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Problema algebraicum de inveniendis quatuor numeris ex datis totidem productis uniuscuiusque horum numerorum in summas trium reliquorum

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XII.

Problema algebraicum de inveniendis quatuor numeris, ex datis totidem productis uniuscujusque horum numerorum in summas trium reliquorum.

Si quatuor numeri inveniendi ponantur v, x, y, z, habebuntur sequentes quatuor aequationes:

$$v(x+y+z)=a$$
$$x(v+y+z)=b$$
$$y(v+x+z)=c$$
$$z(v+x+y)=d$$

Ex his aequationibus per regulas vulgares successive tres incognitae eliminari, et quarta ad resolutionem aequationis perduci poterit. Verum cum nulla sit ratio, cur unam potius quam aliam quamvis eligamus, quae ultimo determinetur, nullam earum per aequationem finalem determinari convenit, sed ejusmodi introducenda est nova incognita, quae ad singulas aequaliter pertineat, et ex qua incognitae definiri queant. Sumamus ergo summam numerorum inveniendorum in hunc finem

$$v+x+y+z=2t,$$

atque hinc aequationes superiores abibunt in has:

$$v(2t-v)=a=2tv-v^2, \quad \text{inde} \quad v=t-\sqrt{(tt-a)}$$
$$x(2t-x)=b=2tx-x^2 \qquad x=t-\sqrt{(tt-b)}$$
$$y(2t-y)=c=2ty-y^2 \qquad y=t-\sqrt{(tt-c)}$$
$$z(2t-z)=d=2tz-z^2 \qquad z=t-\sqrt{(tt-d)}.$$

Eousque igitur solutionem jam produximus, ut ex unica quantitate t omnes quatuor numeros quaesitos expedite determinare valeamus, quare tantum superest, ut hanc quantitatem t investigemus, quod fiet ex aequatione

$$v+x+y+z=2t$$

substituendo loco v, x, y, z valores per t modo inventos:

$$4t-\sqrt{(tt-a)}-\sqrt{(tt-b)}-\sqrt{(tt-c)}-\sqrt{(tt-d)}=2t$$

unde oritur ista aequatio

$$2t=\sqrt{(tt-a)}+\sqrt{(tt-b)}+\sqrt{(tt-c)}+\sqrt{(tt-d)},$$

quae quidem methodo Newtoniana ad rationalitatem perduci posset, at labor foret maxime molestus. Alio igitur modo resolutionem hujus aequationis tentemus.

Ponamus $\sqrt{(tt-a)}=p$, erit $v=t-p$; simili modo

$$\sqrt{(tt-b)}=q,\qquad x=t-q$$
$$\sqrt{(tt-c)}=r,\qquad y=t-r$$
$$\sqrt{(tt-d)}=s,\qquad z=t-s$$

eritque $p+q+r+s=2t$, quae aequatio ob irrationales p, q, r et s in aliam debet transformari, in qua litterarum p, q, r et s potestates exponentium parium tantum occurrant, quo, per substitutionem loco litterarum p, q, r et s faciendam, nascatur aequatio rationalis, ex qua valor incognitae t definiatur.

In hunc finem formemus hanc aequationem

$$X^4-AX^3+BX^2-CX+D=0,$$

cujus quatuor radices sint quantitates datae a, b, c, d. Erit ergo per naturam aequationum:

$$A=a+b+c+d$$
$$B=ab+ac+ad+bc+bd+cd$$
$$C=abc+abd+acd+bcd$$
$$D=abcd.$$

Ponatur jam $Y=tt-X$, seu $X=-Y+tt$, habebimus facta hac substitutione istam aequationem:

$$\left.\begin{array}{r} Y^4-4ttY^3+6t^4Y^2-4t^6Y+t^8 \\ +AY^3-3AttY^2+3At^4Y-At^6 \\ +BY^2-2BttY+Bt^4 \\ +CY-Ctt \\ +D \end{array}\right\}=0.$$

Cujus aequationis quatuor radices ipsius Y erunt

$$tt-a,\quad tt-b,\quad tt-c,\quad tt-d.$$

Loco hujus aequationis ponamus brevitatis gratia hanc

$$Y^4-PY^3+QY^2-RY+S=0$$

ita, ut sit

$$P=4tt-A$$
$$Q=6t^4-3Att+B$$
$$R=4t^6-3At^4+2Btt-C$$
$$S=t^8-At^6+Bt^4-Ctt+D.$$

Sit porro $Y=Z^2$, seu $Z=\pm\sqrt{Y}$, habebimus

$$Z^8-PZ^6+QZ^4-RZ^2+S=0,$$

eruntque hujus aequationis octo radices sequentes

$$+\sqrt{(tt-a)}=+p\qquad -\sqrt{(tt-a)}=-p$$
$$+\sqrt{(tt-b)}=+q\qquad -\sqrt{(tt-b)}=-q$$
$$+\sqrt{(tt-c)}=+r\qquad -\sqrt{(tt-c)}=-r$$
$$+\sqrt{(tt-d)}=+s\qquad -\sqrt{(tt-d)}=-s.$$

Aequatio haec octo dimensionum resolvatur in binas biquadraticas, quarum alterius radices sint $+p$, $+q$, $+r$, $+s$, alterius autem $-p$, $-q$, $-r$, $-s$, quae sint

$$Z^4 - \alpha Z^3 + \beta Z^2 - \gamma Z + \delta = 0$$
$$Z^4 + \alpha Z^3 + \beta Z^2 + \gamma Z + \delta = 0,$$

in quibus erit per naturam aequationum

$$\alpha = p + q + r + s$$
$$\beta = pq + pr + ps + qr + qs + rs$$
$$\gamma = pqr + pqs + prs + qrs$$
$$\delta = pqrs.$$

Quoniam igitur productum ex his duabus aequationibus biquadraticis illi aequationi octo dimensionum aequale esse debet, erit

$$P = \alpha^2 - 2\beta$$
$$Q = \beta^2 - 2\alpha\gamma + 2\delta$$
$$R = \gamma^2 - 2\beta\delta$$
$$S = \delta^2$$

Et cum sit $\alpha = p + q + r + s$, erit $\alpha = 2t$, ideoque $\alpha^2 = 4tt$, unde fit

$$\alpha^2 - 2\beta = 4tt - 2\beta = P = 4tt - A,$$

ergo $\beta = \frac{A}{2}$. Secunda aequatio $Q = \beta^2 - 2\alpha\gamma + 2\delta$ dabit

$$6t^4 - 3Att + B = \frac{A^2}{4} - 4\gamma t + 2\delta,$$

$$\text{sive} \quad \delta = 3t^4 - \frac{3}{2}Att + 2\gamma t - \frac{A^2}{8} + \frac{B}{2}.$$

Tertia vero aequatio $R = \gamma^2 - 2\beta\delta$ praebebit

$$4t^6 - 3At^4 + 2Btt - C = \gamma^2 - A\delta,$$

$$\text{sive} \quad A\delta = -4t^6 + 3At^4 - 2Btt + C + \gamma^2$$

hinc cum superiori fit

$$\left.\begin{matrix} 4t^6 - \frac{3}{2}A^2tt + 2A\gamma t - \frac{A^3}{8} \\ + 2Btt \qquad\qquad + \frac{AB}{2} \\ \qquad\qquad\qquad - C \end{matrix}\right\} = \gamma^2.$$

Extracta radice quadrata obtinebitur

$$\gamma = At \pm \sqrt{\left(4t^6 + (2B - \tfrac{1}{2}A^2)\,tt - \tfrac{1}{8}A^3 + \tfrac{1}{2}AB - C\right)}$$

hincque

$$\delta = 3t^4 + \tfrac{1}{2}Att - \tfrac{1}{8}A^2 + \tfrac{1}{2}B \pm 2t\sqrt{\left(4t^6 + (2B - \tfrac{1}{2}A^2)\,tt - \tfrac{1}{8}A^3 + \tfrac{1}{2}AB - C\right)},$$

cujus quadratum erit

$$\begin{matrix} 25t^8 + 3At^6 - \frac{5}{2}A^2t^4 - \frac{5}{8}A^3tt + \frac{1}{64}A^4 \\ + 11Bt^4 + \frac{5}{2}ABtt - \frac{1}{8}A^2B \\ - 4Ctt + \frac{1}{4}B^2 \end{matrix} \left\{\begin{matrix} \pm 12t^5 \\ \pm 2At^3 \\ \mp \frac{1}{2}A^2t \\ \pm 2Bt \end{matrix}\right\} \sqrt{\left(4t^6 + (2B - \tfrac{1}{2}A^2)tt - \tfrac{1}{8}A^3 + \tfrac{1}{2}AB - C\right)}$$

quod aequale esse debet ipsi S, seu huic expressioni: $t^8 - At^6 + Bt^4 - Ctt + D$; unde resultat haec aequatio

$$\begin{matrix} 24t^8 + 4At^6 - \frac{5}{2}A^2t^4 - \frac{5}{8}A^3tt + \frac{1}{64}A^4 \\ + 10Bt^4 + \frac{5}{2}ABtt - \frac{1}{8}A^2B \\ - 3Ctt + \frac{1}{4}B^2 \\ - D \end{matrix} = \left\{\begin{matrix} + 12t^5 \\ + 2At^3 \\ - \frac{1}{2}A^2t \\ + 2Bt \end{matrix}\right\} \sqrt{\left(4t^6 + (2B - \tfrac{1}{2}A^2)tt - \tfrac{1}{8}A^3 + \tfrac{1}{2}AB - C\right)},$$

quae ad rationalitatem reducta dabit

$$\left.\begin{matrix} +3A^4 \\ -12A^2B \\ +24AC \\ -48D \end{matrix}\right\} t^8 \quad \left.\begin{matrix} +\frac{5}{4}A^5 \\ -8A^3B \\ +7A^2C \\ +12AB^2 \\ -8AD \\ -12BC \end{matrix}\right\} t^6 \quad \left.\begin{matrix} +\frac{3}{16}A^6 \\ -\frac{27}{16}A^4B \\ +\frac{7}{4}A^3C \\ +\frac{9}{2}A^2B^2 \\ +5A^2D \\ -7ABC \\ -3B^3 \\ +9C^2 \\ -20BD \end{matrix}\right\} t^4 \quad \left.\begin{matrix} +\frac{3}{256}A^7 \\ -\frac{9}{64}A^5B \\ +\frac{5}{32}A^4C \\ +\frac{9}{16}A^3B^2 \\ +\frac{5}{4}A^3D \\ -\frac{5}{4}A^2BC \\ -\frac{3}{4}AB^3 \\ -5ABD \\ +\frac{5}{2}B^2C \\ +6CD \end{matrix}\right\} tt \quad \left.\begin{matrix} +\frac{1}{4096}A^8 \\ -\frac{1}{256}A^6B \\ +\frac{3}{128}A^4B^2 \\ -\frac{1}{32}A^4D \\ -\frac{1}{16}A^2B^3 \\ +\frac{1}{4}A^2BD \\ +\frac{1}{16}B^4 \\ -\frac{1}{2}B^2D \\ +D^2 \end{matrix}\right\} = 0$$

Ponatur brevitatis gratia $E = \frac{1}{4}A^2 - B$ et $u = 2t$, erit

$$\left.\begin{matrix} +3A^2E \\ +6AC \\ -12D \end{matrix}\right\} u^8 \quad \left.\begin{matrix} +2A^3E \\ +4A^2C \\ +12AE^2 \\ -8AD \\ +12CE \end{matrix}\right\} u^6 \quad \left.\begin{matrix} +9A^2E^2 \\ +28ACE \\ +80DE \\ +36C^2 \\ +12E^3 \end{matrix}\right\} u^4 \quad \left.\begin{matrix} +12AE^3 \\ +80ADE \\ +96CD \\ +40CE^2 \end{matrix}\right\} uu \quad \left.\begin{matrix} +4E^4 \\ -32DE^2 \\ +64D^2 \end{matrix}\right\} = 0.$$

Haec ergo aequatio quatuor habet radices affirmativas, totidemque negativas, ipsis aequales; ita resolutio aequationis per aequationem biquadraticam perfici queat. Sunt autem A, B, C, D et E quantitates cognitae ex datis a, b, c, d determinatae; est nempe

$$A = a + b + c + d$$
$$B = ab + ac + ad + bc + bd + cd$$
$$C = abc + abd + acd + bcd$$
$$D = abcd$$

itemque $E = \frac{1}{4}A^2 - B.$

Invento autem quocunque valore pro u erunt quantitates quaesitae

$$v = \frac{u - \sqrt{(uu - 4a)}}{2}, \qquad x = \frac{u - \sqrt{(uu - 4b)}}{2}$$
$$y = \frac{u - \sqrt{(uu - 4c)}}{2}, \qquad z = \frac{u - \sqrt{(uu - 4d)}}{2}.$$

Alia Solutio.

Problema etiam hoc modo solvi potest: Ex primis aequationibus est

$$a - b = (v - x)(y + z) \qquad b - c = (x - y)(v + z)$$
$$a - c = (v - y)(x + z) \qquad b - d = (x - z)(v + y)$$
$$a - d = (v - z)(x + y) \qquad c - d = (y - z)(v + x)$$

ex aequationibus prima et ultima nanciscimur

$$v - x = \frac{a-b}{y+z}, \qquad v + x = \frac{c-d}{y-z}.$$

Sit $\frac{a+b-c-d}{2} = h$, erit $h = vx - yz$ et facto $\frac{a+b+c+d}{2} = k$ erit

$$k = vx + vy + vz + xy + xz + yz, \quad \text{ergo} \quad k - h = 2yz + (v + x)(y + z),$$

seu $k - h = 2yz + \frac{(c-d)(y+z)}{y-z} = c + d$, ergo $2yz = \frac{2dy - 2cz}{y-z}$, seu $yyz - yzz = dy - cz$

quae aequatio posito $yz = t$ abit in hanc

$$(d - t)y - (c - t)z = 0.$$

Ponatur nunc $dy - cz = u$, eritque

$$y = \frac{(c-t)u}{(c-d)t} \quad \text{et} \quad z = \frac{(d-t)u}{(c-d)t},$$

qui valores in $t = yz$ substituti praebent

$$t = \frac{(c-t)(d-t)uu}{(c-d)^2 tt},$$

unde prodit
$$u = \frac{(c-d)t\sqrt{t}}{\sqrt{(cd - (c+d)t + tt)}},$$

quo substituto in valoribus pro y et z supra inventis, habebimus

$$\text{I.}\quad y = \frac{(c-t)\sqrt{t}}{\sqrt{(cd-(c+d)\,t+tt)}} \quad \text{et} \quad \text{II.}\quad z = \frac{(d-t)\sqrt{t}}{\sqrt{(cd-(c+d)\,t+tt)}};$$

ac addendo et subtrahendo

$$y+z = \frac{(c+d-2t)\sqrt{t}}{\sqrt{(cd-(c+d)\,t+tt)}} \quad \text{et} \quad y-z = \frac{(c-d)\sqrt{t}}{\sqrt{(cd-(c+d)t+tt)}}.$$

Hinc porro deducitur

$$v+x = \frac{\sqrt{(cd-(c+d)\,t+tt)}}{\sqrt{t}} \quad \text{et} \quad v-x = \frac{(a-b)\sqrt{(cd-(c+d)\,t+tt)}}{(c+d-2t)\sqrt{t}}$$

unde, denuo addendo et subtrahendo, positisque brevitatis gratia,

$$b+c+d-a = m \quad \text{et} \quad a+c+d-b = n$$

prodeunt sequentes valores pro v et x

$$\text{III.}\quad v = \frac{(n-2t)\sqrt{(cd-(c+d)\,t+tt)}}{2\,(c+d-2t)\sqrt{t}} \quad \text{et} \quad \text{IV.}\quad x = \frac{(m-2t)\sqrt{(cd-(c+d)\,t+tt)}}{2\,(c+d-2t)\sqrt{t}}.$$

Cum autem supra invenerimus $h = vx - yz$, hinc substitutis pro v, x, y, z eorum valoribus et facta evolutione prodit, pro determinando valore ipsius t, haec aequatio quatuor dimensionum

$$4t\,(h+t)\,(c+d-2t)^2 = (m-2t)\,(n-2t)\,(c-t)\,(d-t).$$