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De comparatione arcuum curvarum irrectificabilium

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XXII.

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Sectio prima

continens evolutionem hujus aequationis:

$$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy.$$

I.

Si ex hac aequatione sigillatim utriusque variabilis x et y valor extrahatur, reperietur

$$y=\frac{-\beta-\delta x+\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)x+(\delta\delta-\gamma\gamma)xx)}}{\gamma},$$

$$x=\frac{-\beta-\delta y-\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)y+(\delta\delta-\gamma\gamma)yy)}}{\gamma}.$$

Ponatur brevitatis gratia $\beta\beta-\alpha\gamma=Ap$, $\beta(\delta-\gamma)=Bp$ et $\delta\delta-\gamma\gamma=Cp$, eritque

$$\beta+\gamma y+\delta x=+\sqrt{(A+2Bx+Cxx)p},$$

$$\beta+\gamma x+\delta y=-\sqrt{(A+2By+Cyy)p}.$$

II.

Litteris jam A, B, C pro lubitu assumtis, ex iis litterae α, β, γ, δ et p sequenti modo definientur: Primo ex aequalitate secunda fit $\delta-\gamma=\frac{Bp}{\beta}$, qui valor in tertia $\delta+\gamma=\frac{Cp}{\delta-\gamma}$ substitutus dat $\delta+\gamma=\frac{C\beta}{B}$; ita ut sit

$$\delta=\frac{C\beta}{2B}+\frac{Bp}{2\beta} \quad \text{et} \quad \gamma=\frac{C\beta}{2B}-\frac{Bp}{2\beta}.$$

Hinc autem aequalitas prima abit in hanc

$$\beta\beta-\frac{C\alpha\beta}{2B}+\frac{B\alpha p}{2\beta}=Ap$$

ex qua definietur $p=\frac{\beta\beta(2B\beta-C\alpha)}{B(2A\beta-B\alpha)}$, indeque porro

$$\delta=\frac{\beta(AC\beta+BB\beta-BC\alpha)}{B(2A\beta-B\alpha)} \quad \text{et} \quad \gamma=\frac{\beta\beta(AC-BB)}{B(2A\beta-B\alpha)}.$$

Sic ergo litterae α et β arbitrio nostro relinquuntur, quarum altera quidem unitate exprimi poterit, altera vero constantem arbitrariam, a coëfficientibus A, B, C non pendentem, exhibebit.

III.

Differentietur nunc aequatio proposita, ac prodibit

$$dx\,(\beta + \gamma x + \delta y) + dy\,(\beta + \gamma y + \delta x) = 0,$$

unde conficitur haec aequatio

$$\frac{dx}{\beta + \gamma y + \delta x} = \frac{-dy}{\beta + \gamma x + \delta y},$$

quae substitutis valoribus in articulo I inventis, abibit in hanc aequationem differentialem:

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx)}} - \frac{dy}{\sqrt{(A + 2By + Cyy)}} = 0$$

cujus propterea integralis est ipsa aequatio assumta.

IV.

Proposita ergo vicissim hac aequatione differentiali

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx)}} - \frac{dy}{\sqrt{(A + 2By + Cyy)}} = 0,$$

ejus integrale semper algebraice exhiberi poterit, quippe quod erit

$$0 = \alpha + 2\beta\,(x + y) + \frac{\beta\beta(AC - BB)(xx + yy) + 2\beta(AC\beta + BB\beta - BC\alpha)xy}{B(2A\beta - B\alpha)},$$

et quia hic continetur constans ab arbitrio nostro pendens, erit hoc integrale quoque completum aequationis differentialis propositae. Erit ergo retentis litteris graecis

$$\text{vel} \quad y = \frac{-\beta - \delta x + \sqrt{(A + 2Bx + Cxx)p}}{\gamma},$$

$$\text{vel} \quad x = \frac{-\beta - \delta y - \sqrt{(A + 2By + Cyy)p}}{\gamma}.$$

V.

Quemadmodum autem istarum formularum integralium differentia

$$\int\frac{dx}{\sqrt{(A + 2Bx + Cxx)}} - \int\frac{dy}{\sqrt{(A + 2By + Cyy)}},$$

est constans, siquidem inter x et y ea relatio subsistat, ut sit

$$0 = \alpha + 2\beta\,(x + y) + \gamma\,(xx + yy) + 2\delta xy,$$

ita etiam eadem manente relatione, differentia hujusmodi formularum

$$\int\frac{x^n dx}{\sqrt{(A + 2Bx + Cxx)}} - \int\frac{y^n dy}{\sqrt{(A + 2By + Cyy)}}$$

commode exprimi potest; quos valores indagasse operae pretium erit.

VI.

Posito ergo exponente $n = 1$, statuamus

$$\frac{x\,dx}{\sqrt{(A + 2Bx + Cxx)}} - \frac{y\,dy}{\sqrt{(A + 2By + Cyy)}} = dV,$$

eritque valoribus initio traditis pro his formulis irrationalibus substituendis

$$\frac{x\,dx\sqrt{p}}{\beta+\gamma y+\delta x}+\frac{y\,dy\sqrt{p}}{\beta+\gamma x+\delta y}=dV,$$

seu $x\,dx(\beta+\gamma x+\delta y)+y\,dy(\beta+\gamma y+\delta x)=\frac{dV}{\sqrt{p}}(\beta+\gamma y+\delta x)(\beta+\gamma x+\delta y)$; at est

$$(\beta+\gamma y+\delta x)(\beta+\gamma x+\delta y)=\beta\beta+\beta(\gamma+\delta)(x+y)+\gamma\delta(xx+yy)+(\gamma\gamma+\delta\delta)xy.$$

VII.

Quo hanc formulam facilius expediamus, ponamus $x+y=t$ et $xy=u$, erit

$$xx+yy=tt-2u \quad \text{et} \quad x^3+y^3=t^3-3tu,$$

sicque aequatio abit in hanc formam

$$\beta(x\,dx+y\,dy)+\gamma(xx\,dx+yy\,dy)+\delta xy(dx+dy)=\frac{dV}{\sqrt{p}}(\beta\beta+\beta(\gamma+\delta)t+\gamma\delta tt+(\gamma-\delta)^2u).$$

Ipsa autem aequatio assumta fit: $0=\alpha+2\beta t+\gamma tt+2(\delta-\gamma)u$, et penitus introductis litteris t et u habebimus

$$\beta(t\,dt-du)+\gamma(tt\,dt-t\,du-u\,dt)+\delta u\,dt=\frac{dV}{\sqrt{p}}(\beta\beta-\alpha\delta+\beta(\gamma-\delta)t+(\gamma\gamma-\delta\delta)u),$$

seu $dt(\beta t+\gamma tt-(\gamma-\delta)u)-du(\beta+\gamma t)=\frac{dV}{\sqrt{p}}(\beta\beta-\alpha\delta+\beta(\gamma-\delta)t+(\gamma\gamma-\delta\delta)u).$

VIII.

Ex aequatione autem assumta si differentietur, fit $dt(\beta+\gamma t)=(\gamma-\delta)du$, unde aequationis ultimae prius membrum transformatur in

$$\frac{dt}{\gamma-\delta}(-\beta\beta-\beta(\gamma+\delta)t-\gamma\delta tt-(\gamma-\delta)^2u),$$

quod cum aequale esse debeat huic formulae

$$\frac{dV}{\sqrt{p}}(\beta\beta+\beta(\gamma+\delta)t+\gamma\delta tt+(\gamma-\delta)^2u),$$

commode inde oritur $\quad \frac{dV}{\sqrt{p}}=\frac{-dt}{\gamma-\delta} \quad$ et $\quad V=\frac{-t\sqrt{p}}{\gamma-\delta}.$

IX.

Cum jam sit $t=x+y$, habebimus sequentem aequationem integratam

$$\int\frac{x\,dx}{\sqrt{(A+2Bx+Cxx)}}-\int\frac{y\,dy}{\sqrt{(A+2By+Cyy)}}=\text{Const.}-\frac{(x+y)\sqrt{p}}{\gamma-\delta},$$

existente $0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy$, siquidem relationes supra exhibitae inter litteras A, B, C et α, β, γ, δ ac p locum habeant. Hinc ergo eadem manente determinatione variabilium x et y erit generalius:

$$\int\frac{dx(\mathfrak{A}+\mathfrak{B}x)}{\sqrt{(A+2Bx+Cxx)}}-\int\frac{dy(\mathfrak{A}+\mathfrak{B}y)}{\sqrt{(A+2By+Cyy)}}=\text{Const.}-\frac{\mathfrak{B}(x+y)\sqrt{p}}{\gamma-\delta}.$$

X.

Progrediamur porro, ac statuamus

$$\frac{xx\,dx}{\sqrt{(A+2Bx+Cxx)}}-\frac{yy\,dy}{\sqrt{(A+2By+Cyy)}}=dV,$$

erit posito brevitatis ergo $\beta\beta + \beta(\gamma + \delta)t + \gamma\delta tt + (\gamma - \delta)^2 u = T$, si loco istarum formularum surdarum valores ante reperti substituantur

$xxdx(\beta + \gamma x + \delta y) + yydy(\beta + \gamma y + \delta x) = \frac{TdV}{\sqrt{p}}$, existente ut ante $t = x + y$ et $u = xy$.

XI.

Cum nunc sit $x^4 + y^4 = t^4 - 4ttu + 2uu$, erit eliminatis variabilibus x et y

$$\beta(ttdt - tdu - udt) + \gamma(t^3dt - ttdu - 2tudt + udu) + \delta u(tdt - du) = \frac{TdV}{\sqrt{p}},$$

sive
$$dt(\beta tt - \beta u + \gamma t^3 - 2\gamma tu + \delta tu) - du(\beta t + \gamma tt - \gamma u + \delta u) = \frac{TdV}{\sqrt{p}}.$$

Cum autem sit $du = \frac{dt(\beta + \gamma t)}{\gamma - \delta}$, erit hac facta substitutione

$$\frac{dt}{\gamma - \delta}(-\beta\beta t - \beta(\gamma + \delta)tt - \gamma\delta t^3 - (\gamma - \delta)^2 tu) = \frac{TdV}{\sqrt{p}} = \frac{-Ttdt}{\gamma - \delta},$$

sicque erit $\frac{dV}{\sqrt{p}} = \frac{-tdt}{\gamma - \delta}$ et $V = \frac{-tt\sqrt{p}}{2(\gamma - \delta)}$.

XII.

Hinc ergo adipiscimur sequentem aequationem integratam

$$\int\frac{xxdx}{\sqrt{(A + 2Bx + Cxx)}} - \int\frac{yydy}{\sqrt{(A + 2By + Cyy)}} = \text{Const.} - \frac{(x+y)^2\sqrt{p}}{2(\gamma - \delta)},$$

atque in genere concludimus fore

$$\int\frac{dx(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx)}{\sqrt{(A + 2Bx + Cxx)}} - \int\frac{dy(\mathfrak{A} + \mathfrak{B}y + \mathfrak{C}yy)}{\sqrt{(A + 2By + Cyy)}} = \text{Const.} - \frac{\mathfrak{B}(x+y)\sqrt{p}}{\gamma - \delta} - \frac{\mathfrak{C}(x+y)^2\sqrt{p}}{2(\gamma - \delta)},$$

siquidem fuerit $0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy$. Erit autem ex relationibus supra assignatis $\frac{\sqrt{p}}{\gamma - \delta} = \frac{-\beta}{B\sqrt{p}}$ sive $\frac{\sqrt{p}}{\gamma - \delta} = -\sqrt{\frac{2A\beta - B\alpha}{B(2B\beta - C\alpha)}}$.

XIII.

Ponatur jam in genere

$$\frac{x^n dx}{\sqrt{(A + 2Bx + Cxx)}} - \frac{y^n dy}{\sqrt{(A + 2By + Cyy)}} = dV,$$

eritque ponendo $T = \beta\beta + \beta(\gamma + \delta)t + \gamma\delta tt + (\gamma - \delta)^2 u$,

$$x^n dx(\beta + \gamma x + \delta y) + y^n dy(\beta + \gamma y + \delta x) = \frac{TdV}{\sqrt{p}},$$

at ob $x + y = t$ et $xy = u$ habebimus $x = \frac{t + \sqrt{(tt - 4u)}}{2}$ et $y = \frac{t - \sqrt{(tt - 4u)}}{2}$, ideoque

$$\beta + \gamma x + \delta y = \frac{2\beta + (\gamma + \delta)t + (\gamma - \delta)\sqrt{(tt - 4u)}}{2},$$

$$\beta + \gamma y + \delta x = \frac{2\beta + (\gamma + \delta)t - (\gamma - \delta)\sqrt{(tt - 4u)}}{2}.$$

XIV.

Differentiando autem habebimus

$$dx = \frac{dt\sqrt{(tt - 4u)} + tdt - 2du}{2\sqrt{(tt - 4u)}} \quad \text{et} \quad dy = \frac{dt\sqrt{(tt - 4u)} - tdt + 2du}{2\sqrt{(tt - 4u)}},$$

at ante vidimus esse $du=\frac{dt(\beta+\gamma t)}{\gamma-\delta}$: quo valore substituto prodibit

$$dx=\frac{-dt(2\beta+(\gamma+\delta)t-(\gamma-\delta)\sqrt{(tt-4u)})}{2(\gamma-\delta)\sqrt{(tt-4u)}}$$

$$dy=\frac{dt(2\beta+(\gamma+\delta)t+(\gamma-\delta)\sqrt{(tt-4u)})}{2(\gamma-\delta)\sqrt{(tt-4u)}}.$$

Hisque valoribus substitutis

$$dx(\beta+\gamma x+\delta y)=\frac{-dt(4\beta\beta+4\beta(\gamma+\delta)t+4\gamma\delta tt+4(\gamma-\delta)^2u)}{4(\gamma-\delta)\sqrt{(tt-4u)}}=\frac{-Tdt}{(\gamma-\delta)\sqrt{(tt-4u)}},$$

$$\text{et}\quad dy(\beta+\gamma y+\delta x)=\frac{+Tdt}{(\gamma-\delta)\sqrt{(tt-4u)}}.$$

XV.

Nostra ergo aequatione per T divisa habebimus

$$\frac{-dt(x^n-y^n)}{(\gamma-\delta)\sqrt{(tt-4u)}}=\frac{dV}{\sqrt{p}}\quad\text{et}\quad V=\frac{-\sqrt{p}}{\gamma-\delta}\int\frac{dt(x^n-y^n)}{\sqrt{(tt-4u)}},$$

existente $x=\frac{t+\sqrt{(tt-4u)}}{2}$ et $y=\frac{t-\sqrt{(tt-4u)}}{2}$ atque $u=\frac{\alpha+2\beta t+\gamma tt}{2(\gamma-\delta)}$, unde

$$\sqrt{(tt-4u)}=\sqrt{\frac{2\alpha+4\beta t+(\gamma+\delta)tt}{\delta-\gamma}}.$$

Unde valores ipsius $\frac{x^n-y^n}{\sqrt{(tt-4u)}}$ ex sequente progressione colligi poterunt:

$$\frac{x^0-y^0}{\sqrt{(tt-4u)}}=0,$$

$$\frac{x^1-y^1}{\sqrt{(tt-4u)}}=1,$$

$$\frac{x^2-y^2}{\sqrt{(tt-4u)}}=t,$$

$$\frac{x^3-y^3}{\sqrt{(tt-4u)}}=tt-u=\frac{(\gamma-2\delta)tt-2\beta t-\alpha}{2(\gamma-\delta)},$$

$$\frac{x^4-y^4}{\sqrt{(tt-4u)}}=t^3-2tu=\frac{-2\delta t^3-4\beta tt-2\alpha t}{2(\gamma-\delta)},$$

$$\frac{x^5-y^5}{\sqrt{(tt-4u)}}=t^4-3ttu+uu=\frac{-(\gamma\gamma+2\gamma\delta-4\delta\delta)t^4-4\beta(2\gamma-3\delta)t^3+(4\beta\beta-4\alpha\gamma+6\alpha\delta)tt+4\alpha\beta t+\alpha\alpha}{4(\gamma-\delta)^2}$$

etc. etc.

XVI.

Nanciscemur ergo formulas sequentes integratas

$$\int\frac{x^3dx}{\sqrt{(A+2Bx+Cxx)}}-\int\frac{y^3dy}{\sqrt{(A+2By+Cyy)}}=\text{Const.}-\frac{\sqrt{p}}{2(\gamma-\delta)^2}\left(\frac{1}{3}(\gamma-2\delta)(x+y)^3-\beta(x+y)^2-\alpha(x+y)\right)$$

$$\int\frac{x^4dx}{\sqrt{(A+2Bx+Cxx)}}-\int\frac{y^4dy}{\sqrt{(A+2By+Cyy)}}=\text{Const.}+\frac{\sqrt{p}}{(\gamma-\delta)^2}\left(\frac{1}{4}\delta(x+y)^4+\frac{2}{3}\beta(x+y)^3+\frac{1}{2}\alpha(x+y)^2\right)$$

quae scilicet locum habent, si variabiles x et y ita a se invicem pendent, ut sit

$$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy,$$

atque hi coëfficientes pariter atque p secundum praescriptas formulas ex datis A, B, C determinentur.

XVII.

Hinc ergo infinitae formulae integrales exhiberi possunt, quae etsi ipsae non sint integrabiles, earum tamen differentia vel sit constans, vel geometrice seu algebraice assignari queat. Quae comparatio cum in analysi insignem habeat usum, tum imprimis in arcubus curvarum irrectificabilium inter se comparandis summam affert utilitatem, quam in aliquot exemplis ostendisse juvabit.

De comparatione arcuum Circuli.

1. Sit radius circuli $=1$, in eoque abscissa a centro sumta $=z$; erit arcus ei respondens $=\int\frac{dz}{\sqrt{(1-zz)}}$, cujus propterea sinus est $=z$. Ut igitur nostrae formulae hujusmodi arcus circuli exprimant, poni debet $A=1$, $B=0$, $C=-1$; quo facto habebimus

$$\beta\beta-\alpha\gamma=p,\quad \beta(\delta-\gamma)=0 \quad\text{et}\quad \delta\delta-\gamma\gamma=-p;$$

has enim determinationes ab ipsa origine peti oportet, quia ob $B=0$, valores inventi fiunt incongrui. Jam ex formula secunda sequitur vel $\delta-\gamma=0$, vel $\beta=0$, quorum ille valor $\delta=\gamma$ formulae tertiae adversatur. Erit ergo $\beta=0$, $\delta=\pm\sqrt{(\gamma\gamma-p)}$ et $\alpha=\frac{-p}{\gamma}$. Ambae ergo quantitates constantes γ et p arbitrio nostro relinquuntur.

2. Quo formulae nostrae fiant simpliciores, ponamus $\gamma=1$ et $p=cc$, eritque

$$\alpha=-cc,\quad \beta=0,\quad \gamma=1 \quad\text{et}\quad \delta=-\sqrt{(1-cc)},$$

ac nostra aequatio canonica, relationem variabilium x et y determinans, fiet

$$0=-cc+xx+yy-2xy\sqrt{(1-cc)},$$

ex qua colligitur
$$y=x\sqrt{(1-cc)}\pm c\sqrt{(1-xx)}.$$

3. Quodsi ergo iste valor ipsi y tribuatur, erit

$$\int\frac{dx}{\sqrt{(1-xx)}}-\int\frac{dy}{\sqrt{(1-yy)}}=\text{Const.}$$

Denotemus brevitatis gratia haec integralia ita

$$\int\frac{dx}{\sqrt{(1-xx)}}=\Pi.x \quad\text{et}\quad \int\frac{dx}{\sqrt{(1-yy)}}=\Pi.y$$

atque $\Pi.x$ et $\Pi.y$ indicabunt arcus circuli, abscissis seu sinibus x et y respondentes. Quocirca erit

$$\Pi.x-\Pi.(x\sqrt{(1-cc)}+c\sqrt{(1-xx)})=\text{Const.}$$

4. Ad constantem determinandam ponatur $x=0$, et ob $\Pi.0=0$, fiet Const. $=-\Pi.c$ sicque erit

$$\Pi.c+\Pi.x=\Pi.(x\sqrt{(1-cc)}+c\sqrt{(1-xx)});$$

unde arcus assignari poterit aequalis summae duorum arcuum quorumcunque. Ac si x capiatur negativum ob $\Pi.(-x)=-\Pi.x$, erit

$$\Pi.c-\Pi.x=\Pi.(c\sqrt{(1-xx)}-x\sqrt{(1-cc)}),$$

qua arcus, differentiae duorum arcuum aequalis, definitur.

5. Si in formula priori ponatur $x = c$, erit $2\Pi . c = \Pi . 2c\sqrt{(1-cc)}$. Ac si porro ponatur $x = 2c\sqrt{(1-cc)}$, ut sit $\Pi . x = 2\Pi . c$, erit ob $\sqrt{(1-xx)} = 1 - 2cc$,

$$3\Pi . c = \Pi . (3c - 4c^3).$$

Posito autem ultra $x = 3c - 4c^3$, erit

$$4\Pi . c = \Pi . (x\sqrt{(1-cc)} + c\sqrt{(1-xx)})$$

unde multiplicatio arcuum circularium est manifesta.

De comparatione arcuum Parabolae.

* 6. Existente (Fig. 55.) AB parabolae axe, sumentur abscissae AP in tangente verticis A, sitque parameter parabolae $= 2$; unde vocata abscissa quacunque $AP = z$, erit applicata $Pp = \frac{zz}{2}$, ideoque arcus $Ap = \int dz\sqrt{(1+zz)}$, quae expressio ut ad nostras formulas reducatur, in hanc abit $\int \frac{dz(1+zz)}{\sqrt{(1+zz)}}$. Quare fieri oportet $A = 1$, $B = 0$ et $C = 1$, unde ut ante habebimus

$$\beta = 0, \quad \alpha = \frac{-p}{\gamma} \quad \text{et} \quad \delta = \pm\sqrt{(\gamma\gamma + p)}.$$

Sit ergo $\gamma = 1$ et $p = cc$, atque aequatio relationem inter x et y exhibens erit

$$0 = -cc + xx + yy - 2xy\sqrt{(cc+1)}, \quad \text{seu} \quad y = x\sqrt{(1+cc)} + c\sqrt{(1+xx)}.$$

7. Deinde ob $\sqrt{p} = c$ et $\gamma - \delta = 1 + \sqrt{(1+cc)}$, facto $\mathfrak{A} = 1$, $\mathfrak{B} = 0$ et $\mathfrak{C} = 1$, erit ex formula XII data

$$\int \frac{dx\,(1+xx)}{\sqrt{(1+xx)}} - \int \frac{dy\,(1+yy)}{\sqrt{(1+yy)}} = \text{Const.} - \frac{c(x+y)^2}{2+2\sqrt{(1+cc)}}.$$

At est $x + y = x(1+\sqrt{(1+cc)}) + c\sqrt{(1+xx)}$, ergo

$$(x+y)^2 = 2xx(1+cc+\sqrt{(1+cc)}) + cc + 2cx(1+\sqrt{(1+cc)})\sqrt{(1+xx)}.$$

Quare formularum istarum integralium differentia erit

$$\text{Const.} - cxx\sqrt{(1+cc)} - ccx\sqrt{(1+xx)} = \text{Const.} - cxy.$$

8. Indicetur arcus parabolae abscissae cuicunque z respondens $\int dz\sqrt{(1+zz)}$ per $\Pi . z$, et nostra aequatio hanc induet formam:

$$\Pi . x - \Pi . (x\sqrt{(1+cc)} + c\sqrt{(1+xx)}) = -\Pi . c - cx(x\sqrt{(1+cc)} + c\sqrt{(1+xx)}),$$

sive
$$\Pi . c + \Pi . x = \Pi . (x\sqrt{(1+cc)} + c\sqrt{(1+xx)}) - cx(x\sqrt{(1+cc)} + c\sqrt{(1+xx)}).$$

Datis ergo duobus arcubus quibuscunque, tertius arcus assignari potest, qui a summa illorum deficit quantitate geometrice assignabili. Vel quo indoles hujus aequationis clarius perspiciatur, erit

$$\Pi . c + \Pi . x = \Pi . y - cxy$$

siquidem fuerit
$$y = x\sqrt{(1+cc)} + c\sqrt{(1+xx)}.$$

9. Cum sit $y > x$, sint in figura abscissae $AE = c$, $AF = x$ et $AG = y$, erit arcus $Ae = \Pi . c$ et arcus $fg = \Pi . y - \Pi . x$; hinc ergo habebimus

$$\text{Arc} . Ae = \text{Arc} . fg - cxy, \quad \text{seu} \quad \text{Arc} . fg - \text{Arc} . Ae = cxy$$

existente $y = x\sqrt{(1+cc)} + c\sqrt{(1+xx)}$. Ex his igitur sequentia problemata circa parabolam resolvi poterunt.

10. **Problema 1.** Dato arcu parabolae Ae, in vertice A terminato a puncto quovis f, alium abscindere arcum fg, ita ut differentia horum arcuum $fg - Ae$ geometrice assignari queat.

Solutio. Ponatur arcus dati Ae abscissa $AE = e$, et abscissa, termino dato f arcus quaesiti fg respondens, $AF = f$; abscissa vero alteri termino g arcus quaesiti respondens, $AG = g$, quae ita accipiatur, ut sit $g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)}$; eritque existente parabolae parametro $= 2$, uti constanter assumemus:
$$\text{Arc}.\,fg - \text{Arc}.\,Ae = efg.$$
A puncto autem f quoque retrorsum arcus abscindi potest $f\gamma$, qui superet arcum Ae quantitate algebraica: ob signum radicale $\sqrt{(1+ff)}$ enim ambiguum, capiatur
$$AT = \gamma = f\sqrt{(1+ee)} - e\sqrt{(1+ff)}$$
eritque Arc. $f\gamma$ — Arc. $Ae = ef\gamma$. Q. E. I.

11. **Coroll. 1.** Inventis ergo his duobus punctis g et γ, erit quoque arcuum fg et $f\gamma$ differentia geometrice assignabilis; erit enim
$$\text{Arc}.\,fg - \text{Arc}.\,f\gamma = ef(g-\gamma).$$
At est $g - \gamma = 2e\sqrt{(1+ff)}$; unde $e = \frac{g-\gamma}{2\sqrt{(1+ff)}}$. Tum vero habemus $g + \gamma = 2f\sqrt{(1+ee)}$, sive $\sqrt{(1+ee)} = \frac{g+\gamma}{2f}$; unde eliminanda e fit
$$1 = \frac{(g+\gamma)^2}{4ff} - \frac{(g-\gamma)^2}{4(1+ff)}, \text{ seu } 4ff(1+ff) = (g+\gamma)^2 + 4ffg\gamma.$$
Fit ergo
$$\gamma = -g(1+2ff) + 2f\sqrt{(1+ff)(1+gg)}.$$

12. **Coroll. 2.** Dato ergo arcu quocunque fg, existente $AF = f$ et $AG = g$, a puncto f retrorsum arcus $f\gamma$ abscindi potest, ita ut arcuum fg et $f\gamma$ differentia fiat geometrica. Capiatur scilicet $AT = \gamma = -g(1+2ff) + 2f\sqrt{(1+ff)(1+gg)}$ eritque
$$\text{Arc}.\,fg - \text{Arc}.\,f\gamma = 2f(g\sqrt{(1+ff)} - f\sqrt{(1+gg)})^2\sqrt{(1+ff)}.$$
Horum ergo arcuum differentia evanescere nequit, nisi sit vel $f = 0$, quo casu fit $\gamma = -g$, vel $g = f$, quo casu uterque arcus fg et $f\gamma$ evanescit.

13. **Coroll. 3.** Ut igitur positis $AE = e$, $AF = f$, $AG = g$ differentia arcuum fg et Ae fiat geometrice assignabilis scilicet Arc. fg — Arc. $Ae = efg$, oportet sit
$$g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)},$$
seu ex trium quantitatum e, f, g binis datis tertia ita determinatur, ut sit
$$\text{vel } g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)},$$
$$\text{vel } f = g\sqrt{(1+ee)} - e\sqrt{(1+gg)},$$
$$\text{vel } e = g\sqrt{(1+ff)} - f\sqrt{(1+gg)}.$$

14. **Coroll. 4.** Cum sit $g = f\sqrt{(1+ee)} + e\sqrt{(1+ff)}$, erit
$$\sqrt{(1+gg)} = ef + \sqrt{(1+ee)(1+ff)},$$
unde colligitur
$$g + \sqrt{(1+gg)} = (e + \sqrt{(1+ee)})(f + \sqrt{(1+ff)}).$$
Ergo ut arcus fg superet arcum Ae quantitate algebraica efg, oportet ut sit
$$\frac{g + \sqrt{(1+gg)}}{f + \sqrt{(1+ff)}} = e + \sqrt{(1+ee)}.$$

15. **Coroll. 5.** Haec ultima formula ideo est notatu digna, quod in ea quantitatum e, f et g functiones sint a se invicem separatae. Quod si ergo ponatur

$$e+\sqrt{(1+ee)}=E, \quad f+\sqrt{(1+ff)}=F, \quad g+\sqrt{(1+gg)}=G,$$

erit
$$e=\frac{EE-1}{2E}, \qquad f=\frac{FF-1}{2F}, \qquad g=\frac{GG-1}{2G}.$$

Quare si capiatur $\frac{G}{F}=E$, erit arcuum differentia

$$\text{Arc}.fg-\text{Arc}.Ae=efg=\frac{(EE-1)(FF-1)(GG-1)}{8EFG},$$

seu
$$\text{Arc}.fg-\text{Arc}.Ae=\frac{(FF-1)(GG-1)(GG-FF)}{8FFGG}=\frac{fg(GG-FF)}{2FG}.$$

16. **Problema 2.** Dato arcu parabolae quocunque fg, a puncto parabolae dato p alium abscindere arcum pq ita, ut differentia horum duorum arcuum fg et pq fiat geometrice assignabilis.

Solutio. Pro arcu dato fg ponantur abscissae $AF=f$, $AG=g$; pro arcu autem quaesito pq sint abscissae $AP=p$, $AQ=q$. Jam a vertice parabolae concipiatur arcus Ae respondens abscissae $AE=e$, cujus defectus ab utroque illorum arcuum sit geometrice assignabilis. Ad hoc autem vidimus (14) requiri, ut sit

$$\frac{g+\sqrt{(1+gg)}}{f+\sqrt{(1+ff)}}=e+\sqrt{(1+ee)} \quad \text{et} \quad \frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}}=e+\sqrt{(1+ee)}.$$

Ponamus brevitatis gratia

$$f+\sqrt{(1+ff)}=F \qquad p+\sqrt{(1+pp)}=P$$
$$g+\sqrt{(1+gg)}=G \qquad q+\sqrt{(1+qq)}=Q$$

atque ut problemati satisfiat, necesse est sit $\frac{G}{F}=\frac{Q}{P}$. Porro autem cum sit ex (15)

$$\text{Arc}.fg-\text{Arc}.Ae=\frac{fg(GG-FF)}{2FG} \quad \text{similiterque} \quad \text{Arc}.pq-\text{Arc}.Ae=\frac{pq(QQ-PP)}{2PQ},$$

erit arcuum determinatorum differentia

$$\text{Arc}.pq-\text{Arc}.fg=\frac{pq(QQ-PP)}{2PQ}-\frac{fg(GG-FF)}{2FG}$$

ideoque geometrice assignabilis. Q. E. I.

17. **Coroll. 1.** Cum autem sit $\frac{G}{F}=\frac{Q}{P}$, erit $\frac{QQ-PP}{2PQ}=\frac{GG-FF}{2FG}$, unde differentia arcuum determinatorum prodit

$$\text{Arc}.pq-\text{Arc}.fg=\frac{(pq-fg)(GG-FF)}{2FG}.$$

Est autem $f=\frac{FF-1}{2F}$, $g=\frac{GG-1}{2G}$, $p=\frac{PP-1}{2P}$, $q=\frac{QQ-1}{2Q}$, ideoque ob $Q=\frac{GP}{F}$, erit

$$q=\frac{GGPP-FF}{2FGP}.$$

18. **Coroll. 2.** Erit ergo

$$pq=\frac{(PP-1)(GGPP-FF)}{4FGPP} \quad \text{et} \quad fg=\frac{(FF-1)(GG-1)}{4FG} \quad \text{ideoque}$$

$$pq - fg = \frac{(PP - FF)(GGPP - 1)}{4FGPP}.$$

Hinc arcuum differentia prodit

$$\text{Arc}.pq - \text{Arc}.fg = \frac{(GG - FF)(PP - FF)(GGPP - 1)}{8FFGGPP}.$$

19. **Coroll. 3.** Ut igitur arcus pq arcui fg adeo fiat aequalis, esse oportet vel $GG - FF = 0$, vel $PP - FF = 0$, vel $GGPP - 1 = 0$. Primo autem casu arcus fg ideoque et pq evanescit; altero casu punctum p in f, ideoque et q in g cadit, arcusque ergo pq non prodit diversus ab arcu fg; tertius autem casus dat $P = \frac{1}{G}$, seu $p + \sqrt{(1 + pp)} = \frac{1}{g + \sqrt{(1 + gg)}} = \sqrt{(1 + gg)} - g$, unde fit $p = -g$ et $q = -f$, ita ut pq in alterum ramum parabolae cadat, arcuique fg similis et aequalis prodeat.

20. **Coroll. 4.** Hinc ergo sequitur, in parabola non exhiberi posse duos arcus dissimiles, qui sint inter se aequales. Interim proposito quocunque arcu fg, infinitis modis alius abscindi potest pq, qui illum quantitate algebraica superet, vel ab eo deficiat. Superabit scilicet, si fuerit $P > F$, seu $AP > AF$; deficiet autem, si $P < F$, seu $AP < AF$.

21. **Problema 3.** Dato parabolae arcu quocunque fg, a dato puncto p alium arcum abscindere pr, qui duplum arcus fg superet quantitate geometrice assignabili.

Solutio. Positis ut ante abscissis $AF = f$, $AG = g$, $AP = p$, $AQ = q$, sit $AR = r$ denotentque litterae majusculae F, G, P, Q, R istas functiones $f + \sqrt{(1 + ff)}$, $g + \sqrt{(1 + gg)}$ etc. minuscularum cognominum. Primum igitur si statuatur $\frac{Q}{P} = \frac{G}{F}$, erit

$$\text{Arc}.pq - \text{Arc}.fg = \frac{(pq - fg)(GG - FF)}{2FG}.$$

Simili autem modo si statuatur $\frac{R}{Q} = \frac{G}{F}$, erit

$$\text{Arc}.qr - \text{Arc}.fg = \frac{(qr - fg)(GG - FF)}{2FG}.$$

Addantur ergo invicem hae duae aequationes, erit

$$\text{Arc}.pr - 2\,\text{Arc}.fg = \frac{(pq + qr - 2fg)(GG - FF)}{2FG}.$$

Ut jam ex calculo eliminentur litterae q et Q, erit primo $\frac{R}{P} = \frac{GG}{FF}$; tum vero est $q = \frac{GGPP - FF}{2FGP}$, seu $q = \frac{F(PR - 1)}{2GP}$, et ob $p = \frac{PP - 1}{2P}$ et $r = \frac{G^4P^2 - F^4}{2F^2G^2P}$, erit

$$p + r = \frac{(FF + GG)(GGPP \quad FF)}{2FFGGP},$$

ideoque

$$pq + qr = \frac{(FF + GG)(GGPP - FF)^2}{4F^3G^3PP} \quad \text{et} \quad 2fg = \frac{2(FF - 1)(GG - 1)}{4FG}.$$

Sumto ergo $\frac{R}{P} = \frac{GG}{FF}$, arcus pr superabit duplum arcus fg quantitate algebraica. Q. E. I.

22. **Coroll. 1.** Punctum igitur p ita assumi poterit, ut excessus arcus pr supra duplum arcum $2fg$ sit datae magnitudinis; definietur enim P per aequationem algebraicam, ope extractionis radicis quadratae tantum.

23. **Coroll. 2.** Fieri igitur poterit, ut arcus pr praecise sit duplus arcus dati fg, quod evenit si P definiatur ex hac aequatione

$$(GGPP - FF)^2 = \frac{2(FF-1)(GG-1)FFGGPP}{FF+GG}$$

unde elicitur
$$\frac{GGPP}{FF} = \frac{FFGG + 1 + \sqrt{(F^4-1)(G^4-1)}}{FF+GG}$$

et
$$\frac{GP}{F} = \frac{\sqrt{\tfrac{1}{2}}(FF+1)(GG+1) + \sqrt{\tfrac{1}{2}}(FF-1)(GG-1)}{\sqrt{(FF+GG)}} = \frac{FR}{G}.$$

24. **Coroll. 3.** Haec autem determinatio arcus dupli pr maxime fit obvia, si arcus datus fg in vertice A incipiat; tum enim ob $F=1$ fit $GP=F$, seu $P=\frac{1}{G}=\sqrt{(1+gg)}-g$. Obtinetur ergo $p=-g$ et $R=G$, ideoque $r=g$. Hoc scilicet casu arcus pr in parabola circa verticem A utrinque aequaliter extendetur, sicque manifesto fit duplus arcus propositi.

25. **Coroll. 4.** Fieri quoque potest, ut arcus pr in ipso puncto g terminetur, sicque ambo arcus, simplus fg et duplus pr, evadant contigui. Hoc nempe evenit si $P=G$, quo casu haec habetur aequatio

$$F^6 + F^4G^2 - 2F^4G^6 + F^2G^8 - 2F^2G^4 + G^{10} = 0,$$

quae per $FF-GG$ divisa praebet

$$F^4 - 2FFG^6 + 2FFGG - G^8 = 0$$

unde elicitur

$$FF = GG(G^4-1) + GG\sqrt{(G^8-G^4+1)} \quad \text{ideoque} \quad F = G\sqrt{(G^4-1+\sqrt{(G^8-G^4+1)})}$$

$$\text{et} \quad R = \frac{G^3}{FF} \quad \text{seu} \quad R = \frac{\sqrt{(G^8-G^4+1)} - G^4 + 1}{G^3}.$$

26. **Coroll. 5.** Quantitas ergo G, seu parabolae punctum g pro lubitu assumi licet, in quo duo arcus terminabuntur, quorum alter alterius exacte erit duplus. Cum autem sumto g affirmativo, ideoque $G>1$, prodeat $F>G$, punctum f a vertice magis erit remotum quam punctum g; tum vero reperitur

$$r = \frac{RR-1}{2R} = \frac{-(GG-1)\sqrt{(G^8-G^4+1)} - G^6 - G^4 + GG + 1}{2G^3},$$

cujus valor cum sit negativus, punctum r in alterum parabolae ramum incidit. Arcus ergo ita erunt
* dispositi, ut habet figura 56, eritque

$$\text{Arcus } gr = 2\,\text{Arc}.\,fg.$$

27. **Coroll. 6.** Sit g valde parvum, erit $G = 1 + g + \frac{1}{2}gg$, hincque $G^2 = 1 + 2g + 2gg$, $G^3 = 1 + 3g + \frac{9}{2}gg$, $G^4 = 1 + 4g + 8gg$ et $G^8 = 1 + 8g + 32gg$, unde

$$F = (1 + g + \tfrac{1}{2}gg)(1 + 3g + \tfrac{9}{2}gg) = 1 + 4g + 8gg,$$

* ergo $f = \frac{FF-1}{2F} = 4g$; porro $R = 1 - 5g + \frac{25}{2}gg$, unde $r = -5g$. Quare (Fig. 56) si Ag valde parvum, erit proxime $AF = 4AG$ et $AR = 5AG$, ita ut sit quoque $GR = 2GF$.

28. **Scholion.** Antequam ad ulteriorem arcuum parabolicorum multiplicationem progrediamur, etiamsi ea ex formulis datis non difficulter erui queat, tamen expediet differentiam algebraicam arcuum parabolicorum commodius exprimere. Cum igitur (Fig. 55) positis abscissis $AE=e$, $AF=f$, $AG=g$ invenerimus (13) $\text{Arc}.Ag-\text{Arc}.Af-\text{Arc}.Ae=efg$, existente $e=g\sqrt{(1+ff)}-f\sqrt{(1+gg)}$, videndum est, num quantitas efg non possit transformari in terna membra, quae sint singula functiones certae ipsarum e, f et g, ita ut sit $efg=\text{funct}.g-\text{funct}.f-\text{funct}.e$; sic enim quaelibet harum functionum cum arcu cognomine comparari posset. Cum autem sit

$$efg=fgg\sqrt{(1+ff)}-ffg\sqrt{(1+gg)} \quad \text{et} \quad \sqrt{(1+ee)}=\sqrt{(1+ff)}(1+gg)-fg,$$

erit
$$e\sqrt{(1+ee)}=g\sqrt{(1+gg)}+2ffg\sqrt{(1+gg)}-f\sqrt{(1+ff)}-2fgg\sqrt{(1+ff)},$$
hincque

$$fgg\sqrt{(1+ff)}-ffg\sqrt{(1+gg)}=efg=\frac{1}{2}g\sqrt{(1+gg)}-\frac{1}{2}f\sqrt{(1+ff)}-\frac{1}{2}e\sqrt{(1+ee)};$$

quae est expressio talis qualis desideratur. Quare si istas abscissarum e, f, g functiones brevitatis gratia ponamus $\frac{1}{2}e\sqrt{(1+ee)}=\mathfrak{E}$, $\frac{1}{2}f\sqrt{(1+ff)}=\mathfrak{F}$ et $\frac{1}{2}g\sqrt{(1+gg)}=\mathfrak{G}$, habebimus

$$\text{Arc}.Ag-\text{Arc}.Af-\text{Arc}.Ae=\mathfrak{G}-\mathfrak{F}-\mathfrak{E}=\text{Arc}.fg-\text{Arc}.Ae.$$

Si porro hae functiones cum illis, quibus ante usi sumus, comparemus, scilicet

$$e+\sqrt{(1+ee)}=E, \quad f+\sqrt{(1+ff)}=F, \quad g+\sqrt{(1+gg)}=G,$$

erit
$$\mathfrak{E}=\frac{E^4-1}{8EE}, \qquad \mathfrak{F}=\frac{F^4-1}{8FF}, \qquad \mathfrak{G}=\frac{G^4-1}{8GG}$$

et ex natura horum arcuum est $\frac{G}{F}=E$. Si jam simili modo pro arcu pq procedamus, et ex abscissis $AP=p$ et $AQ=q$ has formemus functiones

$$p+\sqrt{(1+pp)}=P \qquad \frac{1}{2}p\sqrt{(1+pp)}=\mathfrak{P}$$
$$q+\sqrt{(1+qq)}=Q \qquad \frac{1}{2}q\sqrt{(1+qq)}=\mathfrak{Q},$$

erit simili modo $\text{Arc}.pq-\text{Arc}.Ae=\mathfrak{Q}-\mathfrak{P}-\mathfrak{E}$, existente $\frac{Q}{P}=E$. Hinc si illa aequatio ab hac subtrahatur, remanebit $\text{Arc}.pq-\text{Arc}.fg=(\mathfrak{Q}-\mathfrak{P})-(\mathfrak{G}-\mathfrak{F})$, si modo fuerit $\frac{Q}{P}=\frac{G}{F}$.

29. **Problema 4.** Dato arcu parabolae quocunque fg, abscindere arcum alium pz, qui ad arcum fg sit in data ratione $n:1$.

Solutio. Positis abscissis $AF=f$, $AG=g$, capiantur plures abscissae $AP=p$, $AQ=q$, $AR=r$, $AS=s$ et ultima $AZ=z$, ex quibus formentur geminae functiones, litteris majusculis cum latinis tum germanicis cognominibus denotandae, scilicet

$$f+\sqrt{(1+ff)}=F, \qquad g+\sqrt{(1+gg)}=G, \qquad p+\sqrt{(1+pp)}=P \text{ etc.}$$
$$\frac{1}{2}f\sqrt{(1+ff)}=\mathfrak{F}, \qquad \frac{1}{2}g\sqrt{(1+gg)}=\mathfrak{G}, \qquad \frac{1}{2}p\sqrt{(1+pp)}=\mathfrak{P} \text{ etc.}$$

sitque primo $\frac{Q}{P}=\frac{G}{F}$, erit

$$\text{Arc}.pq-\text{Arc}.fg=(\mathfrak{Q}-\mathfrak{P})-(\mathfrak{G}-\mathfrak{F}).$$

Deinde sit $\frac{R}{Q}=\frac{G}{F}$, seu $\frac{R}{P}=\frac{G^2}{F^2}$, erit

$$\text{Arc}.qr-\text{Arc}.fg=(\mathfrak{R}-\mathfrak{Q})-(\mathfrak{G}-\mathfrak{F})$$

qua aequatione ad priorem addita fit

$$\text{Arc}.pr-2\,\text{Arc}.fg=(\mathfrak{R}-\mathfrak{P})-2(\mathfrak{G}-\mathfrak{F}).$$

Sit porro $\frac{S}{R}=\frac{G}{F}$, seu $\frac{S}{P}=\frac{G^3}{F^3}$, erit

$$\text{Arc}.rs-\text{Arc}\,fg=(\mathfrak{S}-\mathfrak{R})-(\mathfrak{G}-\mathfrak{F}),$$

qua iterum ad praecedentem adjecta obtinebitur

$$\text{Arc}.ps-3\,\text{Arc}.fg=(\mathfrak{S}-\mathfrak{P})-3(\mathfrak{G}-\mathfrak{F}).$$

Simili modo si ulterius panatur $\frac{T}{S}=\frac{G}{F}$, seu $\frac{T}{P}=\frac{G^4}{F^4}$, erit

$$\text{Arc}.pt-4\,\text{Arc}.fg=(\mathfrak{T}-\mathfrak{P})-4(\mathfrak{G}-\mathfrak{F}).$$

Unde perspicitur, si z sit ultimum punctum arcus pz qui quaeritur, et posita $AZ=z$ fit

$$Z=z+\sqrt{(1+zz)}\quad\text{et}\quad\mathfrak{Z}=\tfrac{1}{2}z\sqrt{(1+zz)},$$

poni debere $\frac{Z}{P}=\frac{G^n}{F^n}$, tumque fore

$$\text{Arc}.pz-n\,\text{Arc}.fg=(\mathfrak{Z}-\mathfrak{P})-n(\mathfrak{G}-\mathfrak{F}).$$

Nunc ut sit $\text{Arc}.pz=n\,\text{Arc}.fg$, reddi oportet $\mathfrak{Z}-\mathfrak{P}=n(\mathfrak{G}-\mathfrak{F})$. At est

$$\mathfrak{Z}=\frac{Z^4-1}{8ZZ},\quad\mathfrak{P}=\frac{P^4-1}{8PP},\quad\mathfrak{G}=\frac{G^4-1}{8GG}\quad\text{et}\quad\mathfrak{F}=\frac{F^4-1}{8FF}.$$

Verum ob $Z=\frac{G^nP}{F^n}$, erit $\mathfrak{Z}=\frac{G^{4n}P^4-F^{4n}}{8F^{2n}G^{2n}PP}$. Quibus valoribus substitutis sequens acquiretur aequatio resolvenda

$$\frac{G^{4n}P^4-F^{4n}}{F^{2n}G^{2n}PP}=\frac{P^4-1}{PP}+\frac{n(GG-FF)(1+FFGG)}{FFGG},$$

sive $$0=G^{2n}(G^{2n}-F^{2n})P^4+F^{2n}(G^{2n}-F^{2n})-nF^{2n-2}G^{2n-2}(G^2-F^2)(F^2G^2+1)PP,$$

seu $$P^4=\frac{+nF^{2n}(G^2-F^2)(F^2G^2+1)P^2}{F^2G^2(G^{2n}-F^{2n})}-\frac{F^{2n}}{G^{2n}}.$$

Quocunque ergo assumto multiplicationis indice n, sive numero integro, sive fracto, ex hac aequatione semper definiri potest P, unde arcus quaesiti pz alter terminus p innotescit. Quo invento pro altero termino z erit $Z=\frac{G^nP}{F^n}$, sicque obtinebitur arcus pz, ut sit $pz=n.fg$. Q. E. I.

30. **Coroll. 1.** Si loco P quaerere velimus Z, in ultima aequatione substitui oportet $\frac{F^nZ}{G^n}$, prodibitque

$$Z^4=\frac{nG^{2n}(G^2-F^2)(F^2G^2+1)ZZ}{F^2G^2(G^{2n}-F^{2n})}-\frac{G^{2n}}{F^{2n}},$$

ubi litterae F et G pariter uti P et Z sunt inter se commutatae.

31. **Coroll. 2.** Cum $G^{2n}-F^{2n}$ dividi queat per G^2-F^2, pro variis valoribus ipsius n formulae inventae ita se habebunt

si $n=1$, $P^4=\frac{(F^2G^2+1)P^2}{G^2}-\frac{F^2}{G^2}$ et $Z=\frac{GP}{F}$,

si $n=2$, $P^4=\frac{2F^2(F^2G^2+1)P^2}{G^2(G^2+F^2)}-\frac{F^4}{G^4}$ et $Z=\frac{G^2P}{F^2}$,

si $n=3$, $P^4=\frac{3F^4(F^2G^2+1)P^2}{G^2(G^4+F^2G^2+F^4)}-\frac{F^6}{G^6}$ et $Z=\frac{G^3P}{F^3}$,

si $n=4$, $P^4=\frac{4F^6(F^2G^2+1)P^2}{G^2(G^6+F^2G^4+F^4G^2+F^6)}-\frac{F^8}{G^8}$ et $Z=\frac{G^4P}{F^4}$,

etc. etc.

32. **Coroll. 3.** Ex solutione ceterum apparet pari modo pro arcu dato quocunque fg inveniri posse alium pz, qui illum arcum n vicibus sumtum data quantitate superet, vel ab eo deficiat; ut enim sit Arc. $pz - n$ Arc. $fg = D$, resolvi oportebit hanc aequationem $\mathfrak{Z}-\mathfrak{P}=n(\mathfrak{G}-\mathfrak{F})+D$, quae non habet plus difficultatis, quam si esset $D=0$.

33. **Scholion.** Haec quidem, quae de circulo et parabola hic protuli, jam dudum satis sunt cognita, et quia utriusque rectificatio quasi in potestate est, (quae enim vel a quadratura circuli vel a logarithmis pendent, in ordinem quantitatum algebraicarum propemodum recipiuntur) nulli omnino difficultati sunt subjecta: ea tamen nihilominus aliquanto uberius hic exponere visum est, quod ex methodo prorsus singulari consequuntur. Quod autem imprimis notatu dignum est, haec methodus ad comparationem aliarum quoque curvarum manuducit, quarum rectificatio per calculum solitum nullo modo expediri potest; ita ut ex eodem quasi fonte plurimae eximiae affectiones tam cognitae quam incognitae hauriri queant, ex quo Analysi non contemnenda incrementa accedere censeri debebunt.

Sectio secunda

continens evolutionem hujus aequationis:

$$0=\alpha+\gamma(xx+yy)+2\delta xy+\zeta xxyy.$$

I.

Extrahatur ex hac aequatione sigillatim radix utriusque quantitatis variabilis x et y, ac reperietur

$$y=\frac{-\delta x+\sqrt{(\delta\delta xx-(\alpha+\gamma xx)(\gamma+\zeta xx))}}{\gamma+\zeta xx}$$

$$x=\frac{-\delta y-\sqrt{(\delta\delta yy-(\alpha+\gamma yy)(\gamma+\zeta yy))}}{\gamma+\zeta yy}$$

Ponatur brevitatis gratia $-\alpha\gamma=Ap$, $\delta\delta-\gamma\gamma-\alpha\zeta=Cp$ et $-\gamma\zeta=Ep$, eritque

$$\gamma y+\delta x+\zeta xxy=\sqrt{(A+Cxx+Ex^4)}\,p$$

$$\gamma x+\delta y+\zeta xyy=-\sqrt{(A+Cyy+Ey^4)}\,p.$$

II.

Si igitur coëfficientes A, C, E fuerint dati, ex iis litterarum graecarum valores facile definiuntur. Erit enim

$$\alpha=\frac{-Ap}{\gamma},\quad \zeta=\frac{-Ep}{\gamma}\quad \text{et}\quad \delta=\sqrt{\left(\gamma\gamma+Cp+\frac{AEpp}{\gamma\gamma}\right)}.$$

si $n = 1$, $P^4 = \frac{(F^2G^2+1)P^2}{G^2} - \frac{F^2}{G^2}$ et $Z = \frac{GP}{F}$,

si $n = 2$, $P^4 = \frac{2F^2(F^2G^2+1)P^2}{G^2(G^2+F^2)} - \frac{F^4}{G^4}$ et $Z = \frac{G^2P}{F^2}$,

si $n = 3$, $P^4 = \frac{3F^4(F^2G^2+1)P^2}{G^2(G^4+F^2G^2+F^4)} - \frac{F^6}{G^6}$ et $Z = \frac{G^3P}{F^3}$,

si $n = 4$, $P^4 = \frac{4F^6(F^2G^2+1)P^2}{G^2(G^6+F^2G^4+F^4G^2+F^6)} - \frac{F^8}{G^8}$ et $Z = \frac{G^4P}{F^4}$,

etc. etc.

32. **Coroll. 3.** Ex solutione ceterum apparet pari modo pro arcu dato quocunque fg inveniri posse alium pz, qui illum arcum n vicibus sumtum data quantitate superet, vel ab eo deficiat; ut enim sit $\text{Arc}.pz - n\,\text{Arc}.fg = D$, resolvi oportebit hanc aequationem $\mathfrak{Z} - \mathfrak{P} = n(\mathfrak{G} - \mathfrak{F}) + D$, quae non habet plus difficultatis, quam si esset $D = 0$.

33. **Scholion.** Haec quidem, quae de circulo et parabola hic protuli, jam dudum satis sunt cognita, et quia utriusque rectificatio quasi in potestate est, (quae enim vel a quadratura circuli vel a logarithmis pendent, in ordinem quantitatum algebraicarum propemodum recipiuntur) nulli omnino difficultati sunt subjecta: ea tamen nihilominus aliquanto uberius hic exponere visum est, quod ex methodo prorsus singulari consequuntur. Quod autem imprimis notatu dignum est, haec methodus ad comparationem aliarum quoque curvarum manuducit, quarum rectificatio per calculum solitum nullo modo expediri potest; ita ut ex eodem quasi fonte plurimae eximiae affectiones tam cognitae quam incognitae hauriri queant, ex quo Analysi non contemnenda incrementa accedere censeri debebunt.

Sectio secunda

continens evolutionem hujus aequationis:

$$0 = \alpha + \gamma(xx + yy) + 2\delta xy + \zeta xxyy.$$

I.

Extrahatur ex hac aequatione sigillatim radix utriusque quantitatis variabilis x et y, ac reperietur

$$y = \frac{-\delta x + \sqrt{(\delta\delta xx - (\alpha + \gamma xx)(\gamma + \zeta xx))}}{\gamma + \zeta xx}$$

$$x = \frac{-\delta y - \sqrt{(\delta\delta yy - (\alpha + \gamma yy)(\gamma + \zeta yy))}}{\gamma + \zeta yy}$$

Ponatur brevitatis gratia $-\alpha\gamma = Ap$, $\delta\delta - \gamma\gamma - \alpha\zeta = Cp$ et $-\gamma\zeta = Ep$, eritque

$$\gamma y + \delta x + \zeta xxy = \sqrt{(A + Cxx + Ex^4)}\,p$$

$$\gamma x + \delta y + \zeta xyy = -\sqrt{(A + Cyy + Ey^4)}\,p.$$

II.

Si igitur coëfficientes A, C, E fuerint dati, ex iis litterarum graecarum valores facile definiuntur. Erit enim

$$\alpha = \frac{-Ap}{\gamma}, \quad \zeta = \frac{-Ep}{\gamma} \quad \text{et} \quad \delta = \sqrt{(\gamma\gamma + Cp + \frac{AEpp}{\gamma\gamma})}.$$

Valores ergo γ et p arbitrio nostro relinquuntur, atque alterum quidem sine ulla restrictione ad lubitum determinare licet. Ponatur ergo $\gamma\gamma = A$ et $p = cc$, fietque

$$\alpha = -cc\sqrt{A}, \quad \gamma = \sqrt{A}, \quad \delta = \sqrt{(A + Ccc + Ec^4)} \text{ et } \zeta = \frac{-Ecc}{\sqrt{A}}$$

et aequatio canonica hanc induet formam

$$0 = -Acc + A(xx + yy) + 2xy\sqrt{(A + Ccc + Ec^4)A} - Eccxxyy.$$

III.

Antequam autem his litteris majusculis utamur, differentiemus ipsam aequationem propositam

$$dx(\gamma x + \delta y + \zeta xyy) + dy(\gamma y + \delta x + \zeta xxy) = 0,$$

quae abit in hanc

$$\frac{dx}{\gamma y + \delta x + \zeta xxy} = \frac{-dy}{\gamma x + \delta y + \zeta xyy}.$$

Substituendo ergo loco horum denominatorum valores surdos primo inventos, habebimus per $\sqrt{p}$ multiplicando

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{-dy}{\sqrt{(A + Cyy + Ey^4)}}.$$

IV.

Proposita ergo hac aequatione differentiali

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{-dy}{\sqrt{(A + Cyy + Ey^4)}},$$

ejus aequatio integralis erit

$$0 = -Acc + A(xx + yy) + 2xy\sqrt{(A + Ccc + Ec^4)A} - Eccxxyy,$$

quae cum constantem novam c ab arbitrio nostro pendentem involvat, erit adeo integralis completa. Inde autem oritur

$$y = \frac{-x\sqrt{(A + Ccc + Ec^4)A} \pm c\sqrt{(A + Cxx + Ex^4)A}}{A - Eccxx},$$

ubi quidem signa radicalium pro lubitu mutare licet.

V.

Cum igitur posita nostra aequatione canonica sit

$$\int\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} + \int\frac{dy}{\sqrt{(A + Cyy + Ey^4)}} = \text{Const.}$$

ponamus ad alias integrationes eruendas

$$\int\frac{xx\,dx}{\sqrt{(A + Cxx + Ex^4)}} + \int\frac{yy\,dy}{\sqrt{(A + Cyy + Ey^4)}} = V,$$

erit ergo loco radicalium valores praecedentes restituendo

$$\frac{xx\,dx}{\gamma y + \delta x + \zeta xxy} + \frac{yy\,dy}{\gamma x + \delta y + \zeta xyy} = \frac{dV}{\sqrt{p}}$$

hincque porro

$$xx\,dx(\gamma x + \delta y + \zeta xyy) + yy\,dy(\gamma y + \delta x + \zeta xxy) =$$

$$\frac{dV}{\sqrt{p}}(\gamma\delta(xx + yy) + (\gamma\gamma + \delta\delta)xy + \zeta\zeta x^3y^3 + \gamma\zeta xy(xx + yy) + 2\delta\zeta xxyy).$$

VI.

Ponamus ad hanc aequationem concinniorem reddendam $xx + yy = tt$ et $xy = u$, ut sit

$$0 = \alpha + \gamma tt + 2\delta u + \zeta uu,$$

et aequatio nostra differentialis erit

$$\gamma(x^3 dx + y^3 dy) + \delta u(xdx + ydy) + \zeta uu(xdx + ydy) = \frac{dV}{\sqrt{p}}(\gamma\delta tt + (\gamma\gamma + \delta\delta)u + \gamma\zeta ttu + 2\delta\zeta uu + \zeta\zeta u^3).$$

At est $xdx + ydy = tdt$, et ob $x^4 + y^4 = t^4 - 2uu$, erit $x^3 dx + y^3 dy = t^3 dt - udu$. Porro aequatio canonica differentiata dat

$$\gamma tdt + \delta du + \zeta udu = 0, \quad \text{ideoque} \quad tdt = \frac{-\delta du - \zeta udu}{\gamma},$$

unde fit $\quad xdx + ydy = -\frac{\delta}{\gamma}du - \frac{\zeta}{\gamma}udu \quad$ et $\quad x^3 dx + y^3 dy = -\frac{\delta}{\gamma}ttdu - \frac{\zeta}{\gamma}ttudu - udu.$

VII.

His igitur valoribus substitutis obtinebimus

$$du\left(-\delta tt - \zeta ttu - \gamma u - \frac{\delta\delta}{\gamma}u - \frac{2\delta\zeta}{\gamma}uu - \frac{\zeta\zeta}{\gamma}u^3\right) = \frac{dV}{\sqrt{p}}(\gamma\delta tt + (\gamma\gamma + \delta\delta)u + \zeta ttu + 2\delta\zeta uu + \zeta\zeta u^3),$$

quae sponte abit in $\frac{-du}{\gamma} = \frac{dV}{\sqrt{p}}$, ita ut sit $V = \frac{-u\sqrt{p}}{\gamma}$, seu $V = \frac{-xy\sqrt{p}}{\gamma}$. Facto ergo $p = cc$, erit

$$\int\frac{xxdx}{\sqrt{(A + Cxx + Ex^4)}} - \int\frac{yydy}{\sqrt{(A + Cyy + Ey^4)}} = \text{Const.} - \frac{cxy}{\sqrt{A}}$$

siquidem fuerit $0 = -Acc + A(xx + yy) + 2xy\sqrt{(A + Ccc + Ec^4)A} - Eccxxyy$, seu

$$y = \frac{c\sqrt{(A + Cxx + Ex^4)A} - x\sqrt{(A + Ccc + Ec^4)A}}{A - Eccxx}.$$

VIII.

Quo nunc rem generalius complectamur, ponamus

$$\int\frac{x^n dx}{\sqrt{(A + Cxx + Ex^4)}} - \int\frac{y^n dy}{\sqrt{(A + Cyy + Ey^4)}} = V,$$

erit

$$x^n dx(\gamma x + \delta y + \zeta xyy) + y^n dy(\gamma y + \delta x + \zeta xxy) = \frac{dV}{\sqrt{p}}(\gamma\delta tt + (\gamma\gamma + \delta\delta)u + \gamma\zeta ttu + 2\delta\zeta uu + \zeta\zeta u^3),$$

posito ut ante $xx + yy = tt$ et $xy = u$. Erit ergo $xx - yy = \sqrt{(t^4 - 4uu)}$, unde

$$x = \sqrt{\frac{tt + \sqrt{(t^4 - 4uu)}}{2}} \quad \text{et} \quad y = \sqrt{\frac{tt - \sqrt{(t^4 - 4uu)}}{2}},$$

seu $\quad x = \frac{1}{2}\sqrt{(tt + 2u)} + \frac{1}{2}\sqrt{(tt - 2u)} \quad$ et $\quad y = \frac{1}{2}\sqrt{(tt + 2u)} - \frac{1}{2}\sqrt{(tt - 2u)}.$

Quare differentiando habebitur

$$dx = \frac{tdt + du}{2\sqrt{(tt + 2u)}} + \frac{tdt - du}{2\sqrt{(tt - 2u)}} = \frac{du(\gamma - \delta - \zeta u)}{2\gamma\sqrt{(tt + 2u)}} - \frac{du(\gamma + \delta + \zeta u)}{2\gamma\sqrt{(tt - 2u)}}.$$

*

IX.

Porro vero est $\gamma x+\delta y+\zeta xyy=(\frac{1}{2}(\gamma+\delta)+\frac{1}{2}\zeta u)\sqrt{(tt+2u)}+(\frac{1}{2}(\gamma-\delta)-\frac{1}{2}\zeta u)\sqrt{tt-2u}$, unde colligitur $dx(\gamma x+\delta y+\zeta xyy)=$

$$\frac{du}{4\gamma}(\gamma+\delta+\zeta u)(\gamma-\delta-\zeta u)+\frac{du}{4\gamma}(\gamma-\delta-\zeta u)(\gamma-\delta-\zeta u)\sqrt{\frac{tt-2u}{tt+2u}}$$
$$-\frac{du}{4\gamma}(\gamma-\delta-\zeta u)(\gamma+\delta+\zeta u)-\frac{du}{4\gamma}(\gamma+\delta+\zeta u)(\gamma+\delta+\zeta u)\sqrt{\frac{tt+2u}{tt-2u}},$$

seu $$dx(\gamma x+\delta y+\zeta xyy)=\frac{-du}{\gamma\sqrt{(t^4-4uu)}}(\gamma\delta tt+\gamma\zeta u+(\gamma\gamma+\delta\delta)u+2\delta\zeta uu+\zeta\zeta u^3),$$

et quia $dy(\gamma y+\delta x+\zeta xxy)=-dx(\gamma x+\delta y+\zeta xyy)$, erit

$$\frac{dV}{\sqrt{p}}=\frac{-du(x^n-y^n)}{\gamma\sqrt{(t^4-4uu)}}\quad\text{et}\quad V=\frac{-\sqrt{p}}{\gamma}\int\frac{(x^n-y^n)\,du}{\sqrt{(t^4-4uu)}}.$$

X.

Ut haec formula evadat integrabilis, oportet pro n scribi numerum parem, ut etiam usus hujus formae plerumque exigit. Quare

si $n=0$, erit $x^0-y^0=0$ . . . . $V=\text{Const.}$

$n=2$ $\quad x^2-y^2=\sqrt{(t^4-4uu)}$ . . . . $V=\frac{-u\sqrt{p}}{\gamma}$

$n=4$ $\quad x^4-y^4=tt\sqrt{(t^4-4uu)}$ . . . . $V=\frac{-\sqrt{p}}{\gamma}\int tt\,du$

$n=6$ $\quad x^6-y^6=(t^4-uu)\sqrt{(t^4-4uu)}$ . . . . . $V=\frac{-\sqrt{p}}{\gamma}\int(t^4-uu)\,du$

$n=8$ $\quad x^8-y^8=(t^6-2ttuu)\sqrt{(t^4-4uu)}$ . . . . $V=\frac{-\sqrt{p}}{\gamma}\int(t^6-2ttuu)\,du$

etc. etc.

XI.

Cum vero sit $tt=\frac{-\alpha-2\delta u-\zeta uu}{\gamma}$, erit

$$\int tt\,du=\frac{-\alpha u}{\gamma}-\frac{\delta uu}{\gamma}-\frac{\zeta u^3}{3\gamma}$$
$$\int(t^4-uu)\,du=\frac{\alpha\alpha}{\gamma\gamma}u+\frac{2\alpha\delta}{\gamma\gamma}uu+\frac{(4\delta\delta+2\alpha\zeta-\gamma\gamma)}{3\gamma\gamma}u^3+\frac{\delta\zeta}{\gamma\gamma}u^4+\frac{\zeta\zeta}{5\gamma\gamma}u^5.$$

Ex his introductis litteris majusculis A, C, E una cum constanti arbitraria c, aequatio in fine art. VII data satisfaciet huic aequationi integrali

$$\int\frac{dx(\mathfrak{A}+\mathfrak{C}xx+\mathfrak{E}x^4)}{\sqrt{(A+Cxx+Ex^4)}}-\int\frac{dy(\mathfrak{A}+\mathfrak{C}yy+\mathfrak{E}y^4)}{\sqrt{(A+Cyy+Ey^4)}}=\text{Const.}-\frac{\mathfrak{C}cxy}{\sqrt{A}}-\frac{\mathfrak{E}cxy}{\sqrt{A}}\left(cc-xy\sqrt{\frac{A+Ccc+Ec^4}{A}}+\frac{Eccxxyy}{3A}\right).$$

Unde sequentes curvarum comparationes adipiscimur.

Comparatio arcuum Ellipsis.

1. Expressio simplicissima ad hoc genus pertinens est utique curva lemniscata, sed quia comparationem arcuum ejus jam satis prolixe sum persecutus, hic statim ab ellipsi incipiam. Sit igitur

(Fig. 57) ACB quadrans ellipticus, cujus alter semiaxis $CA = 1$, alter $CB = k$. Eritque posita abscissa quacunque $CP = z$, arcus ei respondens $Bp = \int dz \sqrt{\frac{1-(1-kk)zz}{1-zz}}$. Sit brevitatis gratia $1 - kk = n$, ita ut $\sqrt{n}$ denotet distantiam foci a centro C, hincque fiet Arc. $Bp = \int \frac{dz\sqrt{(1-nzz)}}{\sqrt{(1-zz)}}$.

2. Reddatur formulae hujus numerator rationalis, ut prodeat

$$\text{Arc}.\,Bp = \int \frac{dz\,(1-nzz)}{\sqrt{(1-(n+1)zz+nz^4)}},$$

ad quam formam ut formulae superiores reducantur, poni oportet $A = 1$, $C = -n-1$, $E = n$, $\mathfrak{A} = 1$, $\mathfrak{C} = -n$, $\mathfrak{E} = 0$; quo facto habebimus pro differentia duorum arcuum ellipticorum

$$\int dx \sqrt{\frac{1-nxx}{1-xx}} - \int dy \sqrt{\frac{1-nyy}{1-yy}} = \text{Const.} + ncxy$$

siquidem abscissa y ex abscissa x ita determinetur, ut sit

$$y = \frac{c\sqrt{(1-xx)(1-nxx)} - x\sqrt{(1-cc)(1-ncc)}}{1-nccxx},$$

sive $\quad 0 = -cc + xx + yy + 2xy\sqrt{(1-cc)(1-ncc)} - nccxxyy.$

3. Denotet $\Pi . z$ arcum ellipsis abscissae z respondentem, ac nostra aequatio inventa hanc induet formam

$$\Pi . x - \Pi . y = \text{Const.} + ncxy,$$

posito autem $x = 0$, fit $y = c$, unde Const. $= -\Pi . c$. Ergo

$$\Pi . c + \Pi . x - \Pi . y = ncxy.$$

Sin autem sumto $\sqrt{(1-cc)(1-ncc)}$ negativo, ut sit

$$y = \frac{c\sqrt{(1-xx)(1-nxx)} + x\sqrt{(1-cc)(1-ncc)}}{1-nccxx}$$

fiet $\Pi . y - \Pi . c - \Pi . x = -ncxy$, sive $\Pi . c - (\Pi . y - \Pi . x) = ncxy$, ut ante.

4. Ternae autem quantitates c, x, y ita a se invicem pendent, ut habita signorum ratione inter se permutari possint; si enim ad abbreviandum ponatur

$$\sqrt{(1-cc)(1-ncc)} = C,\quad \sqrt{(1-xx)(1-nxx)} = X,\quad \sqrt{(1-yy)(1-nyy)} = Y,$$

erit $$y = \frac{cX + xC}{1-nccxx},\quad x = \frac{yC - cY}{1-nccyy},\quad c = \frac{yX - xY}{1-nxxyy},$$

ex quibus per combinationem eliciuntur sequentes formulae

$$\begin{aligned} yy - xx &= c\,(yX + xY) & xX + yY &= (nccxy + C)(yX + xY),\\ yy - cc &= x\,(yC + cY) & cC - xX &= (ncxyy - Y)(xC - cX),\\ xx - cc &= y\,(xC - cX) & cC + yY &= (ncxxy + X)(yC + cY) \end{aligned}$$

ac denique

$$\begin{aligned} 2xyC &= xx + yy - cc - nccxxyy\\ 2cyX &= cc + yy - xx - nccxxyy\\ -2cxY &= cc + xx - yy - nccxxyy. \end{aligned}$$

5. **Problema 1.** Dato arcu elliptico Be in vertice B terminato, abscindere a quovis puncto dato f alium arcum fg, ut eorum differentia $fg - Be$ geometrice assignari queat.

Solutio. Sint abscissae datae $CE = e$, $CF = f$ et quaesita $Cg = g$, erit Arc. $Be = \Pi.e$ et Arc. $fg = \Pi.g - \Pi.f$; ut igitur arcuum fg et Be differentia fiat geometrica, necesse est, ut sit $\Pi.e - (\Pi.g - \Pi.f) =$ quantitati algebraicae. Hoc autem, ut vidimus, evenit si

$$g = \frac{e\sqrt{(1-ff)(1-nff)} + f\sqrt{(1-ee)(1-nee)}}{1-neeff}.$$

Quodsi ergo abscissae $CG = g$ hic tribuatur valor, erit Arc. Be — Arc. $fg = nefg$, posito scilicet $CA = 1$ et $CB = k$, atque $n = 1 - kk$. Q. E. I.

6. **Coroll. 1.** Poterit etiam a puncto dato f versus B accedendo ejusmodi arcus $f\gamma$ abscindi, ut differentia $Be - f\gamma$ fiat algebraica. Posita enim abscissa $CT = \gamma$ capiatur

$$\gamma = \frac{f\sqrt{(1-ee)(1-nee)} - e\sqrt{(1-ff)(1-nff)}}{1-neeff}$$

eritque Arc. Be — Arc. $f\gamma = nef\gamma$.

7. **Coroll. 2.** Erit ergo quoque arcuum $f\gamma$ et fg differentia geometrice assignabilis; habebitur enim Arc. $f\gamma$ — Arc. $fg = nef(g-\gamma)$. Est autem

$$g - \gamma = \frac{2e\sqrt{(1-ff)(1-nff)}}{1-neeff}$$

sive cum sit $$2fg\sqrt{(1-ee)(1-nee)} = ff + gg - ee - neeffgg \quad \text{et}$$

$$+2f\gamma\sqrt{(1-ee)(1-nee)} = ff + \gamma\gamma - ee - neeff\gamma\gamma, \quad \text{erit}$$

$$ee = \frac{ff - \gamma g}{1 - nff\gamma g} \quad \text{et} \quad g - \gamma = 2\sqrt{(1-ff)(1-nff)}(ff-\gamma g)(1-nff\gamma g)$$

atque $$\text{Arc.} f\gamma - \text{Arc.} fg = 2nf(ff-\gamma g)\sqrt{(1-ff)(1-nff)}.$$

8. **Coroll. 3.** Cum sit

$$g = \frac{e\sqrt{(1-ff)(1-nff)} + f\sqrt{(1-ee)(1-nee)}}{1-neeff},$$

erit $$\sqrt{(1-gg)} = \frac{\sqrt{(1-ee)(1-ff)} - ef\sqrt{(1-nee)(1-nff)}}{1-neeff}$$

$$\text{et} \quad \sqrt{(1-ngg)} = \frac{\sqrt{(1-nee)(1-nff)} - nef\sqrt{(1-ee)(1-ff)}}{1-neeff}$$

hincque $$\frac{g}{\sqrt{(1-gg)}} = \frac{e\sqrt{(1-ee)(1-nff)} + f\sqrt{(1-ff)(1-nee)}}{1-ee-ff+neeff}$$

$$\frac{\sqrt{(1-ngg)}}{\sqrt{(1-gg)}} = \frac{\sqrt{(1-ee)(1-nee)(1-ff)(1-nff)} + (1-n)ef}{1-ee-ff+neeff}$$

$$\frac{g\sqrt{(1-ngg)}}{\sqrt{(1-gg)}} = \frac{e(1-2nff+nf^4)\sqrt{(1-ee)(1-nee)} + f(1-2nee+ne^4)\sqrt{(1-ff)(1-nff)}}{(1-ee-ff+neeff)(1-neeff)}$$

$$\sqrt{(1-gg)(1-ngg)} = \frac{ef(2n(ee+ff)-(n+1)(1+neeff)) + (1+neeff)\sqrt{(1-ee)(1-nee)(1-ff)(1-nff)}}{(1-neeff)^2}$$

Hujusmodi autem formulae inveniuntur, si simpliciores in verso quoque exprimantur; sic erit

$$\frac{1}{g} = \frac{f\sqrt{(1-ee)(1-nee)} - e\sqrt{(1-ff)(1-nff)}}{ff-ee}$$

$$\frac{1}{\sqrt{(1-gg)}} = \frac{\sqrt{(1-ee)(1-ff)} + ef\sqrt{(1-nee)(1-nff)}}{1-ee-ff+neeff}$$

$$\frac{1}{\sqrt{(1-ngg)}} = \frac{\sqrt{(1-nee)(1-nff)} + nef\sqrt{(1-ee)(1-ff)}}{1-nee-nff+neeff}.$$

9. **Coroll. 4.** Has formulas ideo evolvere visum est, ut si fieri posset, ex iis ejusmodi relatio inter e, f, g determinaretur, ut functio quaepiam ipsius g fieret aequalis producto ex functionibus similibus ipsarum e et f. Verum hujusmodi expressio, qualis pro parabola est reperta, hic pro ellipsi non tam facile erui posse videtur. Simpliciores autem harum formularum combinationes dant

$$\sqrt{(1-gg)} + ef\sqrt{(1-ngg)} = \sqrt{(1-ee)(1-ff)}$$
$$\sqrt{(1-ngg)} + nef\sqrt{(1-gg)} = \sqrt{(1-nee)(1-nff)}.$$

10. **Coroll. 5.** Ut igitur sit $\mathrm{Arc}.Be - \mathrm{Arc}.fg = nefg$, relatio inter abscissas e, f, g ita debet esse comparata, ut sit

$$\text{vel} \quad g = \frac{e\sqrt{(1-ff)(1-nff)} + f\sqrt{(1-ee)(1-nee)}}{1-neeff},$$

$$\text{vel} \quad f = \frac{g\sqrt{(1-ee)(1-nee)} - e\sqrt{(1-gg)(1-ngg)}}{1-neegg},$$

$$\text{vel} \quad e = \frac{g\sqrt{(1-ff)(1-nff)} - f\sqrt{(1-gg)(1-ngg)}}{1-nffgg}.$$

11. **Coroll. 6.** Si punctum g statuatur in vertice A, erit $g = 1$ et $f = \sqrt{\frac{1-ee}{1-nee}}$, qui est casus a Com. Fagnani datus. Nunc igitur hoc problema de duobus arcubus ellipseos, quorum differentia sit geometrice assignabilis, multo generalius est solutum, cum dato arcu Be, alter terminus arcus quaesiti ubi libuerit, accipi queat.

12. **Coroll. 7.** Effici autem omnino nequit, ut horum arcuum differentia evanescat; ita ut duo arcus dissimiles ellipsis inter se aequales exhiberi queant; ut enim hoc eveniret, vel e, vel f vel g evanescere deberet, unde vel arcus evanescentes vel similes prodituri essent.

13. **Problema 2.** Dato ellipsis arcu quocunque fg, a puncto quovis dato p, alium arcum pq abscindere, ita ut horum duorum arcuum differentia sit geometrice assignabilis.

Solutio. Positis abscissis pro arcu dato $CF = f$, $CG = g$, et pro quaesito $CP = p$ et $CQ = q$, quarum quidem altera, vel p vel q, pro lubitu assumi poterit. In subsidium nunc vocetur arcus Be abscissae $CE = e$ respondens, qui per problema 1 ita sit comparatus, ut fiat

$$\mathrm{Arc}.Be - \mathrm{Arc}.fg = nefg \quad \text{et} \quad \mathrm{Arc}.Be - \mathrm{Arc}.pq = nepq.$$

Hoc autem ut eveniat, necesse est ut sit

$$e = \frac{g\sqrt{(1-ff)(1-nff)} - f\sqrt{(1-gg)(1-ngg)}}{1-nffgg}$$

$$\text{pariterque} \quad e = \frac{q\sqrt{(1-pp)(1-npp)} - p\sqrt{(1-qq)(1-nqq)}}{1-nppqq}.$$

His igitur duobus valoribus inter se aequatis determinabitur q per f, g et p, uti problema exigit et quia abscissa e est cognita, erit

$$\text{Arc}.fg - \text{Arc}.pq = ne(pq - fg).$$

Sicque differentia arcuum fg et pq est geometrica, et arcus quaesiti pq alter terminus ab arbitrio nostro pendet. Q. E. I.

14. **Coroll. 1.** Datis ergo punctis f, g et p, quartum punctum q, seu ejus abscissa $CQ = q$ ex hac aequatione debet definiri

$$\frac{g\sqrt{(1-ff)(1-nff)} - f\sqrt{(1-gg)(1-ngg)}}{1-nffgg} = \frac{q\sqrt{(1-pp)(1-npp)} - p\sqrt{(1-qq)(1-nqq)}}{1-nppqq},$$

vel, quia haec formula non parum est complicata, quantitas e ex hujusmodi aequationibus simplicioribus eliminari poterit

$$\sqrt{(1-ee)} - fg\sqrt{(1-nee)} = \sqrt{(1-ff)(1-gg)} \text{ et } \sqrt{(1-ee)} - pq\sqrt{(1-nee)} = \sqrt{(1-pp)(1-qq)}$$
$$\sqrt{(1-nee)} - nfg\sqrt{(1-ee)} = \sqrt{(1-nff)(1-ngg)} \text{ et } \sqrt{(1-nee)} - npq\sqrt{(1-ee)} = \sqrt{(1-npp)(1-nqq)},$$

unde elicitur

$$\sqrt{(1-ff)(1-gg)} - pq\sqrt{(1-nff)(1-ngg)} = \sqrt{(1-pp)(1-qq)} - fg\sqrt{(1-npp)(1-nqq)},$$

vel etiam

$$\sqrt{(1-nff)(1-ngg)} - npq\sqrt{(1-ff)(1-gg)} = \sqrt{(1-npp)(1-nqq)} - nfg\sqrt{(1-pp)(1-qq)}.$$

15. **Coroll. 2.** Ut ambo hi arcus fg et pq fiant inter se aequales, oportet sit $pq = fg$. Ponatur $pp + qq = t$, et ambae postremae aequationes dabunt

$$\sqrt{(1-ff)(1-gg)} - fg\sqrt{(1-nff)(1-ngg)} = \sqrt{(1-t+ffgg)} - fg\sqrt{(1-nt+nnffgg)}$$
$$\sqrt{(1-nff)(1-ngg)} - nfg\sqrt{(1-ff)(1-gg)} = \sqrt{(1-nt+nnffgg)} - nfg\sqrt{(1-t+ffgg)},$$

quarum haec per fg multiplicata ad illam addatur, ut prodeat

$$(1-nffgg)\sqrt{(1-ff)(1-gg)} = (1-nffgg)\sqrt{(1-t+ffgg)};$$

seu $1 - ff - gg + ffgg = 1 - t + ffgg$, ideoque $t = ff + gg = pp + qq$. Unde sequitur arcum pq similem et aequalem futurum esse arcui fg.

16. **Problema 3.** Dato arcu ellipsis quocunque fg, abscindere a dato puncto p alium arcum pqr qui deficiat a duplo illius arcus fg quantitate algebraica, seu ut sit $2\,\text{Arc}.fg - \text{Arc}.pqr =$ lineae rectae.

Solutio. Sint abscissae ut ante $CE = e$, $CF = f$, $CG = g$, $CP = p$, $CQ = q$ et $CR = r$, ubi Be est arcus a vertice B abscissus, ab arcu fg dato geometrice discrepans; a quo etiam arcus pq et qr discrepent quantitatibus geometrice assignabilibus. Erit ergo

$$\text{I.}\quad e = \frac{g\sqrt{(1-ff)(1-nff)} - f\sqrt{(1-gg)(1-ngg)}}{1-nffgg},$$

$$\text{II.}\quad e = \frac{q\sqrt{(1-pp)(1-npp)} - p\sqrt{(1-qq)(1-nqq)}}{1-nppqq},$$

$$\text{III.}\quad e = \frac{r\sqrt{(1-qq)(1-nqq)} - q\sqrt{(1-rr)(1-nrr)}}{1-nqqrr}.$$

Hinc si primum definiatur abscissa e, ex eaque porro abscissae q et r, erit

$$\text{Arc}.fg - \text{Arc}.pq = ne\,(pq - fg)$$

$$\text{Arc}.fg - \text{Arc}.qr = ne\,(qr - fg),$$

quibus aequationibus additis habebitur

$$2\,\text{Arc}.fg - \text{Arc}.pqr = ne\,(pq + qr - 2fg). \quad \text{Q. E. I.}$$

17. **Coroll. 1.** Quoniam dato arcu fg etiam arcus Be datur, spectemus e tanquam quantitatem cognitam, eritque

$$p = \frac{q\sqrt{(1-ee)(1-nee)} - e\sqrt{(1-qq)(1-nqq)}}{1-neeqq}$$

$$r = \frac{q\sqrt{(1-ee)(1-nee)} + e\sqrt{(1-qq)(1-nqq)}}{1-neeqq}$$

unde fit

$$p + r = \frac{2q\sqrt{(1-ee)(1-nee)}}{1-neeqq}.$$

18. **Coroll. 2.** Differentia ergo arcuum $2fg$ et pqr hoc modo determinatorum erit

$$2\,\text{Arc}.fg - \text{Arc}.pqr = 2ne\,\left(\frac{qq\sqrt{(1-ee)(1-nee)}}{1-neeqq} - fg\right).$$

Ut ergo arcus pqr exacte aequalis fiat duplo arcus fg, oportet esse

$$fg = \frac{qq\sqrt{(1-ee)(1-nee)}}{1-neeqq}, \quad \text{unde definitur} \quad qq = \frac{fg}{neefg + \sqrt{(1-ee)(1-nee)}}$$

hincque porro inveniuntur p et r.

19. **Coroll. 3.** Relatio autem abscissarum e, f, g hac aequatione exprimitur

$$ff + gg = ee + neeffgg + 2fg\sqrt{(1-ee)(1-nee)};$$

unde facillime duo arcus in ellipsi, quorum alter alterius sit duplus, hoc modo determinabuntur: Sumta pro lubitu abscissa e, capiatur quoque pro lubitu valor producti fg, ex hinc reperietur summa quadratorum $ff + gg$, unde utraque abscissa f et g seorsim reperietur. Inde vero porro colligitur abscissa q, ex eaque denique abscissae p et r, ut arcus pqr fiat duplus arcus fg.

20. **Coroll. 4.** Nihilo tamen minus arcus fg pro arbitrio assumi potest, nec non alter terminus arcus quaesiti vel p vel r, ex quo deinceps definiri poterit alter terminus, ut arcus pqr fiat duplo major quam arcus fg. Sed haec operatio multo fit molestior, et calculum requirit prolixiorem.

21. **Coroll. 5.** Si priore operatione utamur, qua quantitatibus e et fg arbitrarios valores tribuimus, cavendum est, ne inde valor ipsius q prodeat unitate major, seu $CQ > CA$, sic enim perveniretur ad imaginaria. Ut autem prodeat $q < 1$, capi debet $fg < \sqrt{\frac{1-ee}{1-nee}}$; at si capiatur $fg = \sqrt{\frac{1-ee}{1-nee}}$, fit $g = 1$, $f = \sqrt{\frac{1-ee}{1-nee}}$ et $q = 1$; hincque $p + r = 2\sqrt{\frac{1-ee}{1-nee}}$ et $p = r = \sqrt{\frac{1-ee}{1-nee}}$. Hoc ergo casu arcus fg in A terminatur, et arcus pqr utrinque circa A aequaliter protenditur, uti est obvium.

22. **Exemplum.** Ponamus $n = \frac{1}{2}$ et $ee = \frac{1}{2}$, ut semiaxis conjugatus ellipsis prodeat $CB = \sqrt{\frac{1}{2}}$, altero existente $CA = 1$. Quia nunc esse debet $fg < \sqrt{\frac{2}{3}}$, statuatur $fg = \frac{6}{7}\sqrt{\frac{2}{3}} = \frac{2\sqrt{6}}{7}$, ac reperietur $f = \frac{1}{\sqrt{2}}$, $g = \frac{4\sqrt{3}}{7}$, tum vero $q = \frac{2\sqrt{2}}{3}$; porro autem elicitur $p + r = \frac{6\sqrt{3}}{7}$ et $r - p = \frac{\sqrt{10}}{7}$, unde fit $p = \frac{6\sqrt{3} - \sqrt{10}}{14}$ et $r = \frac{6\sqrt{3} + \sqrt{10}}{14}$. Hic casus Fig. 58 repraesentatur, ubi arcus fg terminus *

g fere in verticem A cadit, punctum vero ultra f versus B reperitur, at punctum r capi debet in ellipsis parte inferiori; ita, ut arcus $pfgAr$ alterum arcum fg, cujus ille est duplus, totum in se complectatur.

23. **Scholion.** Si libuerit alia hujusmodi exempla expedire, in quibus radicalia non inter se implicentur, casus prodibunt simplicissimi ponendo $f = e$, unde prodit

$$g = \frac{2e}{1 - ne^4}\sqrt{(1-ee)(1-nee)};$$

tum vero reperitur $qq = \frac{2ee}{1+ne^4}$, ita ut esse oporteat $2ee < 1 + ne^4$, seu $ee > \frac{1-\sqrt{(1-n)}}{n}$, alioquin loca p, q, r fuerint imaginaria. Hinc itaque pro terminis arcus quaesiti pqr elicitur

$$r + p = \frac{2e}{1-ne^4}\sqrt{2(1-ee)(1-nee)(1+ne^4)}$$

$$r - p = \frac{2e}{1-ne^4}\sqrt{(1-2ee+ne^4)(1-2nee+ne^4)}$$

eritque ut desideratur $\text{Arc}.pqr = 2\,\text{Arc}.fg$. Si ponamus semiaxem conjugatum

$$CB = k = \frac{2(1-ee)}{1-2ee}, \quad \text{ut sit} \quad n = 1 - kk = \frac{-3+4ee}{(1-2ee)^2}$$

pleraeque irrationalitates evanescunt, fiet enim

$$f = e, \quad g = \frac{2e(1-2ee)}{1-3ee+4e^4}, \quad qq = \frac{2ee(1-2ee)^2}{1-4ee+e^4+4e^6}$$

atque
$$r + p = \frac{2e\sqrt{(2-8ee+2e^4+8e^6)}}{1-3ee+4e^4}$$

$$r - p = \frac{2e(1-ee)\sqrt{(1-16e^4)}}{1-3ee+4e^4}.$$

Debet ergo sumi $4ee < 1$, ne loca p et r fiant imaginaria. Imprimis autem notari meretur casus, quem in problemate sequente evolvam.

24. **Problema 4.** In quadrante elliptico ACB abscindere arcum fg, qui sit semissis totius arcus quadrantis $BfgA$.

Solutio. Cum arcus fg duplum esse debeat ipse quadrans BA, quantitates problematis ita debent definiri, ut punctum p in B, et punctum r in A cadat. Erit ergo $p = 0$ et $r = 1$, unde fit $e = q$ et $e = \sqrt{\frac{1-qq}{1-nqq}} = \sqrt{\frac{1-ee}{1-nee}}$, seu $1 - 2ee + ne^4 = 0$, ideoque $ee = \frac{1-\sqrt{(1-n)}}{n}$. Cum autem posito $CB = k$ sit $n = 1 - kk$, erit $ee = \frac{1-k}{1-kk} = \frac{1}{1+k}$, sicque habebimus $e = q = \frac{1}{\sqrt{(1+k)}}$. Tum vero quia esse oportet $2fg = pq + qr$, erit

$$2fg = e = \frac{1}{\sqrt{(1+k)}}, \text{ atque } ff + gg = ee + \frac{1}{4}ne^4 + e\sqrt{(1-ee)(1-nee)},$$

sive $ff + gg = \frac{5+3k}{4+4k}$, ergo ob $2fg = \frac{4\sqrt{(1+k)}}{4+4k}$, fiet

$$(f+g)^2=\frac{5+3k+4\sqrt{(1+k)}}{4+4k} \quad \text{et} \quad (g-f)^2=\frac{5+3k-4\sqrt{(1+k)}}{4+4k}, \quad \text{ergo}$$

$$f=\sqrt{\frac{5+3k-\sqrt{(9+14k+9kk)}}{8+8k}} \quad \text{et} \quad g=\sqrt{\frac{5+3k+\sqrt{(9+14k+9kk)}}{8+8k}},$$

sicque puncta f et g determinantur, ut arcus fg sit semissis quadrantis AB. Q. E. I.

25. **Coroll. 1.** Quo hae formulae simpliciores evadant, ponatur semiaxis conjugatus

$$CB=k=\frac{1-4m}{1+4m}, \quad \text{seu} \quad 4m=\frac{1-k}{1+k}$$

eritque
$$f=CF=\sqrt{\frac{1+m-\sqrt{(mm+\frac{1}{2})}}{2}} \quad \text{et} \quad g=CG=\sqrt{\frac{1+m+\sqrt{(mm+\frac{1}{2})}}{2}}.$$

26. **Coroll. 2.** Vel in subsidium vocetur angulus quidem φ, cujus sinus sit $=\frac{\sqrt{(2m+\frac{1}{2})}}{m+1}$, seu $\sin\varphi=\frac{4\sqrt{(1+k)}}{5+3k}$; eritque $CF=f=\sin\frac{1}{2}\varphi\sqrt{\frac{5+3k}{4+4k}}$ et $CG=g=\cos\frac{1}{2}\varphi\sqrt{\frac{5+3k}{4+4k}}$.

27. **Coroll. 3.** Si sit $k=1$, quo casu ellipsis abit in circulum, erit $\sin\varphi=\sqrt{\frac{1}{2}}$, ideoque $\varphi=45^0$, et ob $\sqrt{\frac{5+3k}{4+4k}}=1$, erit $CF=f=\sin 22\frac{1}{2}^0$ et $CG=g=\cos 22\frac{1}{2}^0=\sin 67\frac{1}{2}^0$, ita ut arcus fg prodeat 45^0, qui utique est semissis quadrantis.

28. **Coroll. 4.** Si ellipsis semiaxis conjugatus $CB=k$ evanescat, prae $CA=1$, fiet $f=\frac{1}{2}$ et $g=1$; sin autem $CB=k$ sit quasi infinitus respectu $CA=1$, erit $f=0$ et $g=\sqrt{\frac{3}{4}}$, unde applicatae $Ff=k$ et $Gg=\frac{1}{2}k$; ita ut hi duo casus eodem recidant, utroque enim ellipsis confunditur cum linea recta.

29. **Coroll. 5.** Si fuerit $k=\frac{5}{7}$, prodit $f=\sqrt{\frac{1}{6}}$ et $g=\sqrt{\frac{7}{8}}$. At si generalius ponatur $m=\frac{1-2uu}{4u}$, ut sit $k=\frac{2uu+u-1}{1+u-2uu}$, fiet $f=\sqrt{\frac{1-u}{2}}$ et $g=\sqrt{\frac{1+2u}{4u}}$. Jam ut utraque expressio fiat rationalis, sit $u=1-2ff$, fietque

$$k=\frac{1-5ff+4f^4}{3ff-4f^4} \quad \text{et} \quad g=\frac{\sqrt{(3-10ff+8f^4)}}{2(1-2ff)}.$$

Ergo f ita debet determinari, ut $3-10ff+8f^4$ fiat quadratum; quod cum eveniat casu $f=1$, ponatur $f=\frac{1-z}{1+z}$, eritque

$$3-10ff+8f^4=\frac{1-20z+86zz-20z^3+z^4}{(1+z)^4}.$$

Cujus numerator ergo quadratum effici debet, ita tamen ut prodeat $f<1$, seu z affirmativum et unitate minus. Statim quidem apparet quadratum prodire posito $z=-\frac{3}{10}$; quia vero hic valor est negativus, ponatur $z=\frac{y-3}{10}$, eritque numerator ille

$$1-20z+86zz-20z^3+z^4=\frac{y^4-212y^3+10454yy-77108y+391.391}{10000}.$$

Posita hujus radice $=\frac{yy-106y+391}{100}$, fit $y=\frac{1446}{391}$ et $z=\frac{273}{3910}$, $f=\frac{3637}{4183}$ et $g=\frac{yy-106y+391}{200(1-2ff)(1+z)^2}$, seu

$$g=\frac{yy-106y+391}{200(6z-1-zz)}=\frac{100zz-1000z+82}{200(6z-1-zz)}=\frac{647}{5986}.$$

Sicque casus exhiberi potest, in quo tam semiaxes ellipsis quam ambae abscissae f et g numeris rationalibus exprimuntur.

30. **Scholion.** Simili etiam modo, si detur (Fig. 57) arcus ellipsis quicunque fg, a puncto quovis dato p alius assignari poterit arcus pz, qui datum multiplum arcus fg, puta $m.fg$, superet quantitate algebraica; si enim abscissae ponantur $CF=f$, $CG=g$, $CP=p$, $CQ=q$, $CR=r$, $CS=s$, $CT=t$, et ab abscissa CP numerando fuerit $CZ=z$, ultima indici m respondens; tum in subsidium vocando arcum Be, cujus abscissa $Ce=e$, ut sit

$$e=\frac{g\sqrt{(1-ff)(1-nff)}-f\sqrt{(1-gg)(1-ngg)}}{1-nffgg},$$

ex data abscissa p sequentes ita determinentur

$$q=\frac{p\sqrt{(1-ee)(1-nee)}+e\sqrt{(1-pp)(1-npp)}}{1-neepp},$$

$$r=\frac{q\sqrt{(1-ee)(1-nee)}+e\sqrt{(1-qq)(1-nqq)}}{1-neeqq},$$

$$s=\frac{r\sqrt{(1-ee)(1-nee)}+e\sqrt{(1-rr)(1-nrr)}}{1-neerr},$$

etc.

donec perveniatur ad ultimam z, quae a p numerando locum tenet indice m notatum. Quo facto erit

$$m.\mathrm{Arc}.fg-\mathrm{Arc}.pz=ne\,(pq+qr+rs+\ldots+yz-mfg).$$

Hinc igitur quoque punctum p ita definiri poterit, ut haec quantitas algebraica evanescat, seu fiat

$$pq+qr+rs+\ldots+yz=mfg,$$

quo casu arcus pz exacte erit aequalis arcui fg toties sumto, quot numerus m continet unitates, seu erit $\mathrm{Arc}.pz=m.\mathrm{Arc}.fg$. Dato ergo ellipsis arcu quocunque fg, alius assignari poterit pz, qui ad illum datam teneat rationem, puta $m:1$. Quin etiam m poterit esse numerus fractus, seu ista ratio ut numerus ad numerum $\mu:\nu$; nam quaeratur primo arcus pz, ut sit $pz=\mu.fg$, tum quaeratur alius $\pi\omega$, ut sit $\pi\omega=\nu.fg$, eritque $pz:\pi\omega=\mu:\nu$. Verum quo longius hic progrediamur, hae formulae continuo magis fiunt complicatae, ut calculum in genere expedire non liceat.

31. **Problema 5.** In dato ellipseos quadrante AB arcum abscindere fg, qui sit tertia pars totius quadrantis AB.

Solutio. Cum in genere fuerit determinatus arcus $pqrs$, qui sit triplus arcus fg, dum hic arcus tanquam cognitus est spectatus, nunc vicissim calculus ita instruatur, ut punctum p in B, et punctum s in A incidat, seu ut sit $p=0$ et $s=1$. Formulae ergo modo exhibitae abibunt in has

$$q=e,\quad r=\frac{2e\sqrt{(1-ee)(1-nee)}}{1-ne^4}\quad\text{et}\quad 1=\frac{r\sqrt{(1-ee)(1-nee)}+e\sqrt{(1-rr)(1-nrr)}}{1-neerr},$$

seu $r=\sqrt{\dfrac{1-ee}{1-nee}}$, ob $r=\dfrac{s\sqrt{(1-ee)(1-nee)}-e\sqrt{(1-ss)(1-nss)}}{1-neess}$, unde fit $2e(1-nee)=1-ne^4$, seu $1-2e+2ne^3-ne^4=0$, existente semiaxe $CA=1$, $CB=k$ et $n=1-kk$. Primum ergo ex hac aequatione biquadratica definiri debet valor ipsius e, quae resolutio commode ita succedit.

Sit $e=\frac{1}{x}$, ut habeatur $x^4-2x^3+2nx-n=0$; ac ponatur ad secundum terminum tollendum $x=y+\frac{1}{2}$, prodibit

$$y^4-\frac{3}{2}yy+(2n-1)y-\frac{3}{16}=0,$$

cujus factores fingantur $yy+\alpha y+\beta$ et $yy-\alpha y+\gamma$, eritque

$$\beta+\gamma=\alpha\alpha-\frac{3}{2},\quad \gamma-\beta=\frac{2n-1}{\alpha}\quad \text{et}\quad \beta\gamma=-\frac{3}{16}$$

unde elicimus

$$(\beta+\gamma)^2-(\gamma-\beta)^2=\alpha^4-3\alpha^2+\frac{9}{4}-\frac{(2n-1)^2}{\alpha\alpha}=4\beta\gamma=-\frac{3}{4},$$

$$\text{ideoque}\quad \alpha^6-3\alpha^4+3\alpha^2=(2n-1)^2;$$

subtrahatur utrinque 1, ut cubus fiat completus

$$(\alpha\alpha-1)^3=4nn-4n,\quad \text{ergo}\quad \alpha\alpha=1+\sqrt[3]{4n(n-1)}=1-\sqrt[3]{4nkk}\quad \text{et}\quad \alpha=\sqrt{(1-\sqrt[3]{4nkk})}.$$

Invento ergo α erit

$$\beta=\frac{1}{2}\alpha\alpha-\frac{3}{4}-\frac{(2n-1)}{2\alpha}\quad \text{et}\quad \gamma=\frac{1}{2}\alpha\alpha-\frac{3}{4}+\frac{(2n-1)}{2\alpha}$$

indeque $$y=-\frac{1}{2}\alpha\pm\sqrt{\left(\frac{3}{4}-\frac{1}{4}\alpha\alpha\pm\frac{(2n-1)}{2\alpha}\right)}=\frac{-\alpha\alpha\pm\sqrt{(3\alpha\alpha-\alpha^4\pm2(2n-1)\alpha)}}{2\alpha}$$

unde obtinetur $e=\frac{2}{2y+1}$. Porro debet esse $3fg=pq+qr+rs$, seu

$$3fg=(1+e)\sqrt{\frac{1-ee}{1-nee}},\quad \text{ideoque}\quad fg=\frac{1}{3}(1+e)\sqrt{\frac{1-ee}{1-nee}},$$

ex quo obtinemus

$$ff+gg=ee+\frac{1}{9}nee(1+e)^2\cdot\frac{1-ee}{1-nee}+\frac{2}{3}(1+e)(1-ee).$$

Cognitis igitur valoribus fg et $ff+gg$, seorsim abscissae $CF=f$ et $CG=g$ reperientur, quae arcum determinabunt fg praecise subtriplum totius quadrantis AB. Q. E. I.

Comparatio arcuum Hyperbolae.

32. (Fig. 59). Sit C centrum hyperbolae, cujus semiaxis transversus $CA=k$, et semiaxis conjugatus $=1$. Hinc sumta super axe conjugato a centro C abscissa quacunque $CZ=z$, erit applicata $Zz=k\sqrt{(1+zz)}$, unde

$$\text{arcus } Az=\int dz\sqrt{\frac{1+(1+kk)zz}{1+zz}}=\int\frac{dz\,(1+(1+kk)zz)}{\sqrt{(1+(2+kk)zz+(1+kk)z^4)}}.$$

33. Ponatur brevitatis gratia $1+kk=n$, ita ut n sit numerus affirmativus unitate major, eritque arcus hyperbolae quicunque

$$Az=\int\frac{dz\,(1+nzz)}{\sqrt{(1+(n+1)zz+nz^4)}}.$$

Poni igitur in § XI oportet $A=1$, $C=n+1$, $E=n$, $\mathfrak{A}=1$, $\mathfrak{C}=n$ et $\mathfrak{E}=0$. Unde si fuerit

$$y=\frac{c\sqrt{(1+xx)(1+nxx)}-x\sqrt{(1+cc)(1+ncc)}}{1-nccxx}$$

habebimus $$\int dx\sqrt{\frac{1+nxx}{1+xx}}-\int dy\sqrt{\frac{1+nyy}{1+yy}}=\text{Const.}-ncxy,$$

34. Denotet $\Pi.x$ arcum abscissae x respondentem, et $\Pi.y$ arcum abscissae y respondentem. Quia facto $x=0$ fit $y=c$, erit $\Pi.x-\Pi.y=-\Pi.c-ncxy$, seu

$$\Pi.y-\Pi.x-\Pi.c=ncxy.$$

35. Ob $\sqrt{(1+cc)(1+ncc)}$ ambiguum, poni quoque poterit

$$y=\frac{c\sqrt{(1+xx)(1+nxx)}+x\sqrt{(1+cc)(1+ncc)}}{1-nccxx}$$

eritque $\Pi.y-\Pi.x-\Pi.c=ncxy$, secundum ea, quae de ellipsi § 3 sunt exposita; atque hinc sequens problema solvi poterit.

36. **Problema 6.** Dato arcu hyperbolae Ae a vertice sumto, abscindere a quovis dato puncto f alium arcum fg, ut differentia horum arcuum fg et Ae sit geometrice assignabilis.

Solutio. Ponatur arcus propositi Ae abscissa $CE=e$, abscissa data $CF=f$ et quaesita $CG=g$, statuatur porro

$$g=\frac{e\sqrt{(1+ff)(1+nff)}+f\sqrt{(1+ee)(1+nee)}}{1-neeff}$$

eritque $\Pi.g-\Pi.f-\Pi.e=nefg$. At est

$$\Pi.g-\Pi.f=\text{Arc}.fg \quad \text{et} \quad \Pi.e=\text{Arc}.Ae, \quad \text{unde} \quad \text{Arc}.fg-\text{Arc}.Ae=nefg.$$

Puncto ergo g hoc modo definito erit arcuum fg et Ae differentia geometrice assignabilis. Q. E. I.

37. **Coroll. 1.** Si ergo f ita capiatur, ut sit $1-neeff=0$, seu $f=\frac{1}{e\sqrt{n}}$, abscissa $CG=g$ fit infinita, ideoque et arcus fg erit infinitus, qui etiam arcum Ae excedere reperitur quantitate infinita $nefg$ ob $g=\infty$. Ut igitur casus, quemadmodum figura repraesentatur, substituere possit, necesse est ut capiatur $f<\frac{1}{e\sqrt{n}}$.

38. **Coroll. 2.** Sin autem sit $f>\frac{1}{e\sqrt{n}}$, fiet g negativum, et $\Pi.g$ pariter fiet negativum; unde si fuerit

$$g=\frac{e\sqrt{(1+ff)(1+nff)}+f\sqrt{(1+ee)(1+nee)}}{neeff-1}$$

habebimus $$\Pi.e+\Pi.f+\Pi.g=nefg=Ae+Af+Ag.$$

Tres ergo arcus exhiberi possunt Ae, Af et Ag, quorum summa geometrice assignari queat.

39. **Coroll. 3.** Casus hic, quo summa trium arcuum hyperbolicorum rectificabilis prodiit, eo magis est notatu dignus, quod similis casus in ellipsi locum non habet; ibi enim terni arcus $\Pi.y-\Pi.e-\Pi.x=-ncxy$ (3) nunquam ejusdem signi fieri possunt, propterea quod $nccxx$ unitate semper minus existit.

40. **Coroll. 4.** Horum ternorum arcuum duo inter se fieri possunt aequales; sit enim

$$f=e, \quad \text{erit} \quad g=\frac{2e\sqrt{(1+ee)(1+nee)}}{ne^4-1}$$

unde prodit $2\Pi.e+\Pi.g=neeg$, seu $2\,\text{Arc}.Ae+\text{Arc}.Ag=$ quantitati geometricae. Si igitur insuper fiat $g=e$, habebitur arcus hyperbolicus, cujus triplum, ideoque et ipse ille arcus erit rectificabilis, qui casus cum sit maxime memorabilis, eum in sequente problemate data opera evolvamus.

41. **Problema 7.** In hyperbola a vertice A arcum abscindere Ae, cujus longitudo geometrice assignari queat.

Solutio. Posito hyperbolae semiaxe transverso $CA = k$, et conjugato $= 1$, ita ut posita abscissa $CE = e$, sit applicata $Ee = k\sqrt{(1 + ee)}$; brevitatis gratia autem sit $n = 1 + kk$. Sit ergo $CE = e$ abscissa arcus Ae quaesiti, cujus rectificatio desideratur; quem in finem statuatur in § praec. $g = e$, ut sit

$$e = \frac{2e\sqrt{(1 + ee)}(1 + nee)}{ne^4 - 1} \quad \text{eritque} \quad 3\Pi.e = ne^3, \quad \text{seu} \quad \text{Arc}.Ae = \frac{1}{3}ne^3$$

ideoque rectificabilis. Abscissa ergo hujus arcus $CE = e$ determinari debet ex hac aequatione $ne^4 - 1 = 2\sqrt{(1 + ee)}(1 + nee)$, quae abit in hanc

$$nne^8 - 6ne^4 - 4(n + 1)ee - 3 = 0.$$

Ad quam resolvendam faciamus $ee = \frac{x}{n}$, ut prodeat

$$x^4 - 6nxx - 4n(n + 1)x - 3nn = 0,$$

cujus factores fingantur $(xx + \alpha x + \beta)(xx - \alpha x + \gamma) = 0$; unde comparatione instituta orietur

$$\gamma + \beta = \alpha\alpha - 6n, \quad \gamma - \beta = \frac{-4n(n + 1)}{\alpha} \quad \text{et} \quad \beta\gamma = -3nn.$$

Quare cum sit $(\gamma + \beta)^2 - (\gamma - \beta)^2 = 4\beta\gamma = -12nn$, fiet

$$\alpha^4 - 12n\alpha\alpha + 36nn - \frac{16nn(n + 1)^2}{\alpha\alpha} = -12nn,$$

$$\text{sive} \quad \alpha^6 - 12n\alpha^4 + 48nn\alpha\alpha = 16nn(n + 1)^2.$$

Subtrahatur utrinque $64n^3$, ut fiat

$$(\alpha\alpha - 4n)^3 = 16n^2(n - 1)^2, \quad \text{seu} \quad \alpha\alpha = 4n + \sqrt[3]{16nn(n - 1)^2},$$

$$\text{ergo} \quad \alpha = \sqrt{(4n + \sqrt[3]{16nn(n - 1)^2})}.$$

Invento nunc valore ipsius α, erit porro

$$\beta = \frac{1}{2}\alpha\alpha - 3n + \frac{2n(n + 1)}{\alpha} \quad \text{et} \quad \gamma = \frac{1}{2}\alpha\alpha - 3n - \frac{2n(n + 1)}{\alpha}$$

et quatuor radices ipsius x erunt

$$x = \pm\frac{1}{2}\alpha \pm \sqrt{(3n - \frac{1}{4}\alpha\alpha \pm \frac{2n(n + 1)}{\alpha})} = nee,$$

seu cum valor ipsius α tam affirmative quam negative accipi queat, erit

$$e = \sqrt{\left(\frac{\alpha}{2n} \pm \sqrt{(\frac{3}{n} - \frac{\alpha\alpha}{4nn} + \frac{2(n + 1)}{n\alpha})}\right)}.$$

Hic igitur valor si tribuatur abscissae $CE = e$, erit arcus hyperbolae

$$Ae = \frac{1}{3}ne^3 \quad \text{Q. E. I.}$$

42. **Coroll. 1.** Si loco unitatis semiaxis conjugatus ponatur $= b$, ut abscissae cuicunque $CP = x$ respondeat applicata $Pp = k\sqrt{(1 + \frac{xx}{bb})}$, erit

$$\alpha = \sqrt{(4bb(bb+kk) + \sqrt[3]{16b^4k^2(bb+kk)^2})}$$

tumque sumta abscissa

$$CP = x = b\sqrt{\left(\frac{\alpha}{2(bb+kk)} + \sqrt{\left(\frac{2bb}{bb+kk} + \frac{2bb(2bb+kk)}{\alpha(bb+kk)} - \sqrt[3]{\frac{b^4k^4}{4(bb+kk)^4}}\right)}\right)},$$

$$\text{erit arcus } Ap = \frac{(bb+kk)x^3}{3b^4}.$$

43. **Coroll. 2.** Si hyperbola fuerit aequilatera, seu $k = b = 1$, poni debet $n = 2$, fietque $\alpha = 2\sqrt{3}$ et arcus rectificabilis Ae abscissa prodit

$$CE = e = \sqrt{\frac{\sqrt{3} + \sqrt{(3+2\sqrt{3})}}{2}}$$

et ipsa hujus arcus longitudo reperitur

$$Ae = \frac{\sqrt{3} + \sqrt{(3+2\sqrt{3})}}{3}\sqrt{\frac{\sqrt{3} + \sqrt{(3+2\sqrt{3})}}{2}}.$$

44. **Coroll. 3.** Si ponatur $4n(n-1) = s^3$, ut sit $n = \frac{1+\sqrt{(s^3+1)}}{2}$, signa radicalia cubica ex calculo evanescent; prodit enim

$$\alpha = \sqrt{(2 + ss + 2\sqrt{(s^3+1)})} = \sqrt{(1 - s + ss)} + \sqrt{(1+s)},$$

unde fit $\left(\frac{1+\sqrt{(1+s^3)}}{2}\right)ee =$

$$\frac{1}{2}\sqrt{(1+s)} + \frac{1}{2}\sqrt{(1-s+ss)} \pm \sqrt{\left(1 - \frac{1}{4}ss + \sqrt{(1+s^3)} + (1 - \frac{1}{2}s)\sqrt{(1+s)} + (1 + \frac{1}{2}s)\sqrt{(1-s+ss)}\right)}$$

sive
$$ee = \frac{\sqrt{(1+s)} + \sqrt{(1-s+ss)} \pm \sqrt{(4 - ss + 4\sqrt{(1+s^3)} + 2(2-s)\sqrt{(1+s)} + 2(2+s)\sqrt{(1-s+ss)})}}{1+\sqrt{(1+s^3)}}.$$

45. **Coroll. 4.** Pro hyperbola aequilatera, ubi $n = 2$, si radicalia per fractiones decimales evolvantur, reperitur $CE = e = 1{,}4619354$ et $Ae = 1{,}4248368e$, seu Arc. $Ae = 2{,}0830191$, semiaxe transverso existente $CA = 1$, quos numeros ideo adjeci, quo veritas hujus rectificationis facilius perspici queat.

46. **Coroll. 5.** Casus etiam satis simplex prodit si $s = 1$ et $n = \frac{1+\sqrt{2}}{2} = 1 + kk$, ita ut sit $k = \sqrt{\frac{\sqrt{2}-1}{2}}$, hinc enim fit

$$ee = \frac{\sqrt{2} + 1 + \sqrt{(9 + 6\sqrt{2})}}{1+\sqrt{2}} = 1 + \sqrt{3}.$$

Ergo sumta abscissa $CE = \sqrt{(1+\sqrt{3})}$, erit arcus $Ae = \frac{(1+\sqrt{2})(1+\sqrt{3})\sqrt{(1+\sqrt{3})}}{6}$. In fractionibus decimalibus fit $k = 0{,}45509$, $e = 1{,}65289$ et Arc. $Ae = 1{,}81701$.

47. **Coroll. 6.** Si sit $s = 0$, quo casu fit $n = 1$ et $k = 0$, hyperbola autem abit in lineam rectam CE, erit $ee = 3$ et $e = \sqrt{3} = CE$, arcusque Ae evadit $= \sqrt{3} = CE$, uti natura rei postulat.

48. **Problema 8.** Invenire alios arcus hyperbolicos rectificabiles.

Solutio. Sumta abscissa $CE = e$, capiantur aliae duae abscissae $CP = p$ et $CQ = q$, ut sit

$$q = \frac{e\sqrt{(1+pp)(1+npp)} + p\sqrt{(1+ee)(1+nee)}}{1 - neepp},$$

erit $\Pi.q - \Pi.p - \Pi.e = nepq$. Quia ergo $\Pi.q - \Pi.p = \text{Arc}.pq$ et $\Pi.e = \text{Arc}.Ae$, erit

$$\text{Arc}.pq = nepq + \text{Arc}.Ae.$$

Quodsi igitur abscissae e is tribuatur valor, qui in problemate praecedente est definitus, ita ut arcus Ae sit rectificabilis; hunc scilicet in finem posito

$$\alpha = \sqrt{(4n + \sqrt[3]{16nn(n-1)^2})}$$

capiatur

$$e = \sqrt{\left(\frac{\alpha}{2n} + \sqrt{\left(\frac{3}{n} - \frac{\alpha\alpha}{4nn} + \frac{2(n+1)}{n\alpha}\right)}\right)}$$

eritque arcus $Ae = \frac{1}{3}ne^3$. Hinc sumta abscissa p pro lubitu, ex superiori formula ita definietur abscissa q, ut prodeat arcus rectificabilis

$$\text{Arc}.pq = nepq + \frac{1}{3}ne^3.$$

Verumtamen p ita accipi debet, ut sit $neepp < 1$, seu $p < \frac{1}{e\sqrt{n}}$; cum igitur sit $ne^4 > 1$, capienda est abscissa p minor quam e, et quidem oportet sit

$$\frac{1}{p} > \sqrt{\left(\frac{1}{2}\alpha + \sqrt{(3n - \frac{1}{4}\alpha\alpha + \frac{2n(n+1)}{\alpha})}\right)}.$$

Dummodo ergo punctum p non capiatur ultra hunc terminum, semper ab eo abscindi potest arcus pq, cujus longitudo geometrice assignari queat. Q. E. I.

49. **Coroll. 1.** Quodsi capiatur $p = \frac{1}{e\sqrt{n}}$, ob $1 - neepp = 0$, fiet abscissae q valor infinitus, ideoque ipse arcus rectificabilis pq erit infinitus.

50. **Coroll. 2.** In hyperbola ergo aequilatera, ubi $n = 2$ et $e = \sqrt{\frac{\sqrt{3} + \sqrt{(3 + 2\sqrt{3})}}{2}}$, prior abscissa $CP = p$ tam parva accipi debet, ut sit $p < \frac{1}{\sqrt{(\sqrt{3} + \sqrt{(3 + 2\sqrt{3})})}}$, seu $p < 0{,}4836784$. Sumta igitur hac abscissa tam parva, semper alterum punctum q assignari poterit, ut arcus pq sit rectificabilis.

51. **Scholion.** Insigni hac hyperbolae proprietate, qua reliquis sectionibus conicis antecellit, contentus, non immoror investigationi ejusmodi arcuum, quorum differentia sit algebraica, vel qui inter se datam teneant rationem, cujusmodi quaestiones pro ellipsi evolvi; cum enim talia problemata pro hyperbola simili modo resolvi queant, ea ne lectori sim molestus, data opera praetermitto. Hanc igitur dissertationem finiam comparatione arcuum parabolae cubicalis primariae, cujus rectificationem constat pariter fines analyseos transgredi.

Comparatio arcuum Parabolae cubicalis primariae.

52. (Fig. 60). Sit $Aefg$ parabola cubicalis primaria, A ejus vertex et $AEFG$ ejus tangens in vertice, super qua sumta abscissa quacunque $AP = z$, sit applicata $Pp = \frac{1}{3}z^3$, unde arcus Ap reperitur

$$= \int dz\sqrt{(1 + z^4)} = \int \frac{dz(1 + z^4)}{\sqrt{(1 + z^4)}}.$$

53. Quo igitur formulas nostras huc accommodemus, poni oportet $A=1$, $C=0$, $E=1$, $\mathfrak{A}=1$, $\mathfrak{C}=0$ et $\mathfrak{E}=1$, ita ut sit $y=\frac{c\sqrt{(1+x^4)}+x\sqrt{(1+c^4)}}{1-ccxx}$; quo facto erit

$$\int dx\sqrt{(1+x^4)}-\int dy\sqrt{(1+y^4)}=\text{Const.}-cxy\left(cc+xy\sqrt{(1+c^4)}+\frac{1}{3}ccxxyy\right)$$

sumto tam $\sqrt{A}$ quam c negativo in formulis N° VII et XI expositis.

54. Quodsi ergo tres capiamus abscissas $AE=e$, $AF=f$ et $AG=g$, ita ut sit

$$g=\frac{e\sqrt{(1+f^4)}+f\sqrt{(1+e^4)}}{1-eeff},$$

erit $$\text{Arc.}Af-\text{Arc.}Ag=-\text{Arc.}Ae-efg\left(ee+fg\sqrt{(1+e^4)}+\frac{1}{3}eeffgg\right),$$ seu

$$\text{Arc.}fg-\text{Arc.}Ae=efg\left(ee+fg\sqrt{(1+e^4)}+\frac{1}{3}eeffgg\right).$$

Dato ergo quovis arcu Ae, a dato puncto f abscindi poterit alius arcus fg, ut horum arcuum differentia sit rectificabilis.

55. Si capiantur arcus e et f negativi, ita ut sit $eeff>1$ et

$$g=\frac{e\sqrt{(1+f^4)}+f\sqrt{(1+e^4)}}{eeff-1}$$

et arcus abscissis e, f, g respondentes denotentur per $\Pi.e$, $\Pi.f$, $\Pi.g$, erit

$$\Pi.e+\Pi.f+\Pi.g=efg\left(ee-fg\sqrt{(1+e^4)}+\frac{1}{3}eeffgg\right).$$

Sin autem sit $$g=\frac{e\sqrt{(1+f^4)}+f\sqrt{(1+e^4)}}{1-eeff},$$

erit $$\Pi.g-\Pi.f-\Pi.e=efg\left(ee+fg\sqrt{(1+e^4)}+\frac{1}{3}eeffgg\right).$$

56. Cum sit hoc posteriori casu $ff+gg=ee+2fg\sqrt{(1+e^4)}+eeffgg$, erit quoque

$$\Pi.g-\Pi.f-\Pi.e=\frac{1}{2}efg\left(ee+ff+gg-\frac{1}{3}eeffgg\right).$$

Casu autem altero pro summa arcuum, quo

$$g=\frac{e\sqrt{(1+f^4)}+f\sqrt{(1+e^4)}}{eeff-1},$$

erit $$\Pi.e+\Pi.f+\Pi.g=\frac{1}{2}efg\left(ee+ff+gg-\frac{1}{3}eeffgg\right).$$

57. **Problema 9.** Dato arcu Ae parabolae cubicalis primariae, in ejus vertice A terminato, ab alio quocunque puncto f abscindere in eadem parabola, arcum fg, ita ut horum arcuum differentia $fg-Ae$ sit rectificabilis.

Solutio. Positis abscissis $AE=e$, $AF=f$, $AG=g$, quarum illae duae dantur, haec vero ita accipiatur, ut sit $g=\frac{e\sqrt{(1+f^4)}+f\sqrt{(1+e^4)}}{1-eeff}$, eritque horum arcuum differentia

$$\text{Arc.}fg-\text{Arc.}Ae=\frac{1}{2}efg\left(ee+ff+gg-\frac{1}{3}eeffgg\right)$$

Verum cum data sit abscissa e, altera abscissa f ita accipi debet, ut sit $eeff < 1$, seu $f < \frac{1}{e}$, ne abscissa $AG = g$ prodeat negativa. Sin autem detur punctum g, inde reperitur

$$f = \frac{g\sqrt{(1+e^4)} - e\sqrt{(1+g^4)}}{1 - eegg},$$

unde si g tam fuerit magna, ut sit $eegg > 1$, seu $g > \frac{1}{e}$, erit

$$f = \frac{e\sqrt{(1+g^4)} - g\sqrt{(1+e^4)}}{eegg - 1},$$

simulque necesse est, ut sit $g > e$, ne f fiat negativum. A dato ergo puncto f siquidem sit $f < \frac{1}{e}$, arcus quaesitus fg in consequentia vergit; a puncto autem g, si sit $g > \frac{1}{e}$ et simul $g > e$, arcus quaesitus fg retro accipietur. Q. E. I.

58. **Coroll. 1.** Cum sit applicata $Ee = \frac{1}{3}e^3$, seu $AE^3 = 3Ee$, erit parameter hujus parabolae $= 3$, ideoque unitas nostra est triens parametri.

59. **Coroll. 2.** Si ergo sit $e = 1$, abscissa data f seu g vel debet esse minor quam 1, vel major quam 1; dummodo ergo punctum datum non in e cadat, ab eo semper vel prorsum vel retrorsum arcus quaesito satisfaciens abscindi poterit: prorsum scilicet, si abscissa data minor sit quam e, retrorsum vero, si major. At si abscissa data esset $= 1$, altera vel infinita vel $= 0$ prodiret.

60. **Coroll. 3.** Si sit $e > 1$, ideoque $e > \frac{1}{e}$, altera abscissarum f vel g, quae datur, vel minor esse debet quam $\frac{1}{e}$, vel major quam e; alioquin arcus problemati satisfaciens abscindi nequit, quod ergo usu venit, si abscissa data inter limites e et $\frac{1}{e}$ contineatur.

61. **Coroll. 4.** Sin autem sit $e < 1$, ideoque $\frac{1}{e} > e$, alteram abscissam datam vel minorem esse oportet quam $\frac{1}{e}$, vel majorem quam $\frac{1}{e}$; dum ergo non sit aequalis ipsi $\frac{1}{e}$, quo casu arcus quaesitus vel fieret infinitus, vel ipsi arcui Ae similis et aequalis, reperietur semper arcus problemati satisfaciens.

62. **Coroll. 5.** Hoc autem casu, quo $e < 1$, fieri potest, ut a dato puncto f in utramque partem arcus problemati satisfaciens abscindi queat; hoc scilicet evenit, si abscissa data intra limites e et $\frac{1}{e}$ contineatur: tum enim ea tam loco f quam loco g scribi poterit.

63. **Coroll. 6.** Si arcus fg debeat esse contiguus arcui Ae, seu si sit $f = e$, reperietur

$$g = \frac{2e\sqrt{(1+e^4)}}{1-e^4};$$

hoc ergo fieri nequit nisi sit $e < 1$. Hoc ergo casu erit arcuum differentia

$$\text{Arc}.fg - \text{Arc}.Ae = \frac{2e^5(9 - 2e^4 + e^8)\sqrt{(1+e^4)}}{3(1-e^4)^3}$$

64. **Problema 10.** Dato in parabola cubicali arcu quocunque fg, alium invenire arcum pq, qui illum superet quantitate geometrice assignabili.

Solutio. Sint abscissae datae $AF=f$, $AG=g$, quaesitae $AP=p$ et $AQ=q$, et in subsidium vocetur arcus Ae, cujus abscissa $AE=e$, sitque

$$g=\frac{e\sqrt{(1+f^4)}+f\sqrt{(1+e^4)}}{1-eeff} \quad \text{et} \quad q=\frac{e\sqrt{(1+p^4)}+p\sqrt{(1+e^4)}}{1-eepp}$$

erit
$$\text{Arc}.fg-\text{Arc}.Ae=\frac{1}{2}efg\left(ee+ff+gg-\frac{1}{3}eeffgg\right)=M$$

et
$$\text{Arc}.pq-\text{Arc}.Ae=\frac{1}{2}epq\left(ee+pp+qq-\frac{1}{3}eeppqq\right)=N,$$

ergo
$$\text{Arc}.pq-\text{Arc}.fg=N-M.$$

Eliminemus autem utrinque e, reperieturque

$$e=\frac{g\sqrt{(1+f^4)}-f\sqrt{(1+g^4)}}{1-ffgg}=\frac{q\sqrt{(1+p^4)}-p\sqrt{(1+q^4)}}{1-ppqq},$$

unde si f, g et p dentur, obtinebitur q hoc modo:

$$q=\Big[g(1-ffgg+ffpp-ggpp)\sqrt{(1+f^4)(1+p^4)}-f(1-ffgg+ggpp-ffpp)\sqrt{(1+g^4)(1+p^4)}$$
$$+p(1-ffpp-ggpp+ffgg)\sqrt{(1+f^4)(1+g^4)}-2fgp(ff+gg+pp+ffggpp)\Big]:$$
$$\Big[(1-ffgg-ffpp-ggpp)^2-4ffggpp(ff+gg+pp)\Big],$$

qui valor quoties non fit negativus, praebebit a dato puncto p arcum pq, ab arcu proposito fg geometrice discrepantem. Q. E. I.

65. **Coroll. 1.** Ambo abscissarum paria ita pendent ab e, ut sit

$$ff+gg=ee(1+ffgg)+2fg\sqrt{(1+e^4)}$$
$$pp+qq=ee(1+ppqq)+2pq\sqrt{(1+e^4)},$$

unde reperietur

$$ee=\frac{pq(ff+gg)-fg(pp+qq)}{(pq-fg)(1-fgpq)} \quad \text{et} \quad \sqrt{(1+e^4)}=\frac{(pp+qq)(1+ffgg)-(ff+gg)(1+ppqq)}{2(pq-fg)(1-fgpq)}$$

et hinc penitus eliminando e habebitur

$$((1-ffgg)(pp+qq)+(1-ppqq)(ff+gg))^2=4(1-fgpq)^2((pq-fg)^2+(ff+gg)(pp+qq)),$$
vel
$$((1-ffgg)(pp+qq)-(1-ppqq)(ff+gg))^2=4(pq-fg)^2((1-fgpq)^2+(ff+gg)(pp+qq)).$$

66. **Coroll. 2.** Hinc ergo dato quocunque arcu fg, infinitis modis alii determinari possunt arcus pq, quorum differentia ab illo fg sit geometrice assignabilis. Erit autem haec differentia

$$\text{Arc}.pq-\text{Arc}.fg=\frac{1}{2}e\left(ee(pq-fg)\left(1-\frac{1}{3}ppqq-\frac{1}{3}fgpq-\frac{1}{3}ffgg\right)+pq(pp+qq)-fg(ff+gg)\right)$$
$$=\frac{e(pq-fg)\left(ff+gg+pp+qq-\frac{1}{3}pq(pq+2fg)(ff+gg)-\frac{1}{3}fg(fg+2pq)(pp+qq)\right)}{2(1-fgpq)}.$$

67. **Coroll. 3.** Casus hic duo peculiares considerandi occurrunt, alter quo $pq=fg$, alter quo $fgpq=1$. Priori casu fit $pp+qq=ff+gg$, ideoque $p=f$ et $q=g$; ita ut arcus pq in ipsum arcum fg incidat, eorumque differentia fiat $=0$. Altero vero casu fit

$$(1 - ffgg)(pp + qq) + \left(1 - \frac{1}{ffgg}\right)(ff + gg) = 0, \quad \text{seu} \quad pp + qq = \frac{ff + gg}{ffgg},$$

unde colligitur $p = \frac{1}{g}$ et $q = \frac{1}{f}$, qui est casus a Celeb. Joh. Bernoullio b. m. primum in Actis Lipsiensibus A. 1698 expositus.

68. **Coroll. 4.** Hoc ergo casu Bernoulliano, quo $p = \frac{1}{g}$, $q = \frac{1}{f}$; ac proinde $pq = \frac{1}{fg}$ et $pp + qq = \frac{ff + gg}{ffgg}$, erit arcuum differentia

$$\text{Arc}.pq - \text{Arc}.fg = \frac{e(1 - ffgg)}{6f^3g^3}(3(ff + gg)(1 + ffgg) - ee(1 - ffgg)^2);$$

at est $e(1 - ffgg) = g\sqrt{(1 + f^4)} - f\sqrt{(1 + g^4)}$, unde colligimus

$$ee(1 - ffgg)^2 = (ff + gg)(1 + ffgg) - 2fg\sqrt{(1 + f^4)(1 + g^4)},$$

quibus valoribus substitutis erit

$$\text{Arc}.pq - \text{Arc}.fg = \frac{(g\sqrt{(1 + f^4)} - f\sqrt{(1 + g^4)})}{3f^3g^3}((ff + gg)(1 + ffgg) + fg\sqrt{(1 + f^4)(1 + g)^4}),$$

quae abit in hanc formam

$$\text{Arc}.pq - \text{Arc}.fg = \frac{(1 + f^4)\sqrt{(1 + f^4)}}{3f^3} - \frac{(1 + g^4)\sqrt{(1 + g^4)}}{3g^3},$$

quae est ipsa horum arcuum differentia a Cel. Bernoullio exhibita.

69. **Scholion.** Simili modo dato quocunque arcu parabolae cubicalis fg, alii arcus inveniri poterunt, qui a duplo vel triplo vel quovis multiplo arcus fg discrepent quantitate algebraica: quin etiam hi arcus ita determinari poterunt, ut differentia evanescat. Hinc ergo proposito arcu quocunque fg, alius in eadem parabola assignari poterit, qui arcus istius sit duplus vel triplus, vel alius quicunque multiplus. Ex quo vicissim pro lubitu infinitis modis ejusmodi arcus assignare licebit, qui inter se datam teneant rationem. Ut autem duo arcus sint inter se in ratione aequalitatis, alii assignari nequeunt, nisi qui sint inter se similes et aequales. Quod quo clarius appareat, sit

$$fg = m, \; pq = \mu, \; ff + gg = n \text{ et } pp + qq = \nu,$$

erit primo $$n = ee(1 + mm) + 2m\sqrt{(1 + e^4)},$$

tum vero $$\nu = ee(1 + \mu\mu) + 2\mu\sqrt{(1 + e^4)}.$$

Unde ut arcus pq et fg inter se fiant aequales, oportet esse

$$ee(\mu - m)\left(1 - \frac{1}{3}\mu\mu - \frac{1}{3}m\mu - \frac{1}{3}mm\right) + \mu\nu - mn = 0.$$

At pro n et ν illis valoribus substitutis fit

$$\mu\nu - mn = ee(\mu - m)(1 + \mu\mu + m\mu + mm) + 2(\mu - m)(\mu + m)\sqrt{(1 + e^4)}$$

unde debet esse, postquam per $\mu - m$ fuerit divisum,

$$2ee\left(1 + \frac{1}{3}\mu\mu + \frac{1}{3}m\mu + \frac{1}{3}mm\right) + 2(\mu + m)\sqrt{(1 + e^4)} = 0,$$

quae quantitates cum sint omnes affirmativae, solus prior factor $\mu - m = 0$ dabit solutionem;

eritque $f=p$ et $g=q$. Ad multo illustriora autem progredior ostensurus in hac curva etiam arcus rectificabiles assignari posse.

70. **Problema II.** In parabola cubicali primaria a vertice A arcum exhibere Ae, cujus longitudo geometrice assignari queat.

Solutio. Assumtis tribus abscissis $AE=e$, $AF=f$ et $AG=g$, supra vidimus, si sit

$$g=\frac{e\sqrt{(1+f^4)}+f\sqrt{(1+e^4)}}{eeff-1},$$

fore

$$\Pi.e+\Pi.f+\Pi.g=\frac{1}{2}efg\,(ee+ff+gg-\frac{1}{3}eeffgg).$$

Statuantur nunc hi tres arcus inter se aequales, seu $e=f=g$, eritque

$$e=\frac{2e\sqrt{(1+e^4)}}{e^4-1},\quad \text{seu}\quad e^8-6e^4-3=0$$

hincque $$e^4=3+2\sqrt{3}.$$

Sumta ergo abscissa $AE=e=\sqrt[4]{(3+2\sqrt{3})}$, erit

$$3\,\mathrm{Arc}.Ae=\frac{1}{2}e^5(3-\frac{1}{3}e^4)=\frac{1}{6}e^5(6-2\sqrt{3}),$$

sive $$\mathrm{Arc}.Ae=\frac{1}{9}(3-\sqrt{3})(3+2\sqrt{3})\sqrt[4]{(3+2\sqrt{3})}=\frac{1}{3}(1+\sqrt{3})\sqrt[4]{(3+2\sqrt{3})}.$$

. . . .

Fig. 61. Pag. 494.

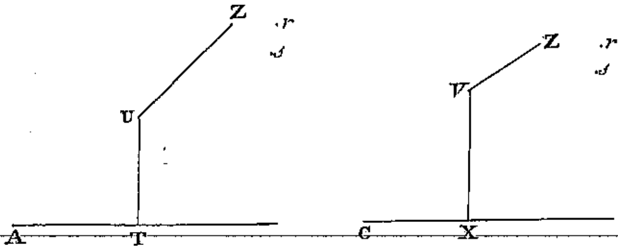

Fig. 55. Pag. 458.

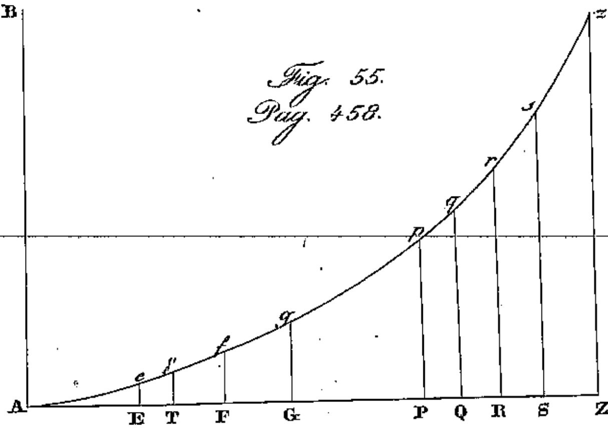

Fig. 56. Pag. 462.

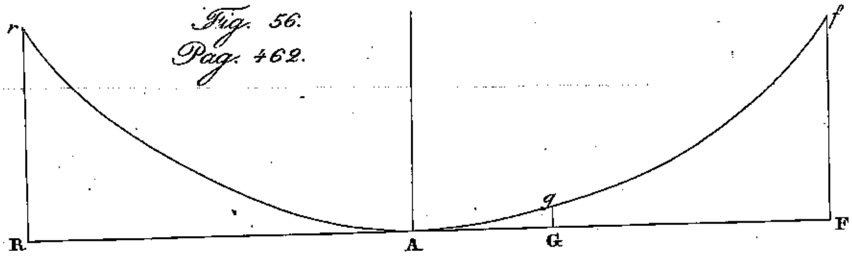

Fig. 57. Pag. 469.

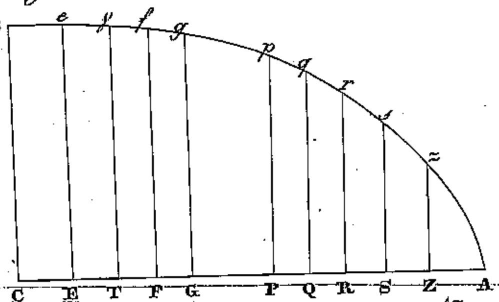

Fig. 58. Pag. 473.

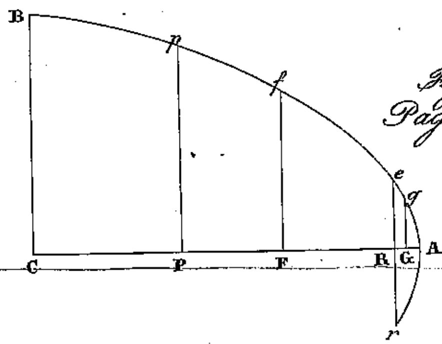

Fig. 59. Pag. 477.

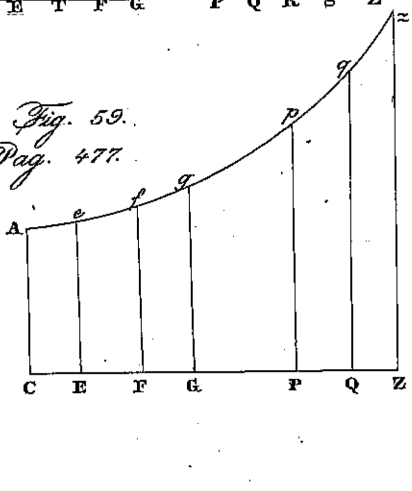

Fig. 60. Pag. 481.