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Solutio duorum problematum, Astronomiam mechanicam spectantium

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1. **Problema.** (Fig. 192). Si corpus sphaeroidicum ex materia homogenea conflatum, attrahatur ad centrum virium O, cujus vis sit reciproce proportionalis quadratis distantiarum, invenire mediam directionem, secundum quam hoc corpus urgebitur.

Solutio. Repraesentet circulus $AGBH$ sectionem hujus corporis per ejus centrum C ad axem normaliter factam, seu sit iste circulus planum aequatoris hujus corporis sphaeroidici propositi, in plano tabulae exhibitum, et recta EF, quae huic plano normaliter insistere concipienda est, referat axem corporis, cujus idcirco poli sint in E et F. Ponatur radius aequatoris $CA = CB = a$, semissis axis $CE = CF = b$. Sit centrum virium ubicunque situm in O, unde ad planum aequatoris demittatur perpendiculum OD; per D et centrum C agatur recta $DACB$, huicque diameter perpendicularis GH. Vocetur distantia $CD = f$ et $OD = g$, ita ut sit $\sqrt{(ff + gg)}$ distantia centri virium O a centro corporis C. Jam consideretur corporis quaecunque particula M, unde ad planum aequatoris demittatur perpendicularis MQ, et per Q diametro AB normalis ducatur corda NPN'. Vocentur nunc coordinatae $CP = x$, $PQ = y$ et $QM = z$; per P quoque axi EF parallela agatur recta RPR', et per M ipsi NN' parallela MRM', atque per R trajiciatur TR ipsi DC parallela, erit

$$DT = PR = QM = z, \quad MR = PQ = y, \quad TR = DP = f - x \quad \text{et} \quad TO = g - z.$$

Hinc fiet $TM = \sqrt{((f - x)^2 + yy)}$, et distantia puncti M a centro virium O, nempe recta

$$MO = \sqrt{(yy + (f - x)^2 + (g - z)^2)},$$

quae brevitatis gratia ponatur $= v$. Urgebitur ergo punctum M in directione MO vi acceleratrice, quadrato v^2 reciproce proportionali; sit ergo haec vis $= \frac{kk}{vv}$, qua punctum M in directione MO sollicitatur. Resolvatur haec vis secundum directiones Mm ipsi DO parallelam, et MT, eritque vis in directione $Mm = \frac{kk(g-z)}{v^3}$, et vis in directione $MT = \frac{kk\sqrt{(yy+(f-x)^2)}}{v^3}$, quae ulterius resol... secundum directiones $M\mu$ ipsi RT vel CD parallelam, et MR, eritque vis in directione $M\mu = \frac{kk(...}{...}$ et vis in directione $MR = \frac{kky}{v^3}$. Sicque quodlibet punctum M tribus urgetur viribus secundum directiones ternis coordinatis x, y, z parallelas, nimirum:

$$\text{secundum directionem } Mm, \text{ vi} = \frac{kk(g-z)}{v^3},$$

$$\text{secundum directionem } M\mu, \text{ vi} = \frac{kk(f-x)}{v^3},$$

$$\text{secundum directionem } MR, \text{ vi} = \frac{kky}{v^3}.$$

Sumta jam $RM' = RM$ consideretur punctum M', quod iisdem coordinatis definietur, quibus punctum M nisi quod sit y negativa; erit enim demisso ex M' in planum aequatoris perpendiculo $M'Q'$, $CP = x$, $PQ' = -y$ et $Q'M' = z$; unde punctum M', quia ejus distantia ab O quoque est $= v$, urgebitur his viribus:

$$\text{secundum directionem } M'm' = \frac{kk(g-z)}{v^3},$$

$$\text{secundum directionem } M'\mu' = \frac{kk(f-x)}{v^3},$$

$$\text{secundum directionem } M'R = \frac{kky}{v^3}.$$

Quodsi ergo haec duo puncta junctim considerentur, vires in directionibus MR et $M'R$ se mutuo destruent, et reliquae revocabuntur ad binas sequentes in puncto R applicatas

$$\text{secundum directionem } Rr, \text{ vis} = \frac{2kk(g-z)}{v^3},$$

$$\text{secundum directionem } RT, \text{ vis} = \frac{2kk(f-x)}{v^3}.$$

Sumantur jam in inferiori hemisphaerio bina puncta M'' et M''' his respondentia, ita ut sit $QM'' = Q'M''' = QM$, ideoque $PR' = PR$, eritque pro his punctis coordinata z negativa. Ponatur eorum distantia a centro virium O

$$\sqrt{(yy + (f-x)^2 + (g+z)^2)} = u,$$

atque ex istis binis punctis nascentur hae duae vires

$$\text{sec. directionem } R'r', \text{ vis} = \frac{2kk(g+z)}{u^3},$$

$$\text{sec. directionem } R'T', \text{ vis} = \frac{2kk(f-x)}{u^3}.$$

nunc abscissa x negativa, seu capiatur $CP' = CP$, atque ex reliquis coordinatis definiantur similimodo quaterna puncta M^{IV}, M^{V}, M^{VI} et M^{VII}, ponaturque

$$\sqrt{(yy + (f+x)^2 + (g-z)^2)} = (\rho) \quad \text{et} \quad \sqrt{(yy + (f+x)^2 + (g+z)^2)} = (u),$$

puncta haec quatuor praebebunt sequentes vires

$$\text{sec. directionem } R''r'', \quad \text{vis} = \frac{2kk\,(g-z)}{(\rho)^3},$$

$$R''T, \quad \text{vis} = \frac{2kk\,(f+x)}{(\rho)^3},$$

$$R'''r''', \text{ vis} = \frac{2kk\,(g+z)}{(u)^3},$$

$$R'''T', \text{ vis} = \frac{2kk\,(f+x)}{(u)^3}.$$

Omnia ergo haec octo puncta, in singulis corporis octantibus similiter posita, conjunctim has praebebunt vires, quibus corpus sollicitabitur:

$$\text{sec. directionem } PR, \quad \text{vis} = 2kkg\,(\rho^{-3} + u^{-3}) - 2kkz\,(\rho^{-3} - u^{-3}),$$

$$P'R'', \text{ vis} = 2kkg\,((\rho)^{-3} + (u)^{-3}) - 2kkz\,((\rho)^{-3} - (u)^{-3}),$$

$$Ss, \quad \text{vis} = 2kkf\,(\rho^{-3} + (\rho)^{-3}) - 2kkx\,(\rho^{-3} - (\rho)^{-3}),$$

$$S's', \quad \text{vis} = 2kkf\,(u^{-3} + (u)^{-3}) - 2kkx\,(u^{-3} - (u)^{-3}).$$

Quemadmodum ergo hae vires sunt natae ex puncto M in primo sphaeroidis octante quadranti ACG sursum imminente assumto: si omnia istius octantis puncta hoc modo colligantur, prodibunt vires, quibus totum sphaeroides sollicitatur, eaeque jam habebuntur reductae ad binas directiones, quarum alterae axi EF, alterae diametro aequatoris AB sint parallelae.

Quo autem hae vires facilius colligi queant, eae, quae directiones habent parallelas, primo summari, tum vero earum momentum exprimi debet. Ita vires PR et $P'R''$ dabunt

$$\text{vim } Yy = 2kkg\,(\rho^{-3} + u^{-3} + (\rho)^{-3} + (u)^{-3}) - 2kkz\,(\rho^{-3} - u^{-3} + (\rho)^{-3} - (u)^{-3}),$$

cujusque momentum respectu centri C seu axis GH sumtum erit

$$Yy \cdot CY = 2kkgx\,(\rho^{-3} + u^{-3} - (\rho)^{-3} - (u)^{-3}) - 2kkxz\,(\rho^{-3} - u^{-3} - (\rho)^{-3} + (u)^{-3}).$$

Deinde vis Ss et $S's'$ coalescent in unam vim

$$Xx = 2kkf\,(\rho^{-3} + u^{-3} + (\rho)^{-3} + (u)^{-3}) - 2kkx\,(\rho^{-3} + u^{-3} - (\rho)^{-3} - (u)^{-3}),$$

cujus momentum respectu ejusdem axis GH erit

$$Xx \cdot CX = 2kkfz\,(\rho^{-3} + (\rho)^{-3} - u^{-3} - (u)^{-3}) - 2kkxz\,(\rho^{-3} - (\rho)^{-3} - u^{-3} + (u)^{-3}).$$

Tribuatur nunc puncto M massa elementaris $dx\,dy\,dz$, per eamque singulae istae expressiones multiplicentur et integratione ter debito modo instituta prodibunt tam vires totales Yy et Xx ex attractione totius sphaeroidis oriundae, quam earum momenta $Yy.CY$ et $Xx.CX$; quae deinceps in unam toti attractioni aequivalentem conjungi poterunt. Quo autem hae integrationes commodius possint, transformemus formulas v^{-3}, u^{-3}, $(v)^{-3}$ et $(u)^{-3}$ in series, quae, si distantia virium $\sqrt{(ff+gg)}$, quam ponamus $=h$, a centro sphaeroidis C fuerit valde magna, conver Cum igitur sit $v=\sqrt{(hh-2fx-2gz+yy+xx+zz)}$, erit

$$v^{-3}=\frac{1}{h^3}+\frac{3fx+3gz}{h^5}-\frac{3yy-3xx-3zz}{2h^5}+\frac{15ffxx+30fgxz+15ggzz}{2h^7},$$

$$u^{-3}=\frac{1}{h^3}+\frac{3fx-3gz}{h^5}-\frac{3yy-3xx-3zz}{2h^5}+\frac{15ffxx-30fgxz+15ggzz}{2h^7},$$

$$(v)^{-3}=\frac{1}{h^3}-\frac{3fx+3gz}{h^5}-\frac{3yy-3xx-3zz}{2h^5}+\frac{15ffxx-30fgxz+15ggzz}{2h^7},$$

$$(u)^{-3}=\frac{1}{h^3}-\frac{3fx-3gz}{h^5}-\frac{3yy-3xx-3zz}{2h^5}+\frac{15ffxx+30fgxz+15ggzz}{2h^7}.$$

Hinc igitur erit vis tota Yy ex attractione totius sphaeroidis orta

$$Yy=\frac{8kkg}{h^3}\int dx\,dy\,dz\left(1-\frac{3yy-3xx-9zz}{2hh}+\frac{15ffxx+15ggzz}{2h^4}\right),$$

et vis tota Xx pro toto sphaeroide orta

$$Xx=\frac{8kkf}{h^3}\int dx\,dy\,dz\left(1-\frac{3yy-9xx-3zz}{2hh}+\frac{15ffxx+15ggzz}{2h^4}\right).$$

Deinde vero erunt momenta totalia

$$Yy.CY=\frac{24kkfg}{h^5}\int xx\,dx\,dy\,dz-\frac{120kkfg}{h^7}\int xxzz\,dx\,dy\,dz,$$

$$Xx.CX=\frac{24kkfg}{h^5}\int zz\,dx\,dy\,dz-\frac{120kkfg}{h^7}\int xxzz\,dx\,dy\,dz.$$

Quoniam triplici integratione opus est, ponantur primo x et z constantes, ut obtineantur vires elementis secundum rectas RM sitis oriunda, eritque

$$Yy=\frac{8kkg}{h^3}\int y\,dx\,dz\left(1-\frac{yy-3xx-9zz}{2hh}+\frac{15ffxx+15ggzz}{2h^4}\right),$$

$$Xx=\frac{8kkf}{h^3}\int y\,dx\,dz\left(1-\frac{yy-9xx-3zz}{2hh}+\frac{15ffxx+15ggzz}{2h^4}\right),$$

$$Yy.CY=\frac{24kkfg}{h^5}\int xxy\,dx\,dz-\frac{120kkfg}{h^7}\int xxzzy\,dx\,dz,$$

$$Xx.CX=\frac{24kkfg}{h^5}\int zzy\,dx\,dz-\frac{120kkfg}{h^7}\int xxzzy\,dx\,dz.$$

Concipiatur jam recta RM usque ad superficiem sphaeroidis producta, atque y determ debebit ex aequatione locali pro hac superficie sphaeroidica, inter coordinatas x, y et z expr quae est $yy = aa - xx - \frac{aazz}{bb}$. Ponatur nunc z constans, ut integrationes pateant ad sectiones sphaeroidis parallelas aequatori secundum MR factas: hunc in finem ponatur $\sqrt{\left(aa - \frac{aazz}{bb}\right)} = p$, ut p sit radius hujus sectionis, atque integrationem eousque extendi oportebit, donec fiat $x = p$. Sit $\frac{aa}{bb} = n$, eritque pro hoc casu

$$\text{vis } Yy = \int \frac{8kkgdz}{h^3} \int dx \left(1 - \frac{aa - 2xx + (n-9)zz}{2hh} + \frac{15ffxx + 15ggzz}{2h^4}\right) \sqrt{(pp - xx)},$$

$$\text{vis } Xx = \int \frac{8kkfdz}{h^3} \int dx \left(1 - \frac{aa - 8xx + (n-3)zz}{2hh} + \frac{15ffxx + 15ggzz}{2h^4}\right) \sqrt{(pp - xx)},$$

$$\text{momentum } Yy \,.\, CY = \int \frac{24kkfgdz}{h^5} \int xxdx \sqrt{(pp - xx)} - \int \frac{120kkfgdz}{h^7} \int xxzzdx \sqrt{(pp - xx)},$$

$$\text{momentum } Xx \,.\, CX = \int \frac{24kkfgdz}{h^5} \int zzdx \sqrt{(pp - xx)} - \int \frac{120kkfgdz}{h^7} \int xxzzdx \sqrt{(pp - xx)}.$$

Posita autem ratione diametri ad peripheriam $= 1 : \pi$, si post integrationem fiat $x = p$, erit

$$\int dx \sqrt{(pp - xx)} = \frac{1}{4} \pi pp, \qquad \int xxdx \sqrt{(pp - xx)} = \frac{1}{16} \pi p^4,$$

quibus valoribus substitutis erit

$$\text{vis } Yy = \int \frac{2\pi kkgppdz}{h^3} \left(1 - \frac{aa - \frac{1}{2}pp + (n-9)zz}{2hh} + \frac{\frac{15}{4}ffpp + 15ggzz}{2h^4}\right),$$

$$\text{vis } Xx = \int \frac{2\pi kkfppdz}{h^3} \left(1 - \frac{aa - 2pp + (n-3)zz}{2hh} + \frac{\frac{15}{4}ffpp + 15ggzz}{2h^4}\right),$$

$$\text{mom. } Yy \,.\, CY = \int \frac{3\pi kkfgp^4}{2h^5} dz - \int \frac{15\pi kkfgp^4zz}{2h^7} dz,$$

$$\text{mom. } Xx \,.\, CX = \int \frac{6\pi kkfgppzz}{h^5} dz - \int \frac{15\pi kkfgp^4zz}{2h^7} dz.$$

Est autem $pp = aa - nzz = aa - \frac{aazz}{bb}$, uti assumsimus, erit ergo $aa = nbb$ et $pp = n(bb - zz)$. Instituatur nunc ultima integratio, ac ponatur $z = b$, quoniam est

$$\int ppdz = n\int dz(bb - zz) = \frac{2}{3} nb^3, \qquad \int p^4 dz = nn\int dz(bb - zz)^2 = \frac{8}{15} nnb^5,$$

$$\int ppzzdz = n\int zzdz(bb - zz) = \frac{2}{15} nb^5, \qquad \int p^4 zzdz = nn\int zzdz(bb - zz)^2 = \frac{8}{105} nnb^7,$$

integralia quaesita ita se habebunt

$$\text{vis } Yy = \frac{2\pi kkg}{h^3}\left(\frac{2}{3}nb^3 - \frac{3nb^5}{5hh} - \frac{2nnb^5}{5hh} - \frac{nnb^5ff + nb^5gg}{h^4}\right),$$

$$\text{vis } Xx = \frac{2\pi kkf}{h^3}\left(\frac{2}{3}nb^3 - \frac{nb^5}{5hh} - \frac{4nnb^5}{5hh} + \frac{nnb^5ff + nb^5gg}{h^4}\right),$$

$$\text{mom. } Yy \,.\, CY = \frac{4\pi nnkkb^5fg}{5h^5} - \frac{4\pi nnkkb^7fg}{7h^7},$$

$$\text{mom. } Xx \,.\, CX = \frac{4\pi nkkb^5fg}{5h^5} - \frac{4\pi nnkkb^7fg}{7h^7},$$

Massa autem totius sphaeroidis est $= \frac{4}{3}\pi aab = \frac{4}{3}\pi nb^3$, quae si dicatur $= M$, eaque in form… inventas introducatur, reperietur

$$\text{vis } Yy = \frac{Mkkg}{h^3}\left(1 - \frac{9bb}{10hh} - \frac{3aa}{5hh} + \frac{3aaff}{2h^4} + \frac{3bbgg}{2h^4}\right),$$

$$\text{vis } Xx = \frac{Mkkf}{h^3}\left(1 - \frac{3bb}{10hh} - \frac{6aa}{5hh} + \frac{3aaff}{2h^4} + \frac{3bbgg}{2h^4}\right),$$

$$\text{mom. } Yy \,.\, CY = \frac{3Mkkaafg}{5h^5} - \frac{3Mkkaabbfg}{7h^7},$$

$$\text{mom. } Xx \,.\, CX = \frac{3Mkkbbfg}{5h^5} - \frac{3Mkkaabbfg}{7h^7}.$$

Neglectis ergo in viribus Yy et Xx terminis praeter primum omnibus, erit

$$CY = \frac{3aaf}{5hh} \qquad \text{et} \qquad CX = \frac{3bbg}{5hh},$$

sicque cognitis punctis Y et X, in quibus applicatae sunt concipiendae vires Yy et Xx, quarum directiones sunt axi sphaeroidis CE et diametro aequatoris BCA respective parallelae, innotescet media directio virium, quibus totum corpus ad centrum virium O sollicitatur. Ad hoc perficiendum concipiatur (fig. 194) sectio sphaeroidis per ejus axem ECF facta, in cujus plano situm sit centrum virium O, et AB sit diameter aequatoris in eodem plano ducta, erit $CE = CF = b$, $CA = CB =$ [illegible] $CD = f$, $OD = g$ et $CO = \sqrt{(ff + gg)} = h$, atque tang $DCO = \frac{g}{f}$. Cum jam directiones binarum virium Xx et Yy se mutuo in z intersecent, media directio earum per punctum z transibit. Transit vero etiam per centrum virium O, eritque ergo haec media directio zO. Quantum autem a centro C distet, fiat haec proportio

$$CD\,(f) : DO\,(g) = CY\left(\frac{3aaf}{5hh}\right) : Yt\left(\frac{3aag}{5hh}\right);$$

erit ergo

$$zt = Yt - CX = \frac{3(aa - bb)g}{5hh} = Cc \quad \text{proxime.}$$

Media ergo directio virium corpus sollicitantium transit non per centrum C, sed per punctum inferius quodpiam c, ut sit $Cc = \frac{3(aa-bb)g}{5hh}$, atque haec directio cO per centrum virium O transibit. Denique tota haec vis erit proxime $cO = \frac{Mkk}{hh}$, seu accuratius: $cO = \frac{Mkk}{hh}\left(1 - \frac{3(aa-bb)(2gg-ff)}{10h^4}\right)$, unde actio vis centripetae determinari poterit. Q. E. I.

2. **Coroll. 1.** Nisi ergo corpus sit sphaericum seu $a = b$, neque directio vis, qua id versus centrum virium O sollicitatur, per centrum corporis C, quod simul est ejus centrum gravitatis, transit, neque vis ipsa cO quadrato distantiae CO amplius est reciproce proportionalis.

3. **Coroll. 2.** Cum igitur motus corporis progressivus perinde se habeat, ac si ipsi in centro gravitatis C applicata esset vis aequalis ipsi

$$cO = \frac{Mkk}{hh}\left(1 - \frac{3(aa-bb)(2gg-ff)}{10h^4}\right),$$

cum directione ipsi cO parallela, haec vis neque per punctum O transibit, neque quadratis distantiarum CO erit reciproce proportionalis. Quamobrem semita corporis non erit ellipsis, in cujus altero foco sit punctum O: haecque aberratio eo erit notabilior, quo magis figura sphaeroidica a sphaerica discrepabit.

4. **Coroll. 3.** Hoc quoque casu axis EF non situm sibi parallelum tenebit, sed a momento vis sollicitantis continuo declinabitur. Quoniam vero momenta $Yy \cdot CY$ et $Xx \cdot CX$ sunt inter se contraria, illud praevalebit si $a > b$, ideoque vis cO momentum ad axem EF versus situm ef inclinandum erit $= \frac{3Mkkfg(aa-bb)}{5h^5}$. Interim tamen haec vis, quia per axem transit, motum vertiginis non afficiet.

5. **Coroll. 4.** Sit nunc angulus, quo axis sphaeroidis ECF ad rectam CO inclinatur, $ECO = \varphi = COD$, permanente distantia $CO = h$, erit $CD = f = h \sin\varphi$ et $OD = g = h\cos\varphi$. Hinc itaque erit intervallum $Cc = \frac{3(aa-bb)\cos\varphi}{5h}$, denotante a semidiametrum aequatoris AC, et b semiaxem sphaeroidis CE.

6. **Coroll. 5.** Angulo porro hoc $ECO = \varphi$ loco rectarum f et g introducto, erit vis, qua sphaeroides in puncto c ad centrum virium O sollicitatur,

$$= \frac{Mkk}{hh}\left(1 - \frac{3(aa-bb)(2\cos^2\varphi - \sin^2\varphi)}{10hh}\right),$$

seu ob $\cos^2\varphi = \frac{1+\cos 2\varphi}{2}$ et $\sin^2\varphi = \frac{1-\cos 2\varphi}{2}$, erit haec vis

$$= \frac{Mkk}{hh}\left(1 - \frac{3(aa-bb)(1+3\cos 2\varphi)}{20hh}\right).$$

7. **Coroll. 6.** Si haec vis in directione parallela centro gravitatis C concipiatur applicata, atque resolvatur secundum directiones CO, et $C\gamma$ ad CO in plano ECO normalem, reperietur

$$\text{vis } CO = \frac{Mkk}{hh}\left(1 - \frac{3(aa-bb)(1+3\cos 2\varphi)}{20hh}\right) \text{ et vis } C\gamma = \frac{Mkk}{hh} \cdot \frac{3(aa-bb)\sin 2\varphi}{10hh}.$$

8. **Coroll. 7.** Ob illam igitur vim CO, quatenus quadratis distantiarum CO non exacte reciproce proportionalis, orbita, quam centrum C describet, aliquantum ab elliptica discrepat, alterius autem vis $C\gamma$ effectus in hoc consistet, ut punctum C non in eodem plano moveatur.

9. **Coroll. 8.** Momentum denique, quo haec vis pollet ad axem corporis EF inclinandum et situm ef compellendum erit

$$= \frac{2Mkk\,(aa - bb)\sin\varphi\cos\varphi}{5h^3} = \frac{Mkk\,(aa - bb)\sin 2\varphi}{5h^3}.$$

Est itaque ceteris paribus reciproce ut cubus distantiae CO. Ratione anguli $ECO = \varphi$ vero hoc momentum erit maximum, si hic angulus ECO fiat semirectus.

10. **Scholion 1.** Cum igitur ex observationibus summa cura ab Illustris Academiae Regiae Parisinae Membris tam in Gallia quam in Lapponia et America institutis certissime evictum sit figuram terrae non esse sphaericam, sed sphaeroidicam compressam, cujus axis per polos ductus minor sit quam diameter aequatoris, hinc non levis mutatio tam in motu terrae quam in axis positione oriri debet. Quae ut definiri possit, non solum veram rationem inter axem terrae et diametrum aequatoris determinari oportet, sed etiam utriusque quantitatem absolutam, quod sequenti modo non difficulter fieri poterit. Sit semidiameter aequatoris $= a$, et semiaxis per polos ductus $= b$, ponatur $b : a = 1 : 1 + \varpi$, ut sit $a = b + \varpi b$, erit ϖ fractio valde parva. Sit in quapiam terrae regione elevatio poli $= p$, erit quantitas gradus meridiani in hac regione

$$= 0{,}017453292\,(b + \tfrac{1}{2}\varpi b - \tfrac{3}{2}\varpi b\cos 2p),$$

$$\text{seu} \quad = \frac{b + \frac{1}{2}\varpi b - \frac{3}{2}\varpi b\cos 2p}{57{,}29577951}.$$

Gradus vero secundum longitudinem in circulo aequatori parallelo mensuratus erit

$$= 0{,}017453292\,(b + \tfrac{3}{2}\varpi b - \tfrac{1}{2}\varpi b\cos 2p)\cos p.$$

Cum jam in Gallia sub elevatione poli $49^0\,21'\,24''$ mensura gradus in meridiano inventa sit 57183 hexapedarum parisinarum; hinc deducitur sequens aequatio

$$b + 0{,}7087569\,.\,\varpi b = 3276344\,\tfrac{1}{2}.$$

Sub circulo autem polari ab Illustri Praeside nostro de Maupertuis gradus meridiani definitus 57438 hexapedarum, pro elevatione poli $66^0\,30'$ („$19'\,34''$"), unde sequitur haec aequatio:

$$b + 1{,}5229976\,\varpi b = 3290955,$$

ex quibus duabus aequatoribus invenitur

$\varpi b = 17943$ hexaped. paris., $b = 3263626$, ac propterea $a = 3281570$ *).

Erit ergo semiaxis terrae $b = 3263626$ hexaped. paris. et semidiameter aequatoris $a = 3281570$ hexaped. paris. illiusque numeri ad hunc ratio proxime erit ut 182 ad 183, ita ut sit $\varpi = \frac{1}{182}$ et $a = \frac{183}{182}b$.

11. **Scholion 2.** Definita ergo figura et quantitate terrae, si vim, qua ad solem urgetur, spectamus, primum ejus orbita aliquantillum ab elliptica recedet, quia vis, qua centrum terrae ad centrum solis sollicitatur, non perfecte est quadratis distantiarum reciproce proportionalis. Erit autem ob $a = \left(1 + \frac{1}{182}\right) b$, $aa = \left(1 + \frac{1}{91}\right) bb$ proxime; ideoque vis, qua centrum terrae C ad solem pellitur, fiet $= \frac{Mkk}{hh}\left(1 - \frac{(1 + 3\cos 2\varphi)\,bb}{607\,hh}\right)$ proxime. Cum autem posita parallaxi solis horizontali sub polis terrae $= 10''$, sit $\frac{b}{h} = \sin 10''$, ideoque $\frac{bb}{hh} = 0{,}00000000235$, erit haec vis $= \frac{Mkk}{hh}\left(1 - \frac{(1 + 3\cos 2\varphi)}{258249300000}\right)$, cujus differentia ab $\frac{Mkk}{hh}$ tantilla est, ut ejus effectus omnino sentiri nequeat. Tum vero adest vis, qua terra de plano eclipticae detorquetur, cujus directio ad hoc planum normalis et sursum, seu boream versus terram sollicitans erit $= \frac{Mkk}{hh} \cdot \frac{\sin 2\varphi}{129124650000}$, quae maxima est quando sol proxime ad polum arcticum accedit, ubi fit $\varphi = 66\frac{1}{2}^{0}$ ac

$$\sin 2\varphi = \sin 47^{0} = 0{,}7313537;$$

sole autem in tropico capricorni versante, pari vi terra de ecliptica deorsum urgebitur; cum autem haec vis sit minima, effectus erit imperceptibilis. Momentum autem, quo axis terrae inclinatur, polusque soli propior ab eo detorquetur, erit $= \frac{Mkk\,(aa - bb)\sin 2\varphi}{5h^3} = \frac{Mkkbb \sin 2\varphi}{455 h^3}$. Quia vero effectus minimus ab hac vi oriundus in accuratissimis observationibus animadverti potest, eam negligere non licebit.

12. **Scholion 3.** Terra deinde quoque ad lunam attrahitur, verum haec vis prae illa, qua ad solem urgetur, tam est exigua, ut in motu terrae vix perceptibilem alterationem efficiat. Quanquam ergo haec vis ad lunam tendens, ob figuram terrae sphaeroidicam, quadratis distantiarum non est reciproce proportionalis, sed ab hac proportione aliquantum recedit, tamen multo minus effectus inde in motu terrae oriundus ullo modo observabilis esse poterit. Aliter vero se res habet in illa vi, qua terra de plano eclipticae detruditur, quae ob lunae vicinitatem multo major est simili illa vi a sole orta. Sit enim distantia lunae a terra $= H$, et vis attractiva acceleratrix $= \frac{KK}{HH}$, erit vis lunae ad terram de plano eclipticae depellendam tendens $= \frac{3MKK\,(aa - bb)\sin 2\varphi}{10 H^4}$; vis solis autem similem

*) *Script. autogr. ad marg.* Sub aequatore lat. 1^{0}: 56725 *tois.* $b - \varpi b = 3250103$, et ex circ. polari $\varpi b = 16192$, $b = 3266295$, $a = 3282487$, ergo

$$a : b = 203 : 202 \quad , \quad a : b = 201 : 200.$$

effectum edens $= \frac{3Mkk(aa-bb)\sin 2\varphi}{10h^4}$. Erit ergo vis lunae ad vim solis in similibus positioni... ut $\frac{KK}{H^4}$ ad $\frac{kk}{h^4}$. Verum ex aestu maris Newtonus conclusit esse vim lunae ad mare movendum... similem vim solis ut 4 ad 1, quam rationem quidem Cel. Dan. Bernoulli multo minorem statuit, scilicet ut $5:2$. Vires autem illae ad mare movendum sunt ut $\frac{KK}{H^3}$ ad $\frac{kk}{h^3}$; facto ergo $\frac{KK}{H^3} = \ldots$ prodibit vis lunae ad terram de plano eclipticae deturbandam ad vim solis ut $\frac{4}{H}$ ad $\frac{1}{h}$, hoc est ut $4h$ ad H, quae ratio proxime erit ut 1333 ad 1, siquidem ponamus $h = 20000$ semid. terrae et $H = 60$; quare haec vis lunae plus quam millies excedit similem vim solis, ejusque ergo effectus non erit negligendus. Tum vero vis lunae ad axem terrae inclinandum impensa erit $= \frac{MKK(aa-bb)\sin 2\ldots}{5H^3}$, quae propterea secundum Newtonum quadruplo major esse deberet quam vis solis; atque ex hoc fonte tam praecessio aequinoctiorum, quam nutatio quaepiam axis terrae sequi debet, quem utrumque effectum, quantum principia Mechanicae etiamnunc cognita id permittunt, determinare conabor.

13. **Problema II.** (Fig. 195). Determinare motum axis terrae, quatenus is a vi solis perturbatur, seu nutationem axis terrae a vi solis oriundam definire.

Solutio. Concipiamus centrum terrae in C quiescere, solemque in ellipsi circa id revolvi; ad praesens enim propositum perinde est, sive motum annuum soli tribuamus sive terrae. Repraesentet ergo planum tabulae planum eclipticae, sitque AOB orbita, in qua sol moveri videtur; sit A ejus apogaeum, B perigaeum, et post tempus quodpiam t sol ex apogaeo pervenerit in situm O; vocetur semiaxis transversus orbitae solaris $= c$, excentricitas $= n$, erit $CA = (1+n)c$ et $CB = (1-n)c$. Anomalia autem vera tempori t respondens, seu angulus ACO sit $= v$, et anomalia media $= \ldots$, distantia $CO = z$. Hoc autem tempore axis terrae teneat situm CE, ita ut sumto E pro polo boreali sit $CE = b$. Ex E in planum eclipticae demittatur perpendiculum EP, ductaque CP vocentur anguli $ACP = \vartheta$ et $ECP = \varphi$, erit $EP = b\sin\varphi$ et $CP = b\cos\varphi$. Jam axis EC cum directione ECO facit angulum ECO, ad quem inveniendum ex P in CO demittatur perpendiculum PQ, eritque etiam EQ ad CO perpendicularis. Cum jam sit ang. $OCP = v - \vartheta$, erit $PQ = b\cos\varphi\sin(v-\vartheta)$ et $CQ = b\cos\varphi\cos(v-\vartheta)$, unde fit $\frac{CQ}{CE} = \cos OCE = \cos\varphi\cos(v-\vartheta)$, qui est ille ipse angulus, quem superius $= \varphi$ vocavimus. Erit autem

$$\sin OCE = \sqrt{(1-\cos^2\varphi\cos^2(v-\vartheta))} \quad \text{et} \quad \sin 2\,OCE = 2\cos\varphi\cos(v-\vartheta)\sqrt{(1-\cos^2\varphi\cos^2(v-\vartheta))}.$$

Quoniam erit momentum vis solis ad hunc angulum OCE augendum

$$= \frac{2Mkk(aa-bb)\cos\varphi\cos(v-\vartheta)\sqrt{(1-\cos^2\varphi\cos^2(v-\vartheta))}}{5z^3},$$

pro quo brevitatis gratia scribatur Mp. Ducatur TEt normalis ad CE, eritque Et directio, secundum quam punctum E ab ista vi detorquebitur. Quantum autem detorqueatur, cognoscetur ex momento inertiae totius terrae, respectu axis ad CE normalis, hoc est respectu diametri aequatoris. Si igitur terra ex materia homogenea statuatur composita, respectu axis per aequatorem ducti reperitur momentum inertiae $= \frac{1}{5}M(aa+bb)$. Quodsi jam angulus OCE brevitatis gratia ponatur $= \ldots$

mutatio ita erit comparata, ut tempusculo dt fiat

$$\frac{2dds}{dt^2}=\frac{Mp}{\frac{1}{5}(aa+bb)M}=\frac{5p}{aa+bb}, \text{ ita ut sit } dds=\frac{5pdt^2}{2(aa+bb)}.$$

At est $p=\frac{2kk(aa-bb)\sin s\cos s}{5z^3}$, ergo $dds=\frac{kkdt^2(aa-bb)\sin s\cos s}{(aa+bb)z^3}$. Capiatur ergo Ee tantum, ut sit tang. $ECe=dds$, erit e punctum, in quod polus E tempusculo dt detorqueretur, si ante quievisset. Cum autem polo motus jam impressus concipi debeat, is ita erit comparatus, ut, si a nullis viribus urgeretur, uniformiter secundum circulum maximum esset progressurus. Quantum ergo hic motus a vi illa solis afficiatur, sequenti modo determinari poterit.

Concipiatur (fig. 196) in superficie sphaerae AO ecliptica, in eaque polus E, sumto A pro apogaeo solis. Ducatur ER ad AO normalis, erit $AR=\vartheta$ et $ER=\varphi$. Progrediatur motu jam concepto polus E tempusculo dt in e, erit $Rr=d\vartheta$ et $eG=d\varphi$, atque si motu uniformi secundum circulum maximum progrederetur, perveniret sequenti tempusculo in e', ut esset $rr'=d\vartheta+2d\varphi d\vartheta$ tang φ et $e'g=d\varphi-d\vartheta^2\sin\varphi\cos\varphi$, quarum formularum demonstrationem deinceps tradam. Jam capiatur $AO=\nu$, junganturque circulo maximo puncta O et e', erit arcus $Oe'=s$, sumto puncto e' pro primo E, et $r'O$ seu $RO=\nu-\vartheta$, atque $r'e'=RE=\varphi$, unde erit $\cos s=\cos\varphi\cos(\nu-\vartheta)$, at erit $\sin Oe'r'=\frac{\sin(\nu-\vartheta)}{\sin s}$, seu tang $Oe'r'=\frac{\text{tang}(\nu-\vartheta)}{\sin\varphi}$, et $\cos Oe'r'=\frac{\sin\varphi\cos(\nu-\vartheta)}{\sin s}$. Nunc quia polus in hoc circulo Oe' pellitur, capiatur

$$e'\varepsilon=dds=\frac{kkdt^2(aa-bb)\sin s\cos s}{(aa+bb)z^3},$$

eritque ε punctum, ad quod polus fine alterius tempusculi dt reperietur; ducatur perpendiculum $\varepsilon\varrho$ et $e'\gamma$ ad $\varepsilon\varrho$ normalis, erit

$$\varepsilon\gamma=\frac{kkdt^2(aa-bb)\cos s\sin\varphi\cos(\nu-\vartheta)}{(aa+bb)z^3} \quad \text{et} \quad e'\gamma=\frac{kkdt^2(aa-bb)\cos s\sin(\nu-\vartheta)}{(aa+bb)z^3}=r'\varrho\cos\varphi,$$

ita ut sit

$$r'\varrho=\frac{kkdt^2(aa-bb)\sin(\nu-\vartheta)\cos(\nu-\vartheta)}{(aa+bb)z^3}.$$

At est $r\varrho=d\vartheta+dd\vartheta=rr'-r'\varrho$ et $e'g+\varepsilon\gamma=d\varphi+dd\varphi$, unde fit

$$dd\vartheta=2d\varphi d\vartheta\ \text{tang}\ \varphi-\frac{kkdt^2(aa-bb)\sin(\nu-\vartheta)\cos(\nu-\vartheta)}{(aa+bb)z^3},$$

et

$$dd\varphi=-d\vartheta^2\sin\varphi\cos\varphi+\frac{kkdt^2(aa-bb)\sin\varphi\cos\varphi\cos^2(\nu-\vartheta)}{(aa+bb)z^3},$$

quibus aequationibus motus poli E continetur, ita ut ex iis ad quodvis tempus positio axis CE definiri queat.

Quodsi vero loco temporis t anomaliam mediam u in calculum introducamus, reperietur $kkdt^2=2c^3du^2$, sicque simul quantitas kk ex calculo egreditur, eritque ergo

$$dd\vartheta = 2\,d\vartheta\,d\varphi \tang \varphi - \frac{2c^3\,du^2\,(aa - bb)\sin(\nu - \vartheta)\cos(\nu - \vartheta)}{(aa + bb)\,z^3},$$

$$dd\varphi = -\,d\vartheta^2 \sin\varphi\cos\varphi + \frac{2c^3\,du^2\,(aa - bb)\sin\varphi\cos\varphi\cos^2(\nu - \vartheta)}{(aa + bb)\,z^3}.$$

Posita autem anomalia media $= u$, quae anomaliae verae $ACO = \nu$ respondeat, ponatur anomalia excentrica $= r$, erit

$$u = r + n\sin r, \qquad \cos\nu = \frac{n + \cos r}{1 + n\cos r},$$

$$z = c\,(1 + n\cos r), \qquad du = dr\,(1 + n\cos r) = \frac{z\,dr}{c},$$

$$\sin\nu = \frac{\sin r\,\sqrt{(1 - nn)}}{1 + n\cos r}, \quad \text{et} \quad d\nu = \frac{dr\,\sqrt{(1 - nn)}}{1 + n\cos r} = \frac{du\,\sqrt{(1 - nn)}}{(1 + n\cos r)^2}.$$

Cum jam du sit constans, erit introducendo r

$$ddr\,(1 + n\cos r) - ndr^2\sin r = 0 \quad \text{seu} \quad ddr = \frac{ndr^2\sin r}{1 + n\cos r},$$

ideoque habebuntur hae duae aequationes

$$dd\vartheta = 2\,d\vartheta\,d\varphi \tang \varphi - \frac{2(aa - bb)\,dr^2\sin(\nu - \vartheta)\cos(\nu - \vartheta)}{(aa + bb)\,(1 + n\cos r)},$$

$$dd\varphi = -\,d\vartheta^2\sin\varphi\cos\varphi + \frac{2(aa - bb)\,dr^2\sin\varphi\cos\varphi\cos^2(\nu - \vartheta)}{(aa + bb)\,(1 + n\cos r)},$$

multiplicetur prior per $d\vartheta\cos^2\varphi$ et posterior per $d\varphi$, ambaeque addantur, prodibit

$$d\vartheta\,dd\vartheta\cos^2\varphi + d\varphi\,dd\varphi - d\varphi\,d\vartheta^2\sin\varphi\cos\varphi = \frac{2(aa-bb)\,dr^2\cos\varphi\cos(\nu-\vartheta)\,(d\varphi\sin\varphi\cos(\nu-\vartheta) - d\vartheta\cos\varphi\sin(\nu-\vartheta))}{(aa + bb)\,(1 + n\cos r)}$$

cujus pars prior est integrabilis; fiet enim

$$\frac{1}{2}d\varphi^2 + \frac{1}{2}d\vartheta^2\cos^2\varphi = \frac{2\,(aa - bb)\,du^2}{aa + bb}\int\frac{\cos\varphi\cos(\nu - \vartheta)\,(d\varphi\sin\varphi\cos(\nu - \vartheta) - d\vartheta\cos\varphi\sin(\nu - \vartheta))}{(1 + n\cos r)^3}$$

Ponatur $\frac{aa - bb}{aa + bb} = m$, eritque

$$dd\vartheta = 2\,d\vartheta\,d\varphi \tang \varphi - \frac{m\,du^2\sin 2\,(\nu - \vartheta)}{(1 + n\cos r)^3}, \quad \frac{dd\varphi}{\sin\varphi\cos\varphi} + d\vartheta^2 = \frac{m\,du^2\,(1 + \cos 2(\nu - \vartheta))}{(1 + n\cos r)^3}.$$

Quo clarius perspiciamus, quemadmodum has aequationes tractari conveniat, assumamus primo axem CE plano eclipticae normaliter insistere; et quia hoc casu angulus OCE est rectus, momentum vi inclinantis evanescit: quare si axis in hoc situ semel quieverit, in eodem perpetuo persistet. Id quod etiam ex aequationibus inventis intelligitur; cum enim sit $\cos\varphi = 0$ et $\tang\varphi = \infty$, prior aequatio dat $d\vartheta\,d\varphi = 0$, et altera $dd\varphi = 0$, quibus satisfit si $d\varphi = 0$, seu si axis CE perpetuo ad planum eclipticae maneat perpendicularis.

Ponamus nunc axem in ipsum planum eclipticae incidere; et quia is ab momento vis solis de hoc plano non depellitur, perpetuo erit $\varphi = 0$, atque motus axis ex priori aequatione sola determinabitur, quae hoc casu abit in $dd\vartheta = \frac{-m\,du^2 \sin 2(v-\vartheta)}{(1+n\cos r)^3}$.

Sit primo orbita circularis, seu $n = 0$ et $v = u$, erit $dd\vartheta + m\,du^2 \sin 2(u-\vartheta) = 0$. Fingatur $d\vartheta = \alpha\,du \cos 2(u-\vartheta) + P\,du$, erit

$$dd\vartheta = -2\alpha du^2 \sin 2(u-\vartheta) + 2\alpha du\,d\vartheta \sin 2(u-\vartheta) + dP\,du, \quad \text{seu}$$

$$dd\vartheta = -2\alpha du^2 \sin 2(u-\vartheta) + \alpha\alpha du^2 \sin 4(u-\vartheta) + 2\alpha P du^2 \sin 2(u-\vartheta) + dP\,du = -m du^2 \sin 2(u-\vartheta).$$

Sit ergo $\alpha = \frac{1}{2}m$, ut sit $\frac{1}{4}mmdu \sin 4(u-\vartheta) + mPdu \sin 2(u-\vartheta) + dP = 0$. Ponatur

$$P = \frac{1}{16}mm\cos 4(u-\vartheta) + Q, \quad \text{ob} \quad du - d\vartheta = du - \frac{1}{2}mdu\cos 2(u-\vartheta) - \frac{1}{16}mmdu\cos 4(u-\vartheta) - Qdu,$$

erit

$$dP = -\frac{1}{4}mmdu \sin 4(u-\vartheta) + \frac{1}{8}m^3du \sin 4(u-\vartheta)\cos 2(u-\vartheta) + \frac{1}{64}m^4du \sin 4(u-\vartheta)\cos 4(u-\vartheta)$$
$$+ \frac{1}{4}mmQdu \sin 4(u-\vartheta) + dQ$$
$$= -\frac{1}{4}mmdu \sin 4(u-\vartheta) - \frac{1}{16}m^3du \sin 2(u-\vartheta)\cos 4(u-\vartheta) - mQdu \sin 2(u-\vartheta),$$

unde apparet Q habiturum esse coefficientem m^3, ideoque ejus valorem tam fore exiguum, ut rejici queat. Erit ergo vero proxime

$$d\vartheta = \frac{1}{2}mdu \cos 2(u-\vartheta) + \frac{1}{16}mmdu \cos 4(u-\vartheta),$$

hincque integrando ponatur

$$\vartheta = C + \frac{1}{4}m \sin 2(u-\vartheta) + \frac{1}{64}mm \sin 4(u-\vartheta) + R,$$

eritque

$$d\vartheta = \frac{1}{2}mdu \cos 2(u-\vartheta) + \frac{1}{16}mmdu \cos 4(u-\vartheta) + dR,$$
$$-\frac{1}{2}md\vartheta \cos 2(u-\vartheta) - \frac{1}{16}mmd\vartheta \cos 4(u-\vartheta),$$

quo valore substituto habebitur

$$dR = \frac{1}{4}mmdu \cos^2 2(u-\vartheta) + \frac{1}{16}m^3du \cos 2(u-\vartheta)\cos 4(u-\vartheta) + \frac{1}{256}m^4du \cos^2 4(u-\vartheta),$$

seu

$$dR = \frac{1}{8}mmdu + \frac{1}{8}mmdu \cos 4(u-\vartheta) + \frac{1}{32}m^3du \cos 2(u-\vartheta)$$
$$+ \frac{1}{32}m^3du \cos 6(u-\vartheta) + \frac{1}{512}m^4du + \frac{1}{512}m^4du \cos 8(u-\vartheta),$$

ergo $R = \frac{1}{8}mmu + \frac{1}{512}m^4u + \frac{1}{32}mm \sin 4(u-\vartheta)$. Consequenter habebitur

$$\vartheta = C + \frac{1}{4}m \sin 2(u-\vartheta) + \frac{3}{64}m^2 \sin 4(u-\vartheta) + \frac{1}{8}m^2u.$$

Potest autem hoc casu aequatio proposita $dd\vartheta + mdu^2 \sin 2(u-\vartheta) = 0$ absolute integrari, si multiplicetur per $2(du - d\vartheta)$, ut sit

$$2du\,dd\vartheta - 2d\vartheta\,dd\vartheta + 2mdu^2(du - d\vartheta)\sin 2(u-\vartheta) = 0,$$

erit enim $2dud\vartheta - d\vartheta^2 = Cdu^2 + mdu^2\cos 2(u-\vartheta)$, vel posito $u - \vartheta = s$, seu $\vartheta = u - s$ habebitur $du^2 - ds^2 = Cdu^2 + mdu^2\cos 2s$, seu $ds^2 = du^2(\alpha - m\cos 2s)$, hincque $du = \frac{ds}{\sqrt{(\alpha - m\cos 2s)}}$, ubi α est constans a motu axis ipsi primum impresso pendens. Quoniam igitur assumimus, momentum vis solis, seu littera m evanescat, axem esse quieturum, posito $m = 0$, erit $ds = du$, ideoque $\alpha = 1$, ita ut sit $du = \frac{ds}{\sqrt{(1 - m\cos 2s)}}$, ex qua aequatione promotionem axis a vi solis oriundam definiri oportet. Cum jam sit m fractio valde parva, erit

$$\frac{1}{\sqrt{(1 - m\cos 2s)}} = 1 + \frac{1}{2}m\cos 2s + \frac{1.3}{2.4}m^2\cos^2 2s + \frac{1.3.5}{2.4.6}m^3\cos^3 2s + \frac{1.3.5.7}{2.4.6.8}m^4\cos^4 2s + \text{etc.}$$

Potestatibus autem $\cos 2s$ ad cosinus angulorum multiplorum reductis, fiet

$$\begin{aligned}\frac{1}{\sqrt{(1 - m\cos 2s)}} = &+ 1 \qquad + \frac{1}{2}m\cos 2s + \frac{1}{2}\cdot\frac{1.3}{2.4}m^2\cos 4s + \frac{1}{4}\cdot\frac{1.3.5}{2.4.6}m^3\cos 6s + \frac{1}{8}\cdot\frac{1.3.5.7}{2.4.6.8}m^4\cos 8s \\ &+ \frac{1}{2}\cdot\frac{1.3}{2.4}m^2 + \frac{3}{4}\cdot\frac{1.3.5}{2.4.6}m^3 + \frac{4}{8}\cdot\frac{1.3.5.7}{2.4.6.8}m^4 \\ &+ \frac{3}{8}\cdot\frac{1.3.5.7}{2.4.6.8}m^4 \\ &\text{etc.}\end{aligned}$$

Integrando ergo habebitur

$$u = g + \left(1 + \frac{3}{16}m^2 + \frac{105}{1024}m^4\right)s + \frac{1}{4}m\left(1 + \frac{15}{32}m^2\right)\sin 2s + \frac{3}{64}m^2\left(1 + \frac{35}{48}m^2\right)\sin 4s$$
$$+ \frac{5}{384}m^3\sin 6s + \frac{35}{8192}m^4\sin 8s,$$

rejiciantur termini, in quibus m ultra duas obtinet dimensiones, eritque

$$u = g + u - \vartheta + \frac{3}{16}m^2u - \frac{3}{16}m^2\vartheta + \frac{1}{4}m\sin 2(u-\vartheta) + \frac{3}{64}m^2\sin 4(u-\vartheta),$$

seu $$\vartheta = g + \frac{3}{16}m^2u + \frac{1}{4}m\sin 2(u-\vartheta) + \frac{3}{64}m^2\sin 4(u-\vartheta),$$

axis ergo durante quavis solis revolutione modo progredietur, modo regredietur per arcum $=\frac{1}{2}m$, ita ut si $m = \frac{1}{200}$, hoc spatium futurum sit $= \frac{1}{400} = 0^0{,}14 = 8'\,24''$. Tum vero qualibet revolutione solis, seu singulis annis, axis in ecliptica progredietur per spatium $= \frac{3}{16}m^2 \cdot 360^0$, quod ergo si $m = \frac{1}{200}$, erit $= \frac{3.360^0}{16.40000} = 6''$.

Aliter autem res se habebit, si axis terrae ad eclipticam fuerit inclinatus; tum enim posita orbita solis circulari, ut sit $n = 0$ et $\varphi = u$, manente $\vartheta = u - s$, hae duo habebuntur aequationes resolvendae

$$dds = 2(ds - du)\,d\varphi\,\text{tang}\,\varphi + mdu^2\sin 2s,$$

$$\frac{dd\varphi}{\sin\varphi\cos\varphi} + (du - ds)^2 = mdu^2(1 + \cos 2s).$$

Multiplicetur aequatio prior per $2q\,ds$ et posterior per $-dq$, ambaeque invicem addantur, erit

$$\left.\begin{matrix} 2q\,ds\,dds - 4q\,ds\,(ds-du)\,d\varphi \operatorname{tang}\varphi \\ -\frac{dq\,dd\varphi}{\sin\varphi\cos\varphi} - dq\,(du-ds)^2 \end{matrix}\right\} = 2mq\,du^2 ds \sin 2s - m\,du^2 dq - m\,du^2 dq \cos 2s$$

et partem posteriorem integrando fiet

$$C - mq\,du^2 - mq\,du^2 \cos 2s = \int \begin{pmatrix} 2q\,ds\,dds - 4q\,ds\,(ds-du)\,d\varphi \operatorname{tang}\varphi \\ \frac{-dq\,dd\varphi}{\sin\varphi\cos\varphi} - dq\,(du-ds)^2 \end{pmatrix}.$$

Sit nunc $q = \cos^2\varphi$, erit $dq = -2d\varphi \sin\varphi\cos\varphi$, ideoque

$$C\,du^2 - m\,du^2 \cos^2\varphi\,(1+\cos 2s) = \int \begin{pmatrix} 2ds\,dds \cos^2\varphi - 4ds\,(ds-du)\,d\varphi \sin\varphi\cos\varphi \\ +2d\varphi\,dd\varphi + 2(du-ds)^2\,d\varphi \sin\varphi\cos\varphi \end{pmatrix}$$

$$= \int (2d\varphi\,dd\varphi + 2ds\,dds\cos^2\varphi - 2ds^2 d\varphi \sin\varphi\cos\varphi + 2du^2 d\varphi\sin\varphi\cos\varphi)$$

$$= d\varphi^2 + ds^2\cos^2\varphi - du^2\cos^2\varphi.$$

Quocirca erit $\quad C\,du^2 = d\varphi^2 + (ds^2 - du^2)\cos^2\varphi + m\,du^2\cos^2\varphi\,(1+\cos 2s).$

Si jam sumamus casu, quo $m=0$, axem quiescere, ut sit $ds = du$ et $d\varphi = 0$, fiet $C=0$ et $\frac{d\varphi^2}{\cos^2\varphi} = du^2 - ds^2 - m\,du^2\,(1+\cos 2s)$, hincque

$$du = \sqrt{\frac{ds^2 + \frac{d\varphi^2}{\cos^2\varphi}}{1 - m\,(1+\cos 2s)}}.$$

Verum constantem C potius convenit definiri ex statu quopiam axis initiali. Si igitur assumamus principia, ubi axis primum a vi solis comitari coepit, fuisse angulum $s = u - \vartheta = \varepsilon$, et inclinationem $\varphi = \gamma$; in hoc statu motum axis nullum statui oportet, seu erit $d\vartheta = 0$ et $d\varphi = 0$, ideoque $ds = du$, quibus substitutis fiet $C\,du^2 = m\,du^2\cos^2\gamma\,(1+\cos 2\varepsilon)$, unde hanc obtinemus aequationem

$$m\,du^2\cos^2\gamma\,(1+\cos 2\varepsilon) = d\varphi^2 + ds^2\cos^2\varphi - du^2\cos^2\varphi + m\,du^2\cos^2\varphi\,(1+\cos 2s),$$

ex qua oritur

$$du^2 = \frac{d\varphi^2 + ds^2\cos^2\varphi}{\cos^2\varphi + m\cos^2\gamma\,(1+\cos 2\varepsilon) - m\cos^2\varphi\,(1+\cos 2s)}.$$

Quoniam inclinatio φ minime a primitiva γ discrepat, ponatur $\varphi = \gamma + \omega$, erit ω quantitas minima, et $d\omega$ prae ds pro evanescente haberi potest. Fiet ergo $d\varphi = d\omega$ et $\cos\varphi = \cos\gamma - \omega\sin\gamma$, atque $\cos^2\varphi = \cos^2\gamma - \omega\sin 2\gamma$, quo valore substituto erit

$$du^2 = \frac{d\omega^2 + ds^2\cos^2\gamma - \omega\,ds^2\sin 2\gamma}{\cos^2\gamma - \omega\sin 2\gamma + m\cos 2\varepsilon + m\omega\sin 2\gamma - m\cos^2\gamma\cos 2s + m\omega\sin 2\gamma\cos 2s},$$

seu

$$du^2 = \frac{ds^2}{1 - m\cos 2s + \frac{m\cos 2\varepsilon + m\omega\sin 2\gamma}{\cos^2\gamma - \omega\sin 2\gamma}} + \frac{d\omega^2}{\cos^2\gamma + m\cos 2\varepsilon - \omega\sin 2\gamma - m\cos^2\gamma\cos 2s},$$

*

vel approximando sit $\frac{\cos 2\varepsilon}{\cos^2\gamma} = \alpha$, erit

$$du^2 = \frac{ds^2}{1 + m\alpha - m\cos 2s + 2m(1+\alpha)\omega \operatorname{tang}\gamma} + \frac{d\omega^2}{\cos^2\gamma + m\cos 2\varepsilon - \omega\sin 2\gamma - m\cos^2\gamma\cos 2s}$$

seu

$$du^2 = \frac{ds^2}{1+m\alpha} + \frac{m\,ds^2\cos 2s}{(1+m\alpha)^2} + \frac{mm\,ds^2\cos^2 2s}{(1+m\alpha)^3} - \frac{2m(1+\alpha)\,\omega\,ds^2\operatorname{tang}\gamma}{(1+m\alpha)^2} + \frac{d\omega^2}{\cos^2\gamma + m\cos 2\varepsilon}.$$

Ponatur $\omega = A\cos 2\varepsilon - A\cos 2s$, quo posito $s = \varepsilon$ fiat $\varphi = \gamma$, erit $d\omega = 2A\,ds\sin 2s$, $dd\omega = 2A\,dds\sin 2s + 4A\,ds^2\cos 2s$. At ob $dd\varphi = dd\omega$ et $\sin\varphi = \sin\gamma + \omega\cos\gamma$,

$$\sin\varphi\cos\varphi = \sin\gamma\cos\gamma + \omega\cos 2\gamma,$$

$$\text{et } \frac{dd\varphi}{\sin\varphi\cos\varphi} = \frac{dd\omega}{\sin\gamma\cos\gamma} - \frac{\omega\,dd\omega\cos 2\gamma}{\sin^2\gamma\cos^2\gamma} = -(du-ds)^2 + m\,du^2(1+\cos 2s).$$

Ergo habebitur

$$-(du-ds)^2\sin\gamma\cos\gamma + m\,du^2\sin\gamma\cos\gamma(1+\cos 2s) = 2A\,dds\sin 2s + 4A\,ds^2\cos 2s.$$

At prior aequatio dat

$$dds = 4A(ds-du)\,ds\sin 2s\left(\operatorname{tang}\gamma + \frac{\omega}{\cos^2\gamma}\right) + m\,du^2\sin 2s,$$

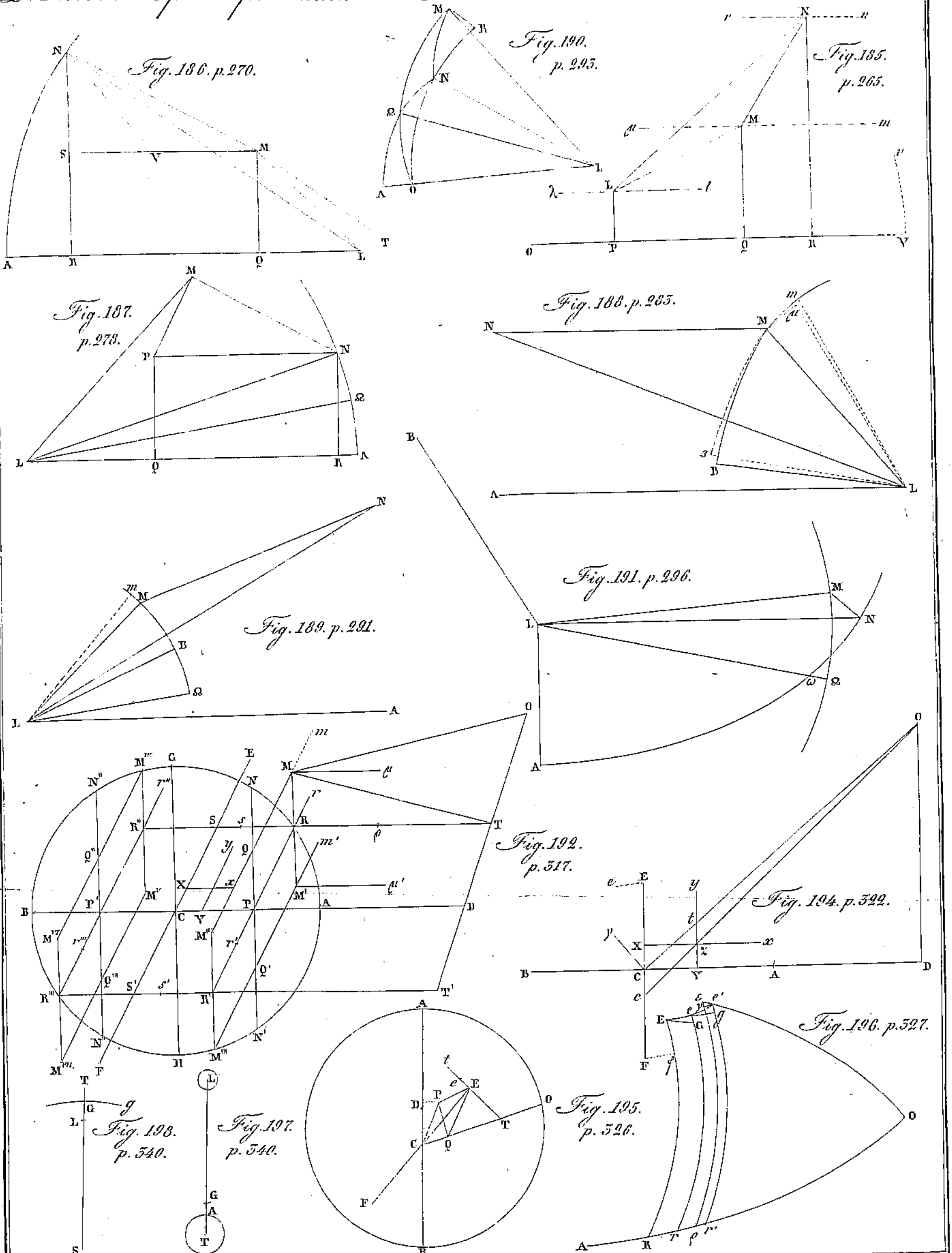
quo valore ibi substituto fiet

$$-(du-ds)^2\sin\gamma\cos\gamma + m\,du^2\sin\gamma\cos\gamma(1+\cos 2s) = 8AA(ds-du)\,ds\sin^2 2s\left(\operatorname{tang}\gamma + \frac{\omega}{\cos^2\gamma}\right)$$
$$+ 2Am\,du^2\sin^2 2s + 4A\,ds^2\cos 2s.$$

At ex superiori aequatione est

$$du^2 = \frac{ds^2}{1+m\alpha} + \frac{m^2 ds^2}{2(1+m\alpha)^3} + \frac{m\,ds^2\cos 2s}{(1+m\alpha)^2} - \frac{2\alpha m(1+\alpha)A\,ds^2\sin\gamma\cos\gamma}{(1+m\alpha)^2}$$
$$+ \frac{2m(1+\alpha)A\,ds^2\operatorname{tang}\gamma\cos 2s}{(1+m\alpha)^2} + \frac{2AA\,ds^2}{(1+m\alpha)\cos^2\gamma} - \frac{2AA\,ds^2\cos 4s}{(1+m\alpha)\cos^2\gamma} + \frac{mm\,ds^2\cos 4s}{2(1+m\alpha)^3}.$$

. . . .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