1862

Fragmentum ex adversariis mathematicis depromtum

Leonhard Euler

XXXIV.

Fragmentum ex Adversariis mathematicis depromtum.

Mechanica.

III.

(N. Fuss.)

Problema. (Fig. 278.). Si corpus super curva ascendat in medio resistente, fueritque $AP = a$ et $AM = s$, celeritas vero in $M = u$, et resistentiae formula $= \Delta + bu + cuu$, ita ut sit $udu + dx + (\Delta + bu + cuu)\,ds = 0$, invenire aequationem inter x et s, ut ista aequatio resolutionem admittat.

Solutio. Primo quidem patet hanc aequationem resolubilem fieri casu $dx = \alpha ds$; tum enim statim habetur $ds = \frac{-udu}{\alpha + \Delta + bu + cuu}$. Praeter hunc vero casum difficile est alios invenire; unde sequens solutio eo magis est notatu digna. Ponatur $ds = \frac{dq}{f - cq}$, ac statuatur $dx = \frac{aqdq - \Delta dq}{f - cq}$; tum enim aequatio induet hanc formam: $(f - cq)\,udu + aqdq + budq + cuudq = 0$. Hic ponatur $u = qv$, ac prodibit

$$\frac{dq}{q\,(f - cq)} = \frac{-vdv}{a + bv + fvv},$$

sicque quantitas q per v definiri poterit; tum vero etiam $u = qv$ per v definitur, at vero x et s dantur per q. Hinc jam porro tempus assignari poterit ex formula $dt = \frac{ds}{u} = \frac{dq}{(f - cq)\,qv}$, unde si loco $\frac{dq}{q\,(f - cq)}$ ejus valor substituatur, erit $dt = \frac{-dv}{a + bv + fvv}$. Sic tempus t etiam functioni ipsius v aequabitur $= \Gamma : \frac{q}{u}$, scilicet t aequabitur functioni nullius dimensionis binarum q et u. Unde patet totum tempus ascensus fore constans. Incipit enim ubi $s = 0$, at s ita definiri potest per q, ut posito $s = 0$, fiat $q = 0$; ergo in determinatione temporis integrale ita sumi debet, ut evanescat sumto $q = 0$. Ascensus vero terminatur, ubi celeritas u evanescit, unde totum tempus ascensus reperitur ex integrali invento ponendo $u = 0$, quod propterea erit quantitas constans, quaecunque fuerit celeritas initialis in a. Evidens igitur est curvam hoc modo inventam simul esse *tautochronam* in hoc medio resistente.

Problema. Proposita aequatione differentiali $udu + dx + ds\,(a + bu + cuu) = 0$, quaeritur qualis functio ipsius s loco x assumi debeat, ut ista aequatio resolutionem admittat.

Solutio. Ponatur $dx + ads = Sds$, ut habeatur haec aequatio:

$$udu + ds\,(S + bu + cuu) = 0.$$

Haec aequatio fingatur integrabilis reddi, si dividatur per hanc formulam $Aqq + Bqu + Cuu$, ubi q designet certam functionem ipsius s. Quare cum in genere formula $Pdu + Qds$ integrationem admittat, si fuerit

$$\left(\frac{dP}{ds}\right) = \left(\frac{dQ}{du}\right),$$

pro nostro casu erit:

$$P = \frac{u}{Aqq + Bqu + Cuu} \quad \text{et} \quad Q = \frac{S + bu + cuu}{Aqq + Bqu + Cuu}.$$

Jam quia q supponitur functio ipsius s, ponatur $dq = rds$, ac reperietur:

$$\left(\frac{dP}{ds}\right) = \frac{-2Aqru - Bruu}{(Aqq + Bqu + Cuu)^2}, \quad \text{deinde} \quad \left(\frac{dQ}{du}\right) = \frac{Abqq + 2Acqqu - BSq + Bcquu - 2CSu - Cbuu}{(Aqq + Bqu + Cuu)^2}.$$

Hinc igitur orietur ista aequatio:

$$Abqq - BSq + 2Aqu\,(r + cq) - 2CSu + Bcquu + Bruu - Cbuu = 0,$$

quorum trium membrorum singula ad nihilum redigi debent. Ex primo fit:

$$Abqq - BSq = 0, \quad \text{unde} \quad S = \frac{Abq}{B}.$$

Secundum membrum dat $2Aqur + 2Acqqu - 2CSu = 0$, quae loco S substituto valore abit in:

$$2Aqr + 2Acqq - \frac{2ACbq}{B} = 0, \quad \text{unde} \quad r = \frac{Cb - Bcq}{B}.$$

Denique tertium membrum praebet $Bcq + Br - Cb = 0$, unde iterum prodit $r = \frac{Cb - Bcq}{B}$, qui ambo valores ipsius r cum sint inter se aequales, nihil amplius determinandum restat, quare cum $r = \frac{dq}{ds}$, habebitur $\frac{dq}{ds} = \frac{Cb - Bcq}{B}$ hincque $ds = \frac{Bdq}{Cb - Bcq}$. Hincque jam fiet $cs = \int \frac{Bcdq}{Cb - Bcq} = l\,\frac{\varDelta}{Cb - Bcq}$.

Quodsi jam velimus, ut sumto $s = 0$ etiam q evanescat, esse debet $\varDelta = Cb$, sicque habebitur:

$$cs = l\,\frac{Cb}{Cb - Bcq}, \quad \text{ideoque} \quad e^{cs} = \frac{Cb}{Cb - Bcq}, \quad \text{unde fit} \quad q = \frac{Cb\,(e^{cs} - 1)}{Bce^{cs}} = \frac{Cb}{Bc}\,(1 - e^{-cs}).$$

Valore autem ipsius q invento, erit $S = \frac{ACbb}{cBB}(1 - e^{-cs})$; quare cum sit:

$$dx + ads = Sds, \quad \text{erit} \quad x = \int Sds - as = \frac{ACbb}{cBB}\left(s + \frac{1}{c}\,e^{-cs}\right) - as + \text{Const.}$$

Unde ut sumto $s = 0$ fiat $x = 0$, fiet:

$$\text{Const.} = \frac{ACbb}{ccBB}, \quad \text{consequenter} \quad x = \frac{ACbb}{ccBB} - as + \frac{ACbb}{cBB}\left(s + \frac{1}{c}\,e^{-cs}\right).$$

Hic ergo litterae A, B, C penitus arbitrio nostro relinquuntur; interim tamen patet, statui non posse $B = 0$; tum vero fieri debet $Cb > Bcq$, donec q certum valorem obtinuerit, in quo terminus scil. q maximum habet valorem. Invento autem divisore $Aqq + Bqu + Cuu$ integratio aequationis nulla amplius laborat difficultate, scil. per logarithmos et arcus circulares.

Problema. Proposita aequatione $udu + dx + (a + bu + cuu)\, ds = 0$, ubi x sit certa functio ipsius s, invenire valorem formulae integralis $t = \int \frac{ds}{u}$.

Solutio. Ponatur brevitatis gratia $udu + dx + (a + bu + cuu)\, ds = dW$, ut sit $dW = 0$. Jam semper certus multiplicator M dabitur, ut formula $\frac{ds}{u} + MdW$ integrationem admittat. Cum igitur sit $dW = 0$, erit $dt = \frac{ds}{u} + MdW$; quae formula cum sit integrabilis, inde definietur ipsum tempus t. Sumatur

$$M = \frac{p}{u\,(\alpha qq + \beta qu + \gamma uu)},$$

ubi p et q sint certae functiones ipsius s, eritque:

$$dt = \frac{ds}{u} + \frac{pudu + pdx + pads + pbuds + cpuuds}{u\,(\alpha qq + \beta qu + \gamma uu)}.$$

hinc ergo numerator erit $\alpha qqds + \beta quds + \gamma uuds + pudu + pdx + apds + bpuds + cpuuds$, denominatore existente $u(\alpha qq + \beta qu + \gamma uu)$. Nunc autem fiat $dt = \frac{qdu - udq}{\alpha qq + \beta qu + \gamma uu}$, quippe quae formula semper potest integrari; ponatur enim $u = q\nu$, eritque $dt = \frac{d\nu}{\alpha + \beta\nu + \gamma\nu\nu}$. Facta autem hac aequalitate:

$$\alpha qqds + \beta quds + \gamma uuds + pudu + pdx + apds + bpuds + cpuuds = qudu - uudq,$$

fieri debet:

$$\alpha qqds + pdx + apds = 0, \quad p = q, \quad \beta quds + bpuds = 0, \quad \gamma uuds + cpuuds = -uudq;$$

unde $ds = \frac{dq}{\gamma + cp}$ et $dx = -\frac{(ap + \alpha qq)\, ds}{p} = -(a + \alpha p)\, ds$. Ubi notetur, cum posito $u = q\nu$ sit $dt = \frac{d\nu}{\alpha + \beta\nu + \gamma\nu\nu}$, ideoque $t = f : \nu = f : \frac{u}{q}$, ideoque t aequabitur functioni nullius dimensionis ipsarum q et u: quare totum tempus exprimetur numero absoluto, ideoque curva erit *tautochrona*.

A. m. T. II. p. 164—166.

FINIS.

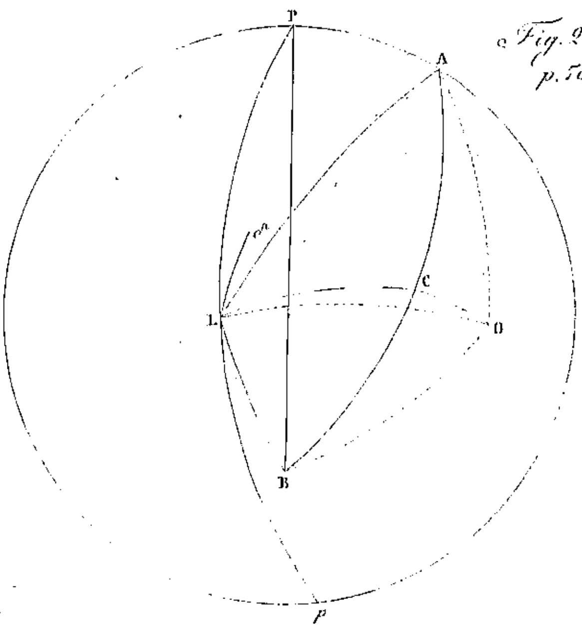


Fig. 273. p. 785.

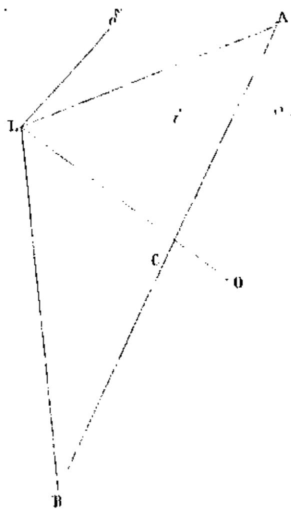


Fig. 274. p. 787.

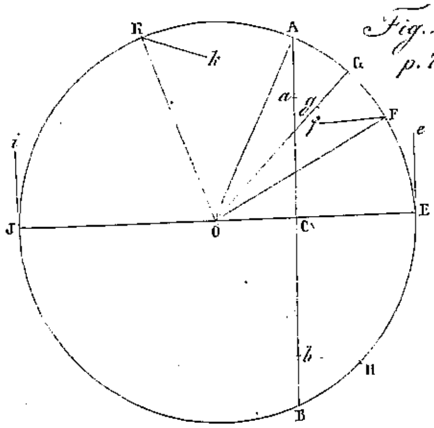


Fig. 275. p. 788.

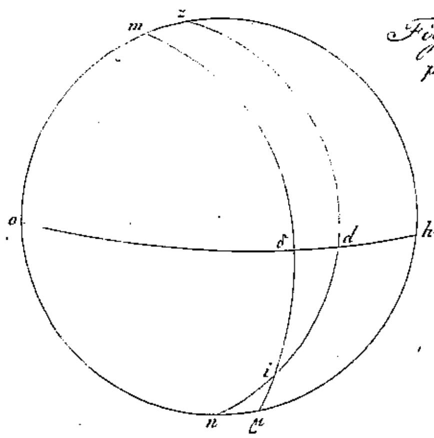
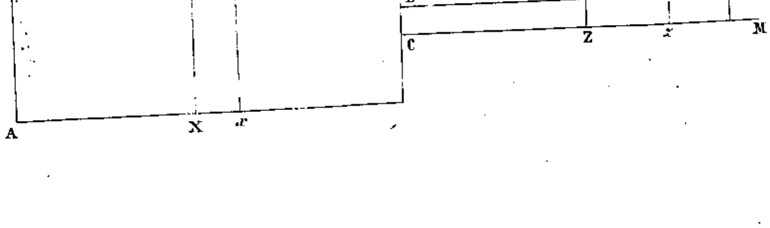


Fig. 276. p. 789.

Fig. 277. p. 816.



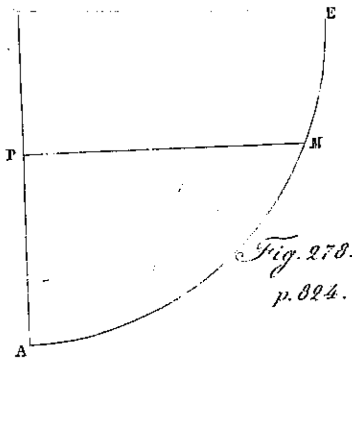


Fig. 278. p. 824.