixed fundamental non-collinear points. Let the perpendicular distances to the line from these fixed points A, B, C be noted by p, q, and r, so that the statement $p = 0$ means that the line passes through A and so on. (Figure 3) The position of L, therefore, will be determined by the ratios $p:q:r$ which are the coordinates of the line. This will be shown more definitely later.

From this come the following dual theorems.

1. The coordinate system of point geometry is determined by three arbitrary lines of reference and an arbitrary unit point.

2. The coordinate system of line geometry is determined by three arbitrary points of reference and an arbitrary unit line.

CHAPTER III

DUAL THEOREMS IN HOMOGENEOUS COORDINATES

1. Tri-linear point coordinates.

Let $X_1 = 0$, $X_2 = 0$, and $X_3 = 0$ (Figure 4) be three fixed straight lines of reference forming a triangle and let k_1, k_2, k_3 be three arbitrarily assumed constants. Let P be any point in the plane ABC and let d_1, d_2, d_3 be three perpendicular distances to P from the three lines of reference. Algebraic signs are to be attached to each of these distances according to the side of the line of reference on which P lies, the positive side of each line being assumed at pleasure.

The coordinates of P are defined as the ratios of three quantities X_1, X_2 and X_3 such that

$$X_1 : X_2 : X_3 = k_1 d_1 : k_2 d_2 : k_3 d_3 \qquad (1.)$$

It is evident that if P is given its coordinates are uniquely determined. Conversely, let real ratios $a_1 : a_2 : a_3$ be assumed for $X_1 : X_2 : X_3$. The ratio $X_1 : X_2 = a_1 : a_2$ furnishes the condition $\frac{d_1}{d_2}$ equals a constant, which is satisfied by any point on a unique line through A. Similarly the ratio $X_2 : X_3 = a_2 : a_3$ is satisfied by any point on a unique line through C. If these lines intersect, the point of intersection is P, which is thus uniquely determined by its coordinates.

In case these two lines are parallel we may extend our coordinate system by saying that the coordinates define a point at infinity. These are, in fact, the limiting ratios approached by $X_1 : X_2 : X_3$ as P recedes indefinitely from the lines of reference. Wecomplete the definition of the coordinates by saying that complex coordinates define imaginary points of the plane, and the coordinates $0:0:0$ are not allowable.

The coordinates of point A joining the lines $X_1 = 0$ and $X_2=0$ are $0:0:1$, those of point B joining the lines $X_1 = 0$ and $X_3=0$ are $0:1:0$, and those of point C joining the lines $X_2=0$ and $X_3=0$ are $1:0:0$. The ratios of k_1, k_2 and k_3 are determined when the point with the coordinates $1:1:1$ is fixed. This point we shall call the unit point, and since the k's are arbitrary it may be taken anywhere. Hencethe coordinate system is determined by three arbitrary lines of reference and an arbitrary unit point.[1]

Example in point geometry:

The element in point geometry has coordinates which are the ratios of the perpendicular distances (each multiplied by an arbitrary constant) from the lines of reference to the point.

Let $X_1 = 0$, $X_2 = 0$, and $X_3 = 0$, be the fixed lines

1 Woods, Higher Geometry, 34.

of reference, and A, B, C and their point of intersection. Let $d_1, d_2,$ and d_3 be the perpendicular distances from the lines of reference. (Figure 5)

Then $\qquad X_1 : X_2 : X_3 = k_1 d_1 : k_2 d_2 : k_3 d_3$

Let $\qquad k_1 = k_2 = k_3 = 1$

For convenience let us set up the ratio between the lines of reference as $d_1 : d_2 : d_3 = 1 : 2 : 3$ and

$$\frac{X_1}{X_2} = \frac{d_1}{d_2} = \frac{1}{2}, \quad \frac{X_2}{X_3} = \frac{d_2}{d_3} = \frac{2}{3}, \quad \frac{X_1}{X_3} = \frac{d_1}{d_3} = \frac{1}{3}$$

First, let us consider the point A joining the lines $X_1 = 0$ and $X_2 = 0$. We, then, consider the ratio $d_1 : d_2 = 1 : 2$. Now let us take any point P' satisfying these conditions. This determines the line passing through A and P' so that any point on this line satisfies the condition of the ratio.

Likewise, let us consider the lines $X_2 = 0$ and $X_3 = 0$ joined by the point C. The ratio here is

$$\frac{X_2}{X_3} = \frac{d_2}{d_3} = \frac{2}{3}$$

Choose any point P'' satisfying this condition and we have determined the line L' passing through C and P'' so that any point on this line satisfies the condition of the ratio.

At the point P where the two lines L' and L'' intersect, we have the point

$$X_1 : X_2 : X_3 = 1 : 2 : 3.$$

2. Dual Theorem in Line Geometry. (Fig. 6)

Let $u_1 = 0$, $u_2 = 0$, $u_3 = 0$ be three fixed points of reference, and let k_1, k_2 and k_3 be three arbitrarily assumed constants. Let p be any line in the plane and let D_1, D_2 and D_3 be the perpendicular distances of the line from the three points of reference.

The coordinates of p are defined as the ratios of three quantities

$$u_1 : u_2 : u_3 = k_1 D_1 : k_2 D_2 : k_3 D_3$$

If p is given, its coordinates are uniquely determined. Conversely, let real ratios $a_1 : a_2 : a_3$ be assumed for $u_1 : u_2 : u_3$. The ratio $u_1 : u_3 = a_1 : a_3$ furnishes the condition $\frac{D_1}{D_3}$ equals a constant, which is satisfied by any line p' through a unique point P on the line b. For in the right triangles ARP and PQC, all the corresponding angles are equal. Therefore the two triangles are similar and the corresponding sides are proportional, so $AP : PC = AR : QC$. Likewise, for any line drawn through P, similar triangles would be formed giving us a similar proportion. Similarly the ratio $u_1 : u_2 = a_1 : a_2$ is satisfied by any line p'' through a unique point P' on the line a. The line joining the points P and P' is p, which is thus uniquely determined by its coordinates.

It will be noticed that P divides the line segment of b between the points A and C in the given ratio. The point P may therefore most easily be found by dividing this segment of line b in the given ratio.

These facts enable us to set up conventions for positiveness and negativeness in line coordinates.

The line L (Figure 7) divides the line AB proportionally internally. From our previous study of geometry we understand that if any line is divided internally by a point, the ratio is positive, and likewise, if any line is divided externally by a point the ratio is negative. Therefore in (Figure 7) we have the two similar triangles ARP and BOP and the sides are in the proportion $AP:PB = AR:BO$. Since the ratio $AP:PB$ is positive the distances D_3 and D_2 are in a positive ratio. But the ratio of these distances is proportional to the line coordinates of a line through the point P_1. And since this ratio is positive we can define the coordinates of any line through the point P as positive. Lidewise the distances D_1 and D_2 are in a positive ratio and in this particular case the coordinates of the line L involve only positive ratios.

In (Figure 8) however, we have the line L dividing the line AB externally, making the ratio of its segments negative. In the similar triangles ARP and BOP, we have $BP:PA = BO:AR$. Since the ratio $BP:PA$ is negative the distances D_2 and D_3 are in a negative ratio, thus making the ratio of the coordinates of any line through

P negative. As was previously shown any line through P' divides BC in a positive ratio and hence would have a positive ratio of coordinates. Therefore the line through the points P and P' would have a set of coordinates involving both positive and negative ratios.

If P and P' are points at infinity, we may say that they fix a line at infinity. These are the limiting ratios approached by $u_1 : u_2 : u_3$ as p recedes indefinitely from the points of reference.

Dual Example in Line Geometry.

The element in line geometry has coordinates which are the ratio of the perpendicular distances, (each multiplied by an arbitrary constant) from the three points of reference.

Let $u_1 = 0$, $u_2 = 0$, and $u_3 = 0$ be the three points of reference and D_1, D_2 and D_3 the perpendicular distances from the points of reference.

Then $u_1 : u_2 : u_3 = K_1 D_1 : K_2 D_2 : K_3 D_3$

Let $K_1 = K_2 = K_3 = 1$ and $D_1 : D_2 : D_3 = 1 : 2 : 3$.

First consider the line a joining the points $u_1 = 0$ and $u_2 = 0$. Draw any line p' satisfying the condition that $D_1 : D_2 = 1 : 2$. Where the line p' intersects the line a, we have the point P so that any line passing through this point satisfies the condition $D_1 : D_2 = 1 : 2$ (Shown by previous theorem.) In practice this point P could be located by dividing the line segment of a between the points $u_1 = 0$ and $u_2 = 0$ in the given ratio. The point P

may therefore most easily be found by dividing this segment of line a in the given ratio.

Likewise, join the points $u_1=0$ and $u_3=0$ by the line b. Through b pass any line p'' satisfying the condition that $D_1 : D_3 = 1:3$. Where this line p'' intersects the line b, we have the point P', so that any line passing through this point satisfies the condition $D_1 : D_3 = 1:3$. The line p joining the points P and P' has the coordinates

$$u_1 : u_2 : u_3 = 1:2:3$$

Example using negative ratios.

Now let us consider an example using negative ratios.

$$u_1 : u_2 : u_3 = K_1 D_1 : K_2 D_2 : K_3 D_3$$

Letting $K_1 = K_2 = K_3 = 1$ as before, we have

$$D_1 : D_2 : D_3 = 2 : 5 : -3$$

First, consider the line ~~[illegible]~~ c (Figure 10). This line will be divided externally into the ratio $5:-3$ by the point P so that the perpendicular distances from the points $u_2=0$ and $u_3=0$ to any line through this point P will have the ratio $5:-3$. (By the previous theorem.) Likewise, considering the line ~~[illegible]~~ a, it will be divided similarly by the point P' into the ratio $2:5$ so that the perpendicular distances from the points $u_1=0$ and $u_2=0$ to any line through this point P' will have the ratio $2:5$. The line joining these two points P and P', will therefore satisfy the condition and have the coordinates $u_1 : u_2 : u_3 = 2:5:-3$.

CHAPTER IV

LINEAR EQUATION IN POINT COORDINATES

It is a fundamental proposition in analytic geometry that any linear equation $AX + BY + C = 0$ represents a straight line. Now let us prove the following theorem. A homogeneous equation of the first degree,

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$

represents a straight line, and conversely.

As a preliminary proposition let us prove that the coordinates of a point on a line joining two fixed points can be expressed in the coordinates of the two fixed points. In elementary geometry the following relations are already known

$$X = \frac{x_1 + \lambda x_2}{1 + \lambda}$$

$$Y = \frac{y_1 + \lambda y_2}{1 + \lambda}$$

or $\quad X : Y = x_1 + \lambda x_2 : y_1 + \lambda y_2$

In trilinear coordinates, we are to prove that

$$x_1 : x_2 : x_3 = y_1 + \lambda z_1 : y_2 + \lambda z_2 : y_3 + \lambda z_3$$

where λ is proportional to m. (Figure 11) $\quad m = \frac{YP}{PZ}$

$$\frac{P_1'}{P_1} = \frac{DY}{DP} \qquad \frac{P_1''}{P_1} = \frac{DZ}{DP}$$

$$\frac{P_1' - P_1}{P_1} = \frac{DY - DP}{DP} \qquad \frac{P_1'' - P_1}{P_1} = \frac{DZ - DP}{DP} = \frac{PZ}{DP}$$

$$\frac{P_1 - P_1'}{P_1} = \frac{DP - DY}{DP} = \frac{YP}{DP} \qquad \frac{P_1 - P_1'}{P_1'' - P_1} = \frac{YP}{PZ} = m.$$

$$P_1 - P' = M P_1'' - M P_1$$

$$P_1 = P_1' + M P_1'' - M P_1$$

$$(1+M) P_1 = P_1' + M P_1''$$

$$P_1 = \frac{P_1' + M P_1''}{1+M}$$

also $P_2 = \dfrac{P_2' - M P_2''}{1+M}$ and $P_3 = \dfrac{P_3' - M P_3''}{1+M}$

But $X_1 : X_2 : X_3 = k_1 P_1 : k_2 P_2 : k_3 P_3$

or

$$\rho X_1 = k_1 P_1 \qquad \rho X_2 = k_2 P_2 \qquad \rho X_3 = k_3 P_3$$

$$\rho' Y_1 = k_1 P_1' \qquad \rho' Y_2 = k_2 P_2' \qquad \rho' Y_3 = k_3 P_3'$$

$$\rho'' Z_1 = k_1 P_1'' \qquad \rho'' Z_2 = k_2 P_2'' \qquad \rho'' Z_3 = k_3 P_3''$$

$$\frac{\rho X_1}{\cancel{k_1}} = \frac{\frac{\rho' Y_1}{\cancel{k_1}} + M \frac{\rho'' Z_1}{\cancel{k_1}}}{1+M}$$

$$\frac{\rho}{\rho'} X_1 = \frac{Y_1 + \frac{M\rho''}{\rho'} Z_1}{1+M}$$

$$\frac{\rho}{\rho'} X_2 = \frac{Y_2 + \frac{M\rho''}{\rho'} Z_2}{1+M}$$

$$\frac{\rho}{\rho'} X_3 = \frac{Y_3 + \frac{M\rho''}{\rho'} Z_3}{1+M}$$

$\therefore$ $X_1 : X_2 : X_3 = y_1 + \frac{M\rho''}{\rho} z_1 : y_2 + \frac{M\rho''}{\rho} z_2 : y_3 + \frac{M\rho''}{\rho} z_3$

or $X_1 : X_2 : X_3 = y_1 + \lambda z_1 : y_2 + \lambda z_2 : y_3 + \lambda z_3.$

as $\lambda = M \frac{\rho''}{\rho}$

$\therefore$ λ is proportional to M

Now, in our original theorem, let us have given

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \qquad (1.)$$

and let $y_1 : y_2 : y_3$ and $z_1 : z_2 : z_3$ be two points on the locus of (1).

Then $$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$

$$a_1 y_1 + a_2 y_2 + a_3 y_3 = 0$$

and $$a_1 z_1 + a_2 z_2 + a_3 z_3 = 0$$

From these three equations we have

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0,$$

and by interchanging rows and columns we get

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

But this determinant might also come from the following homogeneous equations,

$$\lambda_1 x_1 + \lambda_2 y_1 + \lambda_3 z_1 = 0$$

$$\lambda_1 x_2 + \lambda_2 y_2 + \lambda_3 z_2 = 0$$

$$\lambda_1 x_3 + \lambda_2 y_3 + \lambda_3 z_3 = 0.$$

Dividing these equations by $-\lambda_2$ we get

$$\frac{\lambda_1}{\lambda_2} x_1 = \frac{-\lambda_2}{-\lambda_2} y_1 - \frac{\lambda_3}{-\lambda_2} z_1$$

$$\frac{\lambda_1}{\lambda_2} x_2 = \frac{-\lambda_2}{-\lambda_2} y_2 - \frac{\lambda_3}{-\lambda_2} z_2$$

$$\frac{\lambda}{\lambda_2} x_3 = \frac{-\lambda_2}{-\lambda_2} y_3 - \frac{\lambda_3}{-\lambda_2} z_3$$

Letting $\frac{-\lambda_3}{-\lambda_2} = \lambda$ and $\frac{\lambda_1}{-\lambda_2} = \rho$

and rewriting the equations, we get

$$\rho x_1 = y_1 + \lambda z_1$$

$$\rho x_2 = y_2 + \lambda z_2$$

$$\rho x_3 = y_3 + \lambda z_3$$

which in turn gives us

$$x_1 : x_2 : x_3 = y_1 + \lambda z_1 : y_2 + \lambda z_2 : y_3 + \lambda z_3$$

which we know represents the coordinates of any point on the straight line joining $y_1 : y_2 : y_3$ and $z_1 : z_2 : z_3$.[1]

Conversely, for any point on the line joining $y_1 : y_2 : y_3$ and $z_1 : z_2 : z_3$ we have

$$x_1 : x_2 : x_3 = y_1 + \lambda z_1 : y_2 + \lambda z_2 : y_3 + \lambda z_3 .$$

1 Woods, Higher Geometry, 35.

or $\begin{cases} \rho x_1 = y_1 + \lambda z_1 \\ \rho x_2 = y_2 + \lambda z_2 \\ \rho x_3 = y_3 + \lambda z_3 \end{cases}$ or $\begin{cases} \rho x_1 - y_1 - \lambda z_1 = 0 \\ \rho x_2 - y_2 - \lambda z_2 = 0 \\ \rho x_3 - y_3 - \lambda z_3 = 0 \end{cases}$

Hence
$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

and
$$\begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} x_1 - \begin{vmatrix} y_1 & z_1 \\ y_3 & z_3 \end{vmatrix} x_2 + \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} x_3 = 0$$

or
$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$

From these theorems comes the following; if

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$
$$b_1 x_1 + b_2 x_2 + b_3 x_3 = 0$$

are two fixed lines, the equation of any line through their point of intersection is

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \lambda(b_1 x_1 + b_2 x_2 + b_3 x_3) = 0$$

These equations, then, determine the lines of a pencil

$$(a_1 + \lambda b_1) x_1 + (a_2 + \lambda b_2) x_2 + (a_3 + \lambda b_3) x_3 = 0$$

and any line through this pencil has the coefficients,

$$(a_1 + \lambda b_1), \quad (a_2 + \lambda b_2), \quad (a_3 + \lambda b_3).$$

CHAPTER V

LINE COORDINATES DEFINED IN A DIFFERENT MANNER

THE LINEAR EQUATION IN LINE COORDINATES

The coefficients a_1, a_2, a_3 in the equation of a straight line are sufficient to fix the line. For example let $a_1 = 3$, $a_2 = 5$ and $a_3 = -2$. This gives us the equation $3x_1 + 5x_2 - 2x_3 = 0$ a definite line. As the x's vary, any set satisfying the equation will represent a point on the line. Then, suppose we should have a definite point given as $x_1 : x_2 : x_3 = 4 : -3 : 7$ This gives us the equation $4a_1 - 3a_2 + 7a_3 = 0$. Now substituting different values for the a's which satisfy the equation, gives us an infinity of lines passing through the given point P.

These ratios $a_1 : a_2 : a_3$ may be taken as coordinates of a straight line, or line coordinates. Thus the coordinates of the first line above are $3 : 5 : -2$. A variable or general set of line coordinates we shall denote by $u_1 : u_2 : u_3$. A linear equation $u_1 y_1 + u_2 y_2 + u_3 y_3 = 0$. represents all lines through the fixed point of which the point coordinates are $y_1 : y_2 : y_3$. It is the line equation of the point. This will be proved more rigorously later. The general equation $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$ may be considered as the necessary and sufficient condition that the line $u_1 : u_2 : u_3$ and the point $x_1 : x_2 : x_3$ are united; that is, the point lies on the line, and the line passes through the point.

It is a fundamental proposition in analytic geometry that any linear equation $AX+BY+C=0$ represents a straight line. Now let us prove the following theorem: A homogeneous line equation of the first degree, $a_1u_1+a_2u_2+a_3u_3=0$ is the line equation of a point and represents all lines through the point whose point coordinates are $a_1:a_2:a_3$.

We have given now that

$$a_1u_1+a_2u_2+a_3u_3=0 \qquad (1.)$$

Let $v_1:v_2:v_3$ and $w_1:w_2:w_3$ be any two lines through the locus of (1). Then

$$a_1u_1+a_2u_2+a_3u_3=0$$
$$a_1v_1+a_2v_2+a_3v_3=0$$
$$a_1w_1+a_2w_2+a_3w_3=0$$

From these equations we have

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

and by interchanging rows and columns, we get

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0$$

But this determinant might also have come from the following homogeneous equations,

$$\lambda_1 u_1 + \lambda_2 v_1 + \lambda_3 \omega_1 = 0$$
$$\lambda_1 u_2 + \lambda_2 v_2 + \lambda_3 \omega_2 = 0$$
$$\lambda_1 u_3 + \lambda_2 v_3 + \lambda_3 \omega_3 = 0$$

Dividing these equations by $-\lambda_2$, we get

$$\frac{\lambda_1}{-\lambda_2} u_1 = \frac{-\lambda_2}{-\lambda_2} v_1 + \frac{-\lambda_3}{-\lambda_2} \omega_1 .$$
$$\frac{\lambda_1}{-\lambda_2} u_2 = \frac{-\lambda_2}{-\lambda_2} v_2 + \frac{-\lambda_3}{-\lambda_2} \omega_2 .$$
$$\frac{\lambda_1}{-\lambda_2} u_3 = \frac{-\lambda_2}{-\lambda_2} v_3 + \frac{-\lambda_3}{-\lambda_2} \omega_3 .$$

Letting $\frac{-\lambda_3}{-\lambda_2} = \lambda$ and $\frac{+\lambda_1}{-\lambda_2} = \rho$ and rewriting the equations,

we get

$$\rho u_1 = v_1 + \lambda \omega_1$$
$$\rho u_2 = v_2 + \lambda \omega_2$$
$$\rho u_3 = v_3 + \lambda \omega_3$$

which in turn gives us

$$u_1 : u_2 : u_3 = v_1 + \lambda \omega_1 : v_2 + \lambda \omega_2 : v_3 + \lambda \omega_3 .$$

This, as previously shown, represents the coordinates of any line through the intersection of the fixed lines $v_1 : v_2 : v_3$ and $\omega_1 : \omega_2 : \omega_3$. For, since the coordinates $v_1 : v_2 : v_3$ determine a line $v_1 x_1 + v_2 x_2 + v_3 x_3 = 0$ and the coordinates $w_1 : w_2 : w_3$ determine a line $\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = 0$ the ratios $(v_1 + \lambda \omega_1) : (v_2 + \lambda \omega_2) : (v_3 + \lambda \omega_3)$ are the coordinates of some line and we know that such a line must go through the point of intersection of these lines.

And conversely, we know from our previous theorem that

$$u_1 : u_2 : u_3 = v_1 + \lambda w_1 : v_2 + \lambda w_2 : v_3 + \lambda w_3$$

represents the lines of the pencil determined by $v_1 : v_2 : v_3$ and $w_1 : w_2 : w_3$.

or $\begin{cases} \rho u_1 = v_1 + \lambda w_1 \\ \rho u_2 = v_2 + \lambda w_2 \\ \rho u_3 = v_3 + \lambda w_3 \end{cases}$ or $\begin{cases} \rho u_1 - v_1 - \lambda w_1 = 0 \\ \rho u_2 - v_2 - \lambda w_2 = 0 \\ \rho u_3 - v_3 - \lambda w_3 = 0 \end{cases}$

Hence

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} u_1 - \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} u_2 + \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} u_3 = 0$$

or

$$a_1 u_1 + a_2 u_2 + a_3 u_3 = 0.$$

which proves that (1) is the line equation of a point.

CHAPTER VI

EQUIVALENCE OF DEFINITIONS OF LINE COORDINATES

Let us next show, first, that line coordinates are proportional to the segments cut off by the line on the sides of the triangle of reference, each segment being multiplied by a constant factor, and second, that line coordinates are proportional to the three perpendiculars from the vertices of the triangle of reference to the straight line, each perpendicular being multiplied by a constant factor.

The equation of line L (figure 12) is

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \tag{1}$$

The angles α, β, and γ are fixed for any particular coordinate system.

Now in equation (1) let $x_3 = 0$ we then have intercept values for x_1 and x_2 or x_1' and x_2'. Equation (1) now becomes,

$$a_1 x_1' + a_2 x_2' = 0 \quad \text{or} \quad \frac{a_1}{a_2} = \frac{-x_2'}{x_1'} = \frac{-k_2 d_2'}{k_1 d_1'}$$

But $\dfrac{d_2'}{l_2} = \sin\gamma$ or $d_2' = l_2 \sin\gamma$

and $\dfrac{d_1'}{l_1} = \sin\beta$ or $d_1' = l_1 \sin\beta$

$$\frac{d_2'}{d_1'} = \frac{l_2 \sin\gamma}{l_1 \sin\beta}$$

$$\frac{a_1}{a_2} = \frac{-k_2 l_2 \sin\gamma}{k_1 l_1 \sin\beta} = \frac{c_2 \cdot l_2}{c_1 \cdot l_1}$$

Now in equation (1.) let us consider $X_1 = 0$. Our equation then becomes

$a_2 X_2' + a_3 X_3' = 0$ or $\frac{a_2}{a_3} = \frac{-X_3}{X_2} = \frac{-k_3 d_3'}{k_2 d_2''}$

But $\frac{d_2''}{l_4} = \sin\alpha$ or $d_2'' = l_4 \sin\alpha$

and $\frac{d_3'}{l_3} = \sin\beta$ or $d_3' = l_3 \sin\beta$.

$$\frac{d_3'}{d_2''} = \frac{l_3 \sin\beta}{l_4 \sin\alpha}$$

$$\frac{a_2}{a_3} = \frac{-k_3 d_3'}{k_2 d_2''} = \frac{-k_3 l_3 \sin\beta}{k_2 l_4 \sin\alpha} = \frac{C_3 \cdot l_3}{C_4 \cdot l_4}$$

Let us now consider the perpendiculars from the vertices of the triangle to the straight line. We have, first considering again, that $X_3 = 0$ and $a_1 X_1 + a_2 X_2 + a_3 X_3 = 0$

where $\frac{a_1}{a_2} = \frac{-X_2}{X_1} = \frac{-k_2 d_2}{k_1 d_1} = \frac{C_2 l_2}{C_1 l_1}$

But since $\frac{l_1}{l_2} = \frac{D_2}{D_1}$, $\frac{a_1}{a_2} = \frac{C_2 l_2}{C_1 l_1} = \frac{C_2 D_1}{C_1 D_2}$ (1.)

Likewise, since $\frac{l_3}{l_4} = \frac{D_2}{D_3}$, $\frac{a_2}{a_3} = \frac{C_3 l_3}{C_4 l_4} = \frac{C_3 D_2}{C_4 D_3}$ (2.)

Multiplying numerator and denominator of the right hand member of (1.) by C_3 we get $\frac{a_1}{a_2} = \frac{C_2 C_3 D_1}{C_1 C_3 D_2}$

similarly multiplying numerator and denominator of the right hand member of (2.) by C_1 we get $\frac{a_2}{a_3} = \frac{C_1 C_3 D_2}{C_1 C_4 D_3}$

Let $C_2 C_3 = K_1$, $C_1 C_3 = K_2$, and $C_1 C_4 = K_3$.

These equations then give the relations

$$a_1 : a_2 : a_3 = K_1 D_1 : K_2 D_2 : K_3 D_3.$$

Since it was previously shown from the first definition of line coordinates, based upon duality, that

$$u_1 : u_2 : u_3 = K_1 D_1 : K_2 D_2 : K_3 D_3$$

it is now demonstrated that the two methods of defining line coordinates are equivalent.

Now in (figure 12) as we let $a_1 = 0$ our proportion becomes $0 : a_2 : a_3 = 0 : K_2 D_2 : K_3 D_3$. Since the K's are arbitrarily fixed $D_1 = 0$ and the line L passes through the point C. Since the line L was any line the condition that such a line should pass through the point C is that $D_1 = 0$. But the D's are proportional to the coordinates of the line, hence; the equation of the point C is $u_1 = 0$. Similarly the equations of the points A and B are $u_3 = 0$ and $u_2 = 0$. In point coordinates the coordinates of the point C are $1 : 0 : 0$ likewise those of B are $0 : 1 : 0$ and those of A are $0 : 0 : 1$.

We have already shown that $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$ is the condition that the point and the line are united. The line equations of the points A, B, and C of (figure 13.) may be determined as follows. If the point A whose point coordinates are $0 : 0 : 1$ is united with a line as above stated we have the following condition $u_1 \cdot 0 + u_2 \cdot 0 + u_3 \cdot 1 = 0$ or $u_3 = 0$. This is then the line equation of point A. The equation of point B whose coordinates are $0 : 1 : 0$ is then $u_1 \cdot 0 + u_2 \cdot 1 + u_3 \cdot 0 = 0$ or $u_2 = 0$. And the equation of point C whose coordinates are $1 : 0 : 0$ is $1 \cdot u_1 + u_2 \cdot 0 + u_3 \cdot 0 = 0$ or $u_1 = 0$.

It has been shown that in the dual system of line and point coordinates, the elements of reference are now associated in the same triangle. In the case of point coordinates of the point A joining the lines of reference whose equations are $X_1 = 0$ and $X_2 = 0$ are $P(0:0:1)$. Dually the line coordinates of the line joining the points of reference whose equations are $U_1 = 0$ and $U_2 = 0$ are $L(0:0:1)$. Likewise the point coordinates of the point B joining the lines whose equations are $X_1 = 0$ and $X_3 = 0$ are $P(0:1:0)$; while the line coordinates of the line joining the points whose equations are $U_1 = 0$ and $U_3 = 0$ are $L(0:1:0)$. And the point coordinates of the point C joining the lines whose equations are $X_3 = 0$ and $X_2 = 0$ are $P(1:0:0)$.

We restate some of the results thus far obtained in parallel columns so as to show the dualistic relations.

The ratios $X_1 : X_2 : X_3$ determine the point.

A linear equation $a_1 X_1 + a_2 X_2 + a_3 X_3 = 0$ represents all points on the line of which the coordinates are $a_1 : a_2 : a_3$. It is the equation of the line.

If y_i and z_i are fixed points the coordinates of any point on the line

The ratios $U_1 : U_2 : U_3$ determine a straight line.

A linear equation $a_1 U_1 + a_2 U_2 + a_3 U_3 = 0$ represents all lines through the point of which the coordinates are $a_1 : a_2 : a_3$. It is the equation of the point.

If V_i and W_i are fixed lines the coordinates of any line through their connecting them are

$$y_i + \lambda z_i$$

If $a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$

and $b_1 x_1 + b_2 x_2 + b_3 x_3 = 0$

are the equations of two lines, the quation of any line through their point of intersection is

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \lambda(b_1 x_1 + b_2 x_2 + b_3 x_3) = 0$$

Three points y_i, z_i, t_i lie on a straight line when

$$\begin{vmatrix} y_1 & z_1 & t_1 \\ y_2 & z_2 & t_2 \\ y_3 & z_3 & t_3 \end{vmatrix} = 0$$

Three straight lines

$$\sum a_i x_i = 0, \sum b_i x_i = 0$$

$$\sum c_i x_i = 0,$$

meet in a point when

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

point of intersection are

$$v_i + \lambda w_i$$

If $a_1 u_1 + a_2 u_2 + a_3 u_3 = 0$

and $b_1 u_1 + b_2 u_2 + b_3 u_3 = 0$

are the equations of two points, the equation of any point on the line connecting them is

$$a_1 u_1 + a_2 u_2 + a_3 u_3 + \lambda(b_1 u_1 + b_2 u_2 + b_3 u_3) = 0$$

Three lines v_i, w_i, u_i meet in a point when

$$\begin{vmatrix} v_1 & w_1 & u_1 \\ v_2 & w_2 & u_2 \\ v_3 & w_3 & u_3 \end{vmatrix} = 0$$

Three points

$$\sum a_i u_i = 0, \sum b_i u_i = 0,$$

$$\sum c_i u_i = 0.$$

lie on a straight line when

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Woods, <u>Higher Geometry.</u>

CHAPTER VII

CURVES IN LINE COORDINATES

Let us first consider the general method for finding tangents in point coordinates.

We have given the curve $f(x_1\, x_2\, x_3)$ and any secant line P_1P_2 where $P_1(y_1\, y_2\, y_3)$ and $P_2(z_1\, z_2\, z_3)$ are fixed points. Now any point on P_1P_2 can be expressed in the form

$$x_1 : x_2 : x_3 = y_1 + \lambda z_1 : y_2 + \lambda z_2 : y_3 + \lambda z_3$$

where λ is proportional to the ratio in which the point P divides the line P_1P_2. (see Pg. 17) Let Q_1 and Q_2 be two of the points of intersection of P_1P_2 with $f(x_1\, x_2\, x_3)$ Since the straight line and curve intersect at these points the coordinates of points satisfying the straight line here will also satisfy the equation of the curve at Q_1 and Q_2, or, in other words, $f(y_1 + \lambda z_1 : y_2 + \lambda z_2 : y_3 + \lambda z_3) = 0$ will be a true equation for points Q_1 and Q_2.

Expanding by Taylor's Theorem, we have

$$f(y_1 + \lambda z_1 : y_2 + \lambda z_2 : y_3 + \lambda z_3) = f(y_1\, y_2\, y_3) +$$

$$\lambda\left(z_1 \frac{\partial f}{\partial y_1} + z_2 \frac{\partial f}{\partial y_2} + z_3 \frac{\partial f}{\partial y_3}\right) + \frac{\lambda^2}{2!}\left(z_1^2 \frac{\partial^2 f}{\partial y_1^2} + z_2^2 \frac{\partial^2 f}{\partial y_2^2} +\right.$$

$$\left. z_3^2 \frac{\partial^2 f}{\partial y_3^2} + 2 z_1 z_2 \frac{\partial^2 f}{\partial y_1 \partial y_2} + 2 z_1 z_3 \frac{\partial^2 f}{\partial y_1 \partial y_3} + 2 z_2 z_3 \frac{\partial^2 f}{\partial y_2 \partial y_3}\right)$$

$$+ \frac{\lambda^3}{3!}\left(z_1 \frac{\partial}{\partial y_1} + z_2 \frac{\partial}{\partial y_2} + z_3 \frac{\partial}{\partial y_3}\right)^3 f + \text{-------} = 0$$

Where λ now is proportional to the ratio in which Q_1 and

Q_2 divide P_1P_2.

Suppose now that P_1 coincides with Q_1. Then $f(y_1 y_2 y_3) = 0$ and one value of λ becomes 0. Suppose further that Q_2 approaches P_1 (ie. Q_1 since these two are noe coincident) Thus another value of λ is approaching 0 since λ is proportional to the ratio in which Q_2 divides P_1P_2. But analytically the condition that a second value of λ should equal 0 is that

$$z_1 \frac{\partial f}{\partial y_1} + z_2 \frac{\partial f}{\partial y_2} + z_3 \frac{\partial f}{\partial y_3} = 0$$

Hence the condition that a line passing through the point $P_2(z_1 z_2 z_3)$ should have two coincident intersections with a curve at the point $P_1(y_1 y_2 y_3)$ (that is, be tangent to the curve at this point) is

$$z_1 \frac{\partial f}{\partial y_1} + z_2 \frac{\partial f}{\partial y_2} + z_3 \frac{\partial f}{\partial y_3} = 0$$

But no restriction has been placed upon $(z_1 z_2 z_3)$ and hence it may be any point on this tangent line. Hence call $(z_1 z_2 z_3)$ $(x_1 x_2 x_3)$ and we have the equation of the tangent line to any curve $f(x_1 x_2 x_3) = 0$ at the point $y_1 y_2 y_3$ is

$$x_1 \frac{\partial f}{\partial y_1} + x_2 \frac{\partial f}{\partial y_2} + x_3 \frac{\partial f}{\partial y_3} = 0$$

The tangent is uniquely determined unless

$$\frac{\partial f}{\partial y_1} = 0 \qquad \frac{\partial f}{\partial y_2} = 0 \qquad \frac{\partial f}{\partial y_3} = 0$$

In this case we have a singular point at $y_1 y_2 y_3$. This point ~~also~~ lies on the curve. To prove this we recall that

$$y_1 \frac{\partial f}{\partial y_1} + y_2 \frac{\partial f}{\partial y_2} + y_3 \frac{\partial f}{\partial y_3} = N f(y_1 y_2 y_3)$$

from Euler's Theorem for homogeneous functions. But the left hand member we know equals zero because

$$\frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial y_2} = \frac{\partial f}{\partial y_3} = 0 .$$

Therefore $f(y_1 y_2 y_3) = 0$ and thus since $y_1 y_2 y_3$ satisfies $f(x_1 x_2 x_3) = 0$, $y_1 y_2 y_3$ must lie on the curve $f(x_1 x_2 x_3)$. $y_1 y_2 y_3$ therefore represents a singular point on $f(x_1 x_2 x_3) = 0$ and not the center of the curve.

We defined the curve in point geometry as the locus of a point subjected to certain analytic conditions. We now define the curve in line geometry as the locus of a line subjected to certain analytic conditions. We have shown that the equation $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$ unites the point and the line; that is, the point having the coordinates $x_1 : x_2 : x_3$ lies on the line whose coordinates are $u_1 : u_2 : u_3$. We might otherwise think of the curve in point coordinates as the path traveled by the point subjected to given conditions. In line coordinates, then, the curve would be the envelope of all the tangent lines.

The equation of tangent to the curve $f(x_1 x_2 x_3) = 0$ at the fixed point $y_1 y_2 y_3$ is

$$x_1 \frac{\partial f}{\partial y_1} + x_2 \frac{\partial f}{\partial y_2} + x_3 \frac{\partial f}{\partial y_3} = 0$$

Since these partial derivatives are constants and we know these coefficients are proportional to the line coordinates of the given line, then

$$u_1 : u_2 : u_3 = \frac{\partial f}{\partial y_1} : \frac{\partial f}{\partial y_2} : \frac{\partial f}{\partial y_3}$$

or

$$\begin{cases} \rho u_1 = \dfrac{\partial f}{\partial y_1} \\ \rho u_2 = \dfrac{\partial f}{\partial y_2} \\ \rho u_3 = \dfrac{\partial f}{\partial y_3} \end{cases}$$

This gives the line coordinates of the tangent in terms of the three parameters y_1, y_2 and y_3. ρ being merely a proportionality factor. By eliminating these three parameters we get the coordinates of <u>all</u> lines subject to the condition that they shall be tangent to the given curve. In other words, we have found the line equation of the curve. Since $y_1 y_2 y_3$ can now be any point $x_1 x_2 x_3$ on the curve we have the four equations:

$$\rho u_1 = \frac{\partial f}{\partial x_1}$$

$$\rho u_2 = \frac{\partial f}{\partial x_2}$$

$$\rho u_3 = \frac{\partial f}{\partial x_3}$$

$$f(x_1 x_2 x_3) = 0$$

from which to eliminate the three x's.

Let us find for example the line equations of the curve

$$x_1 x_2 + x_2 x_3 + x_1 x_3 = 0 \qquad (1.)$$

$$\frac{\partial f}{\partial X_1} = X_2 + X_3 \qquad \text{or} \qquad \rho u_1 = X_2 + X_3 \qquad (2.)$$

$$\frac{\partial f}{\partial X_2} = X_1 + X_3 \qquad\qquad \rho u_2 = X_1 + X_3 \qquad (3.)$$

$$\frac{\partial f}{\partial X_3} = X_1 + X_2 \qquad\qquad \rho u_3 = X_1 + X_2 \qquad (4.)$$

Subtracting equation (4.) from equation (3.) gives us

$$\rho u_2 - \rho u_3 = X_3 - X_2 \qquad (5.)$$

Adding equation (2.) to equation (5.) gives us.

$$\rho u_1 = X_2 + X_3$$

$$\rho u_1 + \rho u_2 - \rho u_3 = 2X_3$$

$$\frac{\rho u_1 + \rho u_2 - \rho u_3}{2} = X_3$$

Subtracting equation (2.) from equation (5.) gives us.

$$-\rho u_2 + \rho u_3 + \rho u_1 = 2X_2$$

$$\frac{\rho u_1 - \rho u_2 + \rho u_3}{2} = X_2$$

Substituting the value for X_2 gives us

$$\rho u_3 = X_1 + \frac{\rho u_1 - \rho u_2 + \rho u_3}{2}$$

$$X_1 = \frac{\rho u_3 + \rho u_2 - \rho u_1}{2}$$

Substituting these values in equation (1.) gives us the equation

$$\left(\frac{\rho u_3 + \rho u_2 - \rho u_1}{\cancel{2}}\right)\left(\frac{\rho u_1 - \rho u_2 + \rho u_3}{\cancel{2}}\right) + \left(\frac{\rho u_1 - \rho u_2 + \rho u_3}{\cancel{2}}\right)\left(\frac{\rho u_1 + \rho u_2 - \rho u_3}{\cancel{2}}\right)$$

$$+\left(\frac{\rho u_3 + \rho u_2 - \rho u_1}{\cancel{2}}\right)\left(\frac{\rho u_1 + \rho u_2 - \rho u_3}{\cancel{2}}\right) = 0$$

$$u_1^2 + u_2^2 + u_3^2 - 2u_1u_2 - 2u_2u_3 - 2u_1u_3 = 0$$

from which the parameters $X_1 X_2 X_3$ have been eliminates.

Sylvester's Method of Elimination.

Sometimes the above eliminations are exceedingly d difficult, consequently it is well to consider the following method.

We have given the two equations

$$F(x) = a_1 x^2 + b_1 x + c_1 = 0$$
$$\Phi(x) = a_2 x^2 + b_2 x + c_2 = 0$$

Using the following rule,--When we have two equations of n and m degree, if we multiply the m degree equation by x(n-1) times, and the n degree equation by x(m-1) times, we get the equations,

$$\begin{aligned} a_1 x^2 + b_1 x + c_1 &= 0 \\ a_1 x^3 + b_1 x^2 + c_1 x &= 0 \\ a_2 x^2 + b_2 x + c_2 &= 0 \\ a_2 x^3 + b_2 x^2 + c_2 x &= 0 \end{aligned}$$

For consistency of the above equations we have:

$$\begin{vmatrix} 0 & a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 & 0 \\ 0 & a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 & 0 \end{vmatrix} = 0$$

Take for example the curve having the point equation

$$f(x_1 x_2 x_3) = x_1^3 + x_1^2 x_3 - x_2^2 x_3 = 0 \qquad (1)$$

and $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$ (2)

the equation of any line.

We are to find the condition to be imposed upon the u's so that this family of lines shall envelope the curve, in other words to find the line equation of the curve. Solving for $X_3 = -\dfrac{u_2X_2+u_1X_1}{u_3}$ in equation (2) and substituting in equation (1) gives us

$$f(X_1X_2X_3) = X_1^3 - X_1^2\left(\frac{u_1X_1+u_2X_2}{u_3}\right) + X_2^2\left(\frac{u_1X_1+u_2X_2}{u_3}\right) = 0$$

then $u_3X_1^3 - u_1X_1^3 - u_2X_1^2X_2 + u_1X_1X_2^2 + u_2X_2^3 = 0$

or $(u_3-u_1)X_1^3 - u_2X_1^2X_2 + u_1X_2^2X_1 + u_2X_2^3 = 0$

Now letting $\dfrac{X_1}{X_2} = \lambda$ divide each term by X_2^3, and

$$\phi(u_1u_2u_3\lambda) = (u_3-u_1)\lambda^3 - u_2\lambda^2 + u_1\lambda + u_2 = 0$$

$$\frac{\partial\phi}{\partial\lambda} = 3(u_3-u_1)\lambda^2 - 2u_2\lambda + u_1 = 0$$

From Calculus the elimination of λ from the equations $\phi(u_1u_2u_3\lambda)$ and $\dfrac{\partial\phi}{\partial\lambda}$ gives us the envelope. Using Sylvester's method of elimination we get the following set of equations,

$$\begin{array}{lllllll}
 & (u_3-u_1)\lambda^3 & - u_2\lambda^2 & + u_1\lambda & + u_2 & = 0 \\
(u_3-u_1)\lambda^4 & - u_2\lambda^3 & + u_1\lambda^2 & + u_2\lambda & & = 0 \\
 & & 3(u_3-u_1)\lambda^2 & - 2u_2\lambda & + u_1 & = 0 \\
 & 3(u_3-u_1)\lambda^3 & - 2u_2\lambda^2 & + u_1\lambda & & = 0 \\
3(u_3-u_1)\lambda^4 & - 2u_2\lambda^3 & + u_1\lambda^2 & & & = 0
\end{array}$$

$$\begin{vmatrix} 0 & (u_3-u_1) & -u_2 & u_1 & u_2 \\ (u_3-u_1) & -u_2 & u_1 & u_2 & 0 \\ 0 & 0 & 3(u_3-u_1) & -2u_2 & u_1 \\ 0 & 3(u_3-u_1) & -2u_2 & -u_1 & 0 \\ 3(u_3-u_1) & -2u_2 & -u_1 & 0 & 0 \end{vmatrix} = 0$$

the line equation of the curve.

We have been considering thus far homogeneous point and line coordinates and the equation of curves in respect to these elements of reference. Non-homogeneous point coordinates are familiar from elementary analytic geometry and non-homogeneous line coordinates can be established as follows.

The equation $AX+BY+C=0$ is the point equation of a line. Now if we divide each term of this equation through by the constant C it gives us the equation $\frac{A}{C}X+\frac{B}{C}Y+1=0$ or $ax+by+1=0$. The coefficients a and b are the essential constants of this equation and determine the line. They may be taken as line coordinates and the theorems of elimination given in connection with homogeneous coordinates can be applied in non-homogeneous coordinates as well.

Let us consider the line equations in non-homogeneous coordinates of various types of curves, beginning with the circle. The general equation of the circle is $x^2+y^2=r^2$ (1.) The equation of any line cutting the circle (figure 16) is $Ax+By+C=0^{(2.)}$. Dividing each term of this equation by C we get a new set of coefficients which we shall call $ux+vy+1=0$ (3.) . By solving the equations of the line and circle simultaneously, we can find the points of intersection of the line with the circle. Solving equation (3.) for y gives us $y=\frac{-1-ux}{v}$ (4). Substituting this value of y in equation (1.) we have

$$x^2+\left(\frac{1+2ux+u^2x^2}{v^2}\right)=r^2$$

or
$$v^2x^2+1+2ux+u^2x^2-v^2r^2=0$$

$$(v^2+u^2)x^2+2ux+1-v^2r^2=0$$

The condition for tangency of this line with the circle is that the discriminant $B^2-4AC=0$. Equating the discriminant of this equation to zero we have

$$\cancel{4}u^2-\cancel{4}(v^2+u^2)(1-v^2r^2)=0$$

or
$$u^2-v^2+v^4r^2-u^2+u^2v^2r^2=0$$

$$v^2r^2+u^2r^2=1$$

$$v^2+u^2=\frac{1}{r^2}$$

which is the general equation of the circle in line coordinates.

We may also derive the line equation of the circle by the use of Sylvester's method.

$$x^2+y^2=r^2$$
$$ux+vy+1=0$$
$$y=\frac{-ux-1}{v}$$
$$x^2+\left(\frac{-ux-1}{v}\right)^2=r^2$$
$$v^2x^2+u^2x^2+2ux+1=r^2v^2$$
$$F=(v^2+u^2)x^2+2ux+1-v^2r^2=0 \qquad (1.)$$
$$\frac{\partial F}{\partial x}=\quad 2(v^2+u^2)x+2u=0 \qquad (2.)$$
$$2(v^2+u^2)x^2+2ux=0 \qquad (3.)$$

$$\begin{vmatrix} (v^2+u^2) & 2u & (1-v^2r^2) \\ 0 & 2(u^2+v^2) & 2u \\ 2(v^2+u^2) & 2u & 0 \end{vmatrix}=0$$

Dividing rows (2.) and (3.) by 2 , gives us

$$\begin{vmatrix} (u^2+v^2) & 2u & 1-v^2r^2 \\ 0 & u^2+v^2 & u \\ (u^2+v^2) & u & 0 \end{vmatrix}=0$$

Multifplying row (3.) by +1 and subtracting from row (1.) .

$$\begin{vmatrix} 0 & u & 1-v^2r^2 \\ 0 & u^2+v^2 & u \\ u^2+v^2 & u & 0 \end{vmatrix}=0.$$

Solving by minors

$$\cancel{(u^2+v^2)}\begin{vmatrix} u & 1-v^2r^2 \\ u^2+v^2 & u \end{vmatrix} = 0$$

$$\cancel{u^2}-\cancel{u^2}-\cancel{v^2}+v^{\cancel{4}\,2}r^2+u^2\cancel{v^2}r^2=0$$

$$u^2r^2+v^2r^2=1$$

$$u^2+v^2=\frac{1}{r^2}$$

In a like manner we can find the line equation of a parabola from the point equation. The point equation of a parabola is $y^2=4px$ and as before $ux+vy+1=0$ is the equation of any line cutting the parabola. Solving for x we get $x=\frac{-1-vy}{u}$ and substituting this value in our first equation gives us

$$y^2=4p\left(\frac{-1-vy}{u}\right)$$

$$\text{or } uy^2=-4p-4pvy$$

$$uy^2+4pvy+4p=0$$

Now, as before, equating the discriminant of this equation to zero, we have

$$\cancel{16}\,p^{\cancel{2}}v^2-\cancel{16}\,u\cancel{p}=0$$

$$\text{or} \qquad pv^2-u=0$$

$$\text{and} \qquad pv^2=u$$

which is the line equation of a parabola.

By the use of Sylvester's method, let us determine the line equation of the following. We have given the equation

$$f(xy)=x^3-y+1=0 \qquad (1.)$$

and the arbitrary line $ux + vy + 1 = 0.$ (2)

Solving for y in (2) gives us

$$y = \frac{-ux-1}{v}$$

Now, substituting this value in equation (1.)

$$x^3 - \left(\frac{-ux-1}{v}\right) + 1 = 0$$

$$F(u\,v\,x) = vx^3 + ux + 1 + v = 0$$

$$\frac{\partial F}{\partial x} = 3vx^2 + u \qquad = 0$$

The elimination of x between these two will give us the envelope.

$$\begin{array}{lllllllr}
 & vx^3 & & & +ux & +(1+v) & & = 0 \\
vx^4 & & & +ux^2 & +(1+v)x & & & = 0 \\
 & & 3vx^2 & & & & +u & = 0 \\
 & 3vx^3 & & & +ux & & & = 0 \\
3vx^4 & & & +ux^2 & & & & = 0
\end{array}$$

which gives us

$$\begin{vmatrix}
0 & v & 0 & u & (1+v) \\
v & 0 & u & (1+v) & 0 \\
0 & 0 & 3v & 0 & u \\
0 & 3v & 0 & u & 0 \\
3v & 0 & u & 0 & 0
\end{vmatrix} = 0$$

the line equation of the curve.

Multiplying row (2.) by 3 and subtracting from row (5.)

$$\begin{vmatrix} 0 & V & 0 & u & (1+V) \\ V & 0 & u & (1+V) & 0 \\ 0 & 0 & 3V & 0 & u \\ 0 & 3V & 0 & u & 0 \\ 0 & 0 & -2u & -3(1+V) & 0 \end{vmatrix} = 0$$

Reducing by minors

$$\cancel{V}\begin{vmatrix} V & 0 & u & (1+V) \\ 0 & 3V & 0 & u \\ 3V & 0 & u & 0 \\ 0 & -2u & -3(1+V) & 0 \end{vmatrix} = 0$$

Multiplying row (1.) by 3 and subtracting from row (4.)

$$\begin{vmatrix} V & 0 & u & (1+V) \\ 0 & 3V & 0 & u \\ 0 & 0 & -2u & -3(1+V) \\ 0 & -2u & -3(1+V) & 0 \end{vmatrix} = 0$$

Again reducing by minors

$$\cancel{V}\begin{vmatrix} 3V & 0 & u \\ 0 & -2u & -3(1+V) \\ -2u & -3(1+V) & 0 \end{vmatrix} = 0$$

$$-4u^3 - 9V(1+V)^2 = 0$$

$$4u^3 + 9V(1+V)^2 = 0$$

In some cases it is convenient to eliminate the parameter without the use of such general methods as Sylvesters'. We may apply this to the line equation of the circle already obtained by two other methods.

$$x^2+y^2=r^2$$

and the family of lines

$$ux+vy+1=0$$

$$y=\frac{-ux-1}{v}$$

$$x^2+\frac{u^2x^2+2ux+1}{v^2}=r^2$$

or $\phi(u\,v\,x)=(u^2+v^2)x^2+2ux+1-v^2r^2=0 \qquad (1.)$

$$\frac{\partial\phi}{\partial x}=2(u^2+v^2)x+2u \qquad =0 \qquad (2.)$$

Solving for x in equation (2.) gives us

$$x=\frac{-u}{u^2+v^2}$$

Substituting this value for x in (1.) and solving

$$(u^2+v^2)\left(\frac{-u}{u^2+v^2}\right)^2+2u\,\frac{-u}{u^2+v^2}+1-v^2r^2=0$$

$$\frac{u^2}{u^2+v^2}-\frac{2u^2}{u^2+v^2}+\frac{(1-v^2r^2)(u^2+v^2)}{u^2+v^2}=0$$

$$-\cancel{u^2}+\cancel{u^2}-u^2\cancel{x^2}r^2+\cancel{x^2}-v^2\cancel{x}r^2=0$$

$$-u^2r^2+1-v^2r^2=0$$

$$(u^2+v^2)r^2=1$$

$$u^2+v^2=\frac{1}{r^2}.$$

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