1862

Series maxime idoneae pro circuli quadratura proxime invenienda

Leonhard Euler

XIII.

Series maxime idoneae pro circuli quadratura proxime invenienda.

1. Antequam Analyseos infinitorum principia essent perspecta, nulla alia via rationem peripheriae ad diametrum explorandi patebat, praeter considerationem polygonorum circulo cum inscriptorum tum circumscriptorum. Ex quo fonte primum Archimedes notissimam proportionem 22 ad 7, tum vero Metius veritati propiorem 355 ad 113 elicuit; donec tandem Ludolfus a Ceulen hanc proportionem ad 35 figuras in partibus decimalibus produxit, quem stupendum et molestissimum laborem certe vix ulterius prosequi licuisset. Deinde vero, cum, Analysis infinitorum ope, series idoneae rationem diametri ad peripheriam exprimentes essent exhibitae, multo minore labore ratio Ludolfiana multo longius, primo scilicet a Scharpio ad 72, tum vero a Machino ad 100, ac denique a Lagnio ad 128 figuras decimales est continuata; ex qua ratione si circumferentia circuli, cujus diameter distantiam stellarum fixarum maxime remotarum superaret, computaretur, ne millesima quidem pollicis parte a veritate aberraretur.

2. Assidui autem hi calculatores, quorum industria summam meretur laudem et admirationem, omnes usi sunt serie, qua arcus circuli ex tangente definitur, ita ut posita tangente $=t$, radio existente $=1$, arcus respondens sit

$$=t-\frac{1}{3}t^3+\frac{1}{5}t^5-\frac{1}{7}t^7+\frac{1}{9}t^9-\text{etc.},$$

quae series utique maxime convergens reddi posset, si tangentem t pro lubitu diminuere liceret. Verum cum hinc ratio diametri ad peripheriam concludi nequeat, nisi arcus ille ad totam peripheriam assignabilem et cognitam teneat rationem, vix minorem arcum in hunc finem accipere licet quam 30^0, cujus tangens est $\frac{1}{\sqrt{3}}$; unde denotante π peripheriam circuli, cujus diameter est $=1$, fit

$$\frac{\pi}{6}=\frac{1}{\sqrt{3}}\left(1-\frac{1}{3.3}+\frac{1}{5.3^2}-\frac{1}{7.3^3}+\frac{1}{9.3^4}-\frac{1}{11.3^5}+\text{etc.}\right),$$

seu

$$\pi=\left(1-\frac{1}{3.3}+\frac{1}{5.3^2}-\frac{1}{7.3^3}+\frac{1}{9.3^4}-\frac{1}{11.3^5}+\text{etc.}\right)\sqrt{12}.$$

Etsi enim angulus 18^0, cujus tangens est $\sqrt{(1-2\sqrt{\frac{1}{5}})}$, seriem multo magis reddat convergentem, duplex tamen irrationalitas calculum tantopere molestum reddit, ut nullum inde compendium sperari possit, quae molestia pro minoribus angulis multo magis increscit.

3. Exercitatissimus etiam calculator **Lagnius**, qui hunc calculum longissime est prosecutus, angulum 30^0 aliis minoribus in hoc negotio praeferendum censuit; verum antequam ipsius seriei terminos evolvere posset, ex numero 12 radicem quadratam ultra 128 figuras decimales exactam extrahere erat coactus; quem laborem certe 12 horarum spatio expedire haud potuit, quin potius crediderim auctorem ei aliquot adeo dies insudasse, quandoquidem summa, qua opus est, attentio, cum relaxationem tum revisionem pluriumque operationum repetitionem postulat. Hoc autem labore exantlato ipsius seriei 265 terminos ad minimum evolvere debebat; primo igitur numerum $\sqrt{12}$ ad 128 figuras expressum continuo 265 vicibus per ternarium dividi oportebat, ad quod negotium, si cujusque figurae inventioni et scriptioni unum minutum secundum tribuamus, quinque horae vix sufficiebant. Deinde quotos hos singulos respective per numeros impares 3, 5, 7, 9, 11, 13, etc. dividi erat necesse, quae opera, ob divisores continuo majores, ad minimum tempus duplo majus, ideoque 10 horarum postulabat. Denique additio cum terminorum affirmativorum, tum negativorum utraque seorsim breviori quam quinque horarum spatio expediri haud poterat: sicque totus labor intra 37 horas omni adhibita diligentia neutiquam potuerat absolvi. Nullum autem est dubium, quin auctor tempus duplo imo triplo majus impenderit.

4. Quemadmodum autem hic labor mirifice sublevari potuisset, jam pridem ostendi, ubi angulum semirectum in duas pluresve partes dividere docui, quarum tangentes sint rationales. Ita cum sit $\frac{\pi}{4} = \text{Ang. tang } 1 = \text{Ang. tang } \frac{1}{2} + \text{Ang. tang } \frac{1}{3}$, erit per duas series

$$\pi = +2\left(1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \frac{1}{9.4^4} - \text{etc.}\right)$$
$$+ \frac{4}{3}\left(1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \frac{1}{9.9^4} - \text{etc.}\right),$$

quarum adeo prior magis convergit, quam praecedens, ex tangente anguli 30^0 petita; neque hic ulla extractione radicis opus est, quae sola in calculo praecedenti laborem 12 horarum postulaverat. Deinde priores utriusque seriei termini saltem multo minore labore evolvuntur, cum vel paucis constent figuris, vel periodum in iis agnoscant, unde calculus admodum fit expeditus. Etsi autem hic duas series in unam summam colligi oportet, tamen quia magis convergunt, multo paucioribus opus est terminis: ita si fractionem decimalem pro π ad 128 figuras justam desideremus, prioris seriei terminos 210, posterioris vero 132 capi conveniet, qui totus labor, praecedente ratione aestimatus, vix 24 horas requirere videtur.

5. Deinde ex eodem principio, cum sit in genere

$$\text{Ang. tang } \frac{1}{a} = \text{Ang. tang } \frac{1}{b} + \text{Ang. tang } \frac{b-a}{ab+1},$$

erit $\text{Ang. tang } \frac{1}{2} = \text{Ang. tang } \frac{1}{3} + \text{Ang. tang } \frac{1}{7}$, ideoque

$$\frac{\pi}{4} = 2\,\text{Ang. tang } \frac{1}{3} + \text{Ang. tang } \frac{1}{7},$$

unde per duas series magis convergentes fit

$$\pi = +\frac{8}{3}\left(1 - \frac{1}{3.9} + \frac{1}{5.9^2} - \frac{1}{7.9^3} + \frac{1}{9.9^4} - \frac{1}{11.9^5} + \text{etc.}\right)$$
$$+\frac{4}{7}\left(1 - \frac{1}{3.49} + \frac{1}{5.49^2} - \frac{1}{7.49^3} + \frac{1}{9.49^4} - \frac{1}{11.49^5} + \text{etc.}\right).$$

Hinc ergo si valor ipsius π ad 128 figuras justus colligi debeat, prioris seriei 132 terminos, posterioris vero tantum 75 terminos evolvisse sufficiet, horum autem 207 terminorum evolutio certe multo minorem operam requirit, quam calculus a Lagnio subductus, extractione radicis, quae sola tertiam partem insumebat, exclusa. Ex quo totus hic labor vix 18 horarum esset aestimandus, nisi divisio per numerum 49 aliquam molestiam crearet.

6. Simili modo loco Ang. tang $\frac{1}{3}$, si non satis parvus videatur, minores introducere poterimus servatoque altero habebimus Ang. tang $\frac{1}{3}$ = Ang. tang $\frac{1}{7}$ + Ang. tang $\frac{2}{11}$, ideoque

$$\frac{\pi}{4} = 2 \text{ Ang. tang } \frac{2}{11} + 3 \text{ Ang. tang } \frac{1}{7}, \quad \text{et}$$

$$\pi = +\frac{16}{11}\left(1 - \frac{4}{3.121} + \frac{4^2}{5.121^2} - \frac{4^3}{7.121^3} + \frac{4^4}{9.121^4} - \text{etc.}\right)$$
$$+\frac{12}{7}\left(1 - \frac{1}{3.49} + \frac{1}{5.49^2} - \frac{1}{7.49^3} + \frac{1}{9.49^4} - \text{etc.}\right).$$

Verum etsi hic multo pauciores terminos assumsisse sufficiat, divisio tamen per majores numeros 49 et 121 omne fere lucrum adimere videtur. Neque adeo haec transformatio:

$$\text{Ang. tang } \frac{2}{11} = \text{Ang. tang } \frac{1}{7} + \text{Ang. tang } \frac{3}{79}, \quad \text{quae praebet}$$

$$\frac{\pi}{4} = 5 \text{ Ang. tang } \frac{1}{7} + 2 \text{ Ang. tang } \frac{3}{79}$$

calculo contrahendo inserviet; etiamsi enim series pro altero angulo vehementer convergat, tamen indoles fractionis $\frac{3}{79}$ laborem non mediocriter adauget, ita ut praestare videatur seriebus longe minus convergentibus uti.

7. Quando autem calculo numerico est consulendum, non solum ad convergentiam serierum quarum termini in summam colligi debent, respici convenit, sed potissimum ad facilitatem, qua singuli termini per operationes arithmeticas evolvantur: ita si seriei progressio geometrica sit admixta calculus facillime expeditur, si hujus termini in ratione vel decupla, vel centupla, vel millecupla decrescant. Quamobrem seriei, qua angulus, cujus tangens est $= \frac{1}{7}$, ac multo magis ejus, cujus tangens est $= \frac{3}{79}$, termini non sine ingenti labore evolvuntur, qui forte tantus est, ut quilibet maluerit multo plures terminos serierum pro angulis, quorum tangentes sunt $\frac{1}{2}$ et $\frac{1}{3}$ expedire, nequaquam enim major convergentia laborem, quem singulorum terminorum postulat evolutio, compensare videtur. Sin autem ejusmodi angulis uti liceret, quorum tangentes essent $\frac{1}{10}$, $\frac{1}{50}$, $\frac{1}{100}$, etc. nullum est dubium, quin praeter majorem convergentiam, etiam calculus singulorum terminorum mirum in modum sublevaretur.

8. Hunc autem usum egregie praestat alia seriei forma, qua arcum circularem ex data ejus tangente exprimere licet. Deduxi autem hanc seriem ex consideratione formulae differentialis

$$\frac{dx}{\sqrt{(1-xx)}} \quad \text{ponendo ejus integrale} \quad \int \frac{dx}{\sqrt{(1-xx)}} = z\sqrt{(1-xx)}.$$

Hinc enim fiet differentiando

$$dx = dz\,(1-xx) - xzdx, \quad \text{seu} \quad \frac{dz}{dx}(1-xx) - xz - 1 = 0.$$

Statuatur nunc $\qquad z = Ax + Bx^3 + Cx^5 + Dx^7 + Ex^9 + \text{etc.}$

atque hinc colligemus

$$\begin{aligned} \frac{dz}{dx} &= A + 3Bxx + 5Cx^4 + 7Dx^6 + 9Ex^8 + \text{etc.} \\ -\frac{xxdz}{dx} &= \quad - Axx - 3Bx^4 - 5Cx^6 - 7Dx^8 - \text{etc.} \\ -xz &= \quad - Axx - Bx^4 - Cx^6 - Dx^8 - \text{etc.} \\ -1 &= -1. \end{aligned}$$

Singulis ergo terminis ad nihilum redigendis invenitur

$$A = 1, \quad B = \frac{2}{3}A, \quad C = \frac{4}{5}B, \quad D = \frac{6}{7}C, \quad E = \frac{8}{9}D, \quad \text{etc.}$$

ita ut sit $\quad \text{Ang. sin}\, x = x\sqrt{(1-xx)}\,.\left(1 + \frac{2}{3}x^2 + \frac{2.4}{3.5}x^4 + \frac{2.4.6}{3.5.7}x^6 + \frac{2.4.6.8}{3.5.7.9}x^8 + \text{etc.}\right).$

9. Sit jam $\frac{m}{n}$ tangens hujus anguli, cujus sinus positus est $= x$, eritque

$$x = \frac{m}{\sqrt{(mm+nn)}} \quad \text{et} \quad \sqrt{(1-xx)} = \frac{n}{\sqrt{(mm+nn)}},$$

ita ut irrationalitas jam ex calculo excedat fiatque

$$\text{Ang. tang}\,\frac{m}{n} = \frac{mn}{mm+nn}\left(1 + \frac{2mm}{3(mm+nn)} + \frac{2.4.m^4}{3.5(mm+nn)^2} + \frac{2.4.6m^6}{3.5.7(mm+nn)^3} + \text{etc.}\right),$$

quae series non solum magis convergit quam vulgaris ante usitata

$$\text{Ang. tang}\,\frac{m}{n} = \frac{m}{n}\left(1 - \frac{m^2}{3n^2} + \frac{m^4}{5n^4} - \frac{m^6}{7n^6} + \frac{m^8}{9n^8} - \text{etc.}\right),$$

sed etiam singuli termini fere pari facilitate evolvuntur, quoniam continua multiplicatio per fractiones $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$, etc. non difficilius instituitur, quam divisio per numeros 3, 5, 7, 9, etc. Tum vero, in quo maximum commodum cernitur, in eo constat, si numeri $mm + nn$ ad dividendum fuerint aptiores, quam simplices potestates ipsius n, quod commodum imprimis in angulis supra exhibitis locum habet. Haud minimi etiam in hac nova serie est momenti, quod omnes terminos invicem addi conveniat, cum in vulgari alternatim debeant addi et subtrahi.

10. Secundum hanc igitur novam seriem angulos supra exhibitos evolvamus, atque obtinebimus:

I. $\quad \text{Ang. tang}\,\frac{1}{2} = \frac{2}{5}\left(1 + \frac{2}{3}\cdot\frac{1}{5} + \frac{2.4}{3.5}\cdot\frac{1}{5^2} + \frac{2.4.6}{3.5.7}\cdot\frac{1}{5^3} + \text{etc.}\right)$

II. $\quad \text{Ang. tang}\,\frac{1}{3} = \frac{3}{10}\left(1 + \frac{2}{3}\cdot\frac{1}{10} + \frac{2.4}{3.5}\cdot\frac{1}{10^2} + \frac{2.4.6}{3.5.7}\cdot\frac{1}{10^3} + \text{etc.}\right)$

*

$$\text{III.}\quad \text{Ang. tang}\,\frac{1}{7} = \frac{7}{50}\left(1 + \frac{2}{3}\cdot\frac{1}{50} + \frac{2.4}{3.5}\cdot\frac{1}{50^2} + \frac{2.4.6}{3.5.7}\cdot\frac{1}{50^3} + \text{etc.}\right)$$

$$\text{IV.}\quad \text{Ang. tang}\,\frac{3}{79} = \frac{237}{6250}\left(1 + \frac{2}{3}\cdot\frac{9}{6250} + \frac{2.4}{3.5}\cdot\frac{9^2}{6250^2} + \frac{2.4.6}{3.5.7}\cdot\frac{9^3}{6250^3} + \text{etc.}\right),$$

quae series ad calculum arithmeticum manifesto multo magis sunt accommodatae quam praecedentes, cum prima exigat continuam divisionem per 5, secunda per 10, tertia per 50 et quarta per 6250, quae ideo est perquam commoda quod $\frac{9}{6250} = \frac{144}{100000}$: quam ob causam has series praecedentibus longissime anteferendas esse censeo.

11. Denotet more Newtoniano in quavis serie littera P terminum quemque praecedentem totum, quo facilius pateat, quibusnam operationibus inde elici oporteat terminum sequentem; atque prima forma $\pi = 4$ Ang. tang $\frac{1}{2} + 4$ Ang. tang $\frac{1}{3}$ suppeditat has series

$$\begin{aligned}\pi = &+ \frac{8}{5} + \frac{2}{3}\cdot\frac{1}{5}P + \frac{4}{5}\cdot\frac{1}{5}P + \frac{6}{7}\cdot\frac{1}{5}P + \frac{8}{9}\cdot\frac{1}{5}P + \text{etc.}\\ &+ \frac{6}{5} + \frac{2}{3}\cdot\frac{1}{10}P + \frac{4}{5}\cdot\frac{1}{10}P + \frac{6}{7}\cdot\frac{1}{10}P + \frac{8}{9}\cdot\frac{1}{10}P + \text{etc.}\end{aligned}$$

Secunda autem forma $\pi = 8$ Ang. tang $\frac{1}{3} + 4$ Ang. tang $\frac{1}{7}$ dat

$$\begin{aligned}\pi = &+ \frac{24}{10} + \frac{2}{3}\cdot\frac{1}{10}P + \frac{4}{5}\cdot\frac{1}{10}P + \frac{6}{7}\cdot\frac{1}{10}P + \frac{8}{9}\cdot\frac{1}{10}P + \text{etc.}\\ &+ \frac{56}{100} + \frac{2}{3}\cdot\frac{1}{50}P + \frac{4}{5}\cdot\frac{1}{50}P + \frac{6}{7}\cdot\frac{1}{50}P + \frac{8}{9}\cdot\frac{1}{50}P + \text{etc.}\end{aligned}$$

at ex tertia $\pi = 20$ Ang. tang $\frac{1}{7} + 8$ Ang. tang $\frac{3}{79}$ prodit

$$\begin{aligned}\pi = &\quad \frac{28}{10} + \frac{2}{3}\cdot\frac{1}{50}P + \frac{4}{5}\cdot\frac{1}{50}P + \frac{6}{7}\cdot\frac{1}{50}P + \frac{8}{9}\cdot\frac{1}{50}P + \text{etc.}\\ &+ \frac{948}{3125} + \frac{2}{3}\cdot\frac{144}{100000}P + \frac{4}{5}\cdot\frac{144}{100000}P + \frac{6}{7}\cdot\frac{144}{100000}P + \frac{8}{9}\cdot\frac{144}{100000}P + \text{etc.}\end{aligned}$$

In his postremis seriebus prior ita convergit, ut quilibet terminus sit fere quinquagies minor praecedente; posterior vero ita, ut quilibet terminus sit fere septingenties praecedente minor; ex quo hoc commodi assequimur, ut non sit opus in terminis primum sequentibus cyphras antecedentes scribere, quoniam nullum est periculum, ut in locis decimalibus, ubi quivis terminus incipere debet, fallamur, hincque calculus non mediocriter sublevatur.

12. His perpensis non dubito pronunciare rationem peripheriae ad diametrum, seu valorem ipsius π commodissime et promtissime obtineri ex his duabus seriebus

$$\begin{aligned}\pi = &\quad 2{,}8 + P\cdot\frac{2}{3}\cdot\frac{2}{100} + P\cdot\frac{4}{5}\cdot\frac{2}{100} + P\cdot\frac{6}{7}\cdot\frac{2}{100} + \text{etc.}\\ &+ 0{,}30336 + P\cdot\frac{2}{3}\cdot\frac{144}{100000} + P\cdot\frac{4}{5}\cdot\frac{144}{100000} + P\cdot\frac{6}{7}\cdot\frac{144}{100000} + \text{etc.},\end{aligned}$$

neque enim certe aliae exhiberi possunt series, quae tantopere convergant, simulque singuli termini tam facile per calculum arithmeticum evolvantur. Hinc ergo speciminis loco valorem π tantum ad

20 notas decimales deducam, et quo calculus certior reddatur, eum ad 22 notas extendam, in singulis autem terminis finem tantum notabo, ut nota 22^{da} sit ultima, quoniam hinc initium sponte patet. Prioris ergo seriei terminorum evolutio ita se habebit

I. . . .	2,800000000000000000000	div. per 3
	933333333333333333333	
	1866666666666666666666	mult. per $\frac{2}{100}$
II.	37333333333333333333	div. per 5
	7466666666666666666	
	29866666666666666666	mult. per $\frac{2}{100}$
III.	597333333333333333	div. per 7
	85333333333333333	
	512000000000000000	mult. per $\frac{2}{100}$
IV.	10240000000000000	div. per 9
	1137777777777777	
	9102222222222222	mult. per $\frac{2}{100}$
V.	182044444444444	div. per 11
	165494949494949	
	16549494949494	mult. per $\frac{2}{100}$
VI.	33098989898989	div. per 13
	2546076146076	
	30552913752913	mult. per $\frac{2}{100}$
VII.	611058275058	div. per 15
	40737218337	
	570321056721	mult. per $\frac{2}{100}$
VIII.	11406421134	div. per 17
	670965949	
	10735455185	mult. per $\frac{2}{100}$
IX.	214709103	div. per 19
	11300479	
	203408624	mult. per $\frac{2}{100}$
X.	4068172	div. per 21
	193722	
	3874450	mult. per $\frac{2}{100}$
XI.	77489	div. per 23
	3369	
	74120	mult. per $\frac{2}{100}$
XII.	1482	div. per 25
	59	
	1423	mult. per $\frac{2}{100}$
XIII.	28	

Hujus ergo seriei 13 termini sufficiunt pro viginti duabus notis expediendis, unde concludere licet, si calculum ad $22n$ notas continuari oporteret, omnino $13n$ notas sufficere: ex quo calculus ad 128 notas continuandus postulabit 76 terminos. Postremi autem proxime constituunt progressionem geometricam in ratione $1:50$ decrescentem, unde plures eorum evolvi non est opus.

13. Altera autem series sequenti calculo computabitur:

I. . . .	0,3033600000000000000000	div. per 3
	1011200000000000000000	
	2022400000000000000000	mult. per $\frac{144}{100000}$
II.	2912256000000000000	div. per 5
	582451200000000000	
	2329804800000000000	mult. per $\frac{144}{100000}$
III.	3354918912000000	div. per 7
	479274130285714	
	2875644781714285	mult. per $\frac{144}{100000}$
IV.	4140928485668	div. per 9
	460103165074	
	3680825320594	mult. per $\frac{144}{100000}$
V.	5300388461	div. per 11
	481853496	
	4818534965	mult. per $\frac{144}{100000}$
VI.	6938690	div. per 13
	533745	
	6404945	mult. per $\frac{144}{100000}$
VII.	9223	div. per 15
	615	
	8608	mult. per $\frac{144}{100000}$
VIII.	12	

Hujus ergo seriei pro viginti duabus notis tantum opus est octo terminis, unde $22n$ notae circiter postulabunt evolutionem $8n$ terminorum, hincque pro 128 notis sufficiet evolvisse 47 terminos.

14. Utriusque ergo seriei terminos inventos seorsim in summam colligamus, ac prior quidem summatio ita se habebit:

I.	2,8000000000000000000000
II.	373333333333333333333
III.	5973333333333333333
IV.	102400000000000000
V.	1820444444444444
VI.	33098989898989
	2,8379410920210101010101
VII.	611058275058
VIII.	11406421134
IX.	214709103
X.	4068172
XI.	77489
XII.	1482
XIII.	28
	2,8379410920832784562570

Simili modo addantur termini alterius seriei

I.	0,3033600000000000000000
II.	2912256000000000000
III.	3354918912000000
IV.	4140928485668
V.	5300388461
VI.	6938690
VII.	9223
VIII.	12
	0,3036515615065147822055
prior	2,8379410920832784562570
$\pi =$	3,1415926535897932384625

qui numerus sola ultima nota excepta justus deprehenditur, totusque hic calculus laborem unius circiter horae consumsit; ex quo intelligere licet, si quis tantum laborem, quantum Lagnius impendere velit, eum valorem peripheriae π facile ad 200 figuras decimales esse extensurum.

15. Ceterum ad calculi ulterius continuandi compendium notasse juvabit, in utriusque seriei terminis prioribus revolutiones notarum occurrere, quibus semel observatis hos terminos quousque libuerit, facillime continuare licebit, ita prioris seriei termini priores omissis in quoque cyphris initialibus, sequenti modo procedent, ubi notas periodicas deinceps continuo repetendas uncinulis inclusi:

I.	2,800 etc.
II.	37333 etc.
III.	597333 etc.
IV.	102400 etc.
V.	1820444 etc.
VI.	330 (98) (98) etc.
VII.	611 (058275) (058275) etc.
VIII.	11406 (421134) (421134) etc.
IX.	21470 (910370675076557429498605969194204488322135380958) (9103 etc.

Seriei autem posterioris termini priores in infinitum continuati sunt:

I.	0,3033600 etc.
II.	291225600 etc.
III.	335491891200 etc.
IV.	414092848566 (857142) (857142) etc.
V.	53003884616557 (714285) (714285) etc.
VI.	693869034980391 (896103) (896103) etc.
VII.	92231207111239784 (343656) (343656) etc.
VIII.	1239587423575062701 57 (874125) etc.

16. Colligamus nunc octo priores terminos in infinitum continuatos in unam summam, ut ea statim qui calculum ulterius continuare voluerit uti queat, et pariter revolutiones periodicas in utraque summa indicemus:

Summa 8 priorum terminorum seriei prioris

I.	2,80
II.	373333333333333 3333333333
III.	5973333333333 3333333333
IV.	10240000000 0000000000
V.	182044444 4444444444
VI.	33098989 8989898989 8
	2,8379410920210101 0101010101 0
VII.	611058 27505827505
VIII.	11406 42113444113
	2,8379410920832565,70629370629
seu	2,8379410920832565,706293,706293, etc.

Pro posteriori serie

I.	0,30336
II.	2912256
III.	3354918912
IV.	414092848566857142 etc.
	0,3036515615059840485668571428571428571 42 etc.
V.	530038846165577142857142857
	0,30365156150651408741302272000000000000 0
VI.	69386903498039189610389 61 038961 038961 03
VII.	9223120711123978434365 634365 634365 63
VIII.	1239587423570627015 7 874125 874125 87
	69479258663892786457434 84 547452 547452 54
	0,30365156150651408741302272000000000000000 000000 000000 00
	0,30365156150651478220560935892786457434 84,547452,547452,54

hinc summa octo priorum terminorum est

0,303651561506514782205609358927864574 3484,547452, etc.

at terminus sequens nonus sub nota demum vigesima quarta quae est 9 incipit, unde facile accu-tiores approximationes indagare licet.

17. In evolutione quidem terminorum ulteriorum divisio per majores numeros impares moram facessere potest, ita ut ad has operationes multo majus tempus sit impendendum, quam supra ad minores spectans divisores aestimavi. Verumtamen haec difficultas in serie vulgari ex angulo 30^0 petita multo est major, propterea quod ob plures terminos evolvendos, etiam majoribus divisoribus opus est conficiendum, praeterquam quod antequam haec operatio suscipi queat, tam taediosam radicis extractionem absolvi oportet. Quamobcausam dubium plane nullum superesse potest, quin calculator his binis novis seriebus utens longe facilius et promtius rationem peripheriae ad diametrum pro quovis praecisionis gradu definire queat, quam si calculum more consueto instituerct; et quantumvis temporis spatium in hunc laborem insumere cogatur, certum est more solito tempus plus quam duplo majus requiri.

18. Ceterum observasse adhuc juvabit seriem hic pro arcu ex tangente traditam etiam directe ex serie consueta

$$s = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \text{etc.}$$

ut s sit arcus cujus tangens $= t$, elici posse; cum enim sit

$$tts = t^3 - \frac{1}{3}t^5 + \frac{1}{5}t^7 - \frac{1}{7}t^9 + \text{etc.}$$

erit addendo $\quad (1+tt)s = t + \frac{2}{3}t^3 - \frac{2}{3.5}t^5 + \frac{2}{5.7}t^7 - \frac{2}{7.9}t^9 + \text{etc.}$

Porro $\quad tt(1+tt)\,s = t^3 + \frac{2}{3}t^5 - \frac{2}{3.5}t^7 + \frac{2}{5.7}t^9 - \text{etc.}$

et addendo $\quad (1+tt)^2 s = t(1+tt) + \frac{2}{3}t^3 + \frac{2.4}{3.5}t^5 - \frac{2.4}{3.5.7}t^7 + \frac{2.4}{5.7.9}t^9 - \text{etc.}$

simili modo reperitur ulterius:

$$(1+tt)^3 s = t(1+tt)^2 + \frac{2}{3}t^3(1+tt) + \frac{2.4}{3.5}t^5 + \frac{2.4.6}{3.5.7}t^7 - \frac{2.4.6}{5.7.9}t^9 + \text{etc.}$$

$$(1+tt)^4 s = t(1+tt)^3 + \frac{2}{3}t^3(1+tt)^2 + \frac{2.4}{3.5}t^5(1+tt) + \frac{2.4.6}{3.5.7}t^7 + \frac{2.4.6.8}{3.5.7.9}t^9 - \text{etc.}$$

sicque continuo progrediendo evidens est hinc obtineri:

$$s = \frac{t}{1+tt} + \frac{2}{3}\cdot\frac{t^3}{(1+tt)^2} + \frac{2.4}{3.5}\cdot\frac{t^5}{(1+tt)^3} + \frac{2.4.6}{3.5.7}\cdot\frac{t^7}{(1+tt)^4} + \text{etc.},$$

seu $\quad s = \frac{t}{1+tt}\left(1 + \frac{2}{3}\cdot\frac{tt}{1+tt} + \frac{2.4}{3.5}\cdot\frac{t^4}{(1+tt)^2} + \frac{2.4.6}{3.5.7}\cdot\frac{t^6}{(1+tt)^3} + \text{etc.}\right)$

quae series ponendo $t = \frac{m}{n}$ cum ante exhibita congruit.

19. Quoniam seriei prioris terminum nonum exhibui, cujus revolutiones periodicae 48 figuras complectuntur, seriei quoque posterioris terminum nonum hic subjungam

IX. 16800055435025555673161 (293294940353763883175647881530234471410941999177) (293 etc.

cui 23 cyphrae sunt praefigendae, antequam ad comma, partes decimales a loco integrorum separans, perveniatur, ita ut prima hujus termini periodus in loco nonagesimo quarto terminetur. Si loco notarum periodicarum velimus fractionem ordinariam adjicere, hic terminus nonus ita finite exprimetur

$$\text{IX.}\quad 16800055435025555673161\,\frac{713}{2431}\left[=\frac{9}{11}-\frac{3}{13}-\frac{5}{17}\right].$$

Simili autem modo prioris seriei terminus nonus expressus est

$$21470\,\frac{21002}{21879}.$$

Deinde summas octo terminorum supra exhibitas ita repraesentare licet

Prioris seriei

$$\begin{array}{lrl}
\text{Summa I}\ldots\text{VIII} & 2{,}8379410920832565\quad \frac{101}{143} & \left[=\frac{1}{11}+\frac{8}{13}\right]\\
\text{terminus IX} & 21470\,\frac{21002}{21879} & \left[=\frac{5}{9}+\frac{4}{11}-\frac{2}{13}+\frac{2}{17}\right]\\
\hline
\text{summa I}\ldots\text{IX} & 2{,}83794109208327804\,1\frac{12893}{21879} &
\end{array}$$

Posterioris seriei

$$\begin{array}{lr}
\text{I}\ldots\text{VIII} & 0{,}30365156150651478220560935892786457434 84\,\frac{548}{1001}\\
\text{IX} & 16800055435025555673161\,\frac{713}{2431}\\
\hline
\text{I}\ldots\text{IX} & 0{,}30365156150651478220561103893340807690402206 13\,\frac{14307}{17017}
\end{array}$$

ubi notetur esse

$$\frac{14307}{17017}=\frac{3}{7}+\frac{1}{11}+\frac{8}{13}-\frac{5}{17},$$

unde hujus fractionis evolutio est in promtu. Parique modo est prior fractio

$$\frac{12893}{21879}=\frac{5}{9}+\frac{5}{11}-\frac{7}{13}+\frac{2}{17}.$$