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Fragmenta arithmetica ex Adversariis mathematicis deprompta

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X.

Fragmenta arithmetica ex Adversariis mathematicis (*) depromta.

A. Divisores numerorum.

a) *De numeris formae* mxx + nyy *eorumque divisoribus.*

1.

(J. A. Euler.)

THEOREMA. Si formula $mxx+nyy$ casu $x=a$ et $y=b$ praebeat numerum primum α, tunc omnes numeri primi in formula $\alpha \pm 4mnp$ contenti simul erunt numeri formae $mxx+nyy$. Quin etiam omnes numeri primi in hac formula $\alpha qq \pm 4mnp$ contenti simul erunt numeri formae $mxx+nyy$.

NB. Demonstratio adhuc desideratur. A. m. T. I. p. 13.

2.

(Lexell.)

Si formula $mxx+nyy$ divisibilis fuerit per numerum integrum i, infinitae aliae similes formulae per eundem divisibiles exhiberi possunt.

In genere enim haec formula $m(\alpha x \pm \beta i)^2 + n(\alpha y \pm \gamma i)^2$ per i erit divisibilis, quicunque numeri integri pro α, β, γ accipiantur; semper autem numeros α, β, γ ita accipere licebit, ut quadratorum radices ambae $\alpha x - \beta i$ et $\alpha y - \gamma i$ infra $\frac{1}{2}i$ deprimantur, quin etiam altera $\alpha x - \beta i$ ad unitatem revocari poterit, quum enim numeri x et i dentur pro fractione $\frac{i}{x}$, quaeratur in numeris minoribus fractio illi proxime aequalis $\frac{\alpha}{\beta}$, ita ut sit $\alpha x - \beta i = \pm 1$, quo casu invento sit altera radix $\alpha y - \gamma i = r$, atque hi duo valores $x=1$ et $y=r$ quasi principales spectentur, tum vero reliqui ordine in hac tabella exhibentur:

x	y
1	r
2	$2r \pm \delta i$
3	$3r \pm \delta i$
4	$4r - \delta i$
5	$5r - \delta i$

Jam pulchra hic occurrit quaestio, quinam horum valorum pro x et y producturi sint minimam formulam $mxx+nyy$, quae cum minor sit quam $\frac{1}{4}(m+n)ii$, quotus certe minor erit quam $\frac{1}{4}i(m+n)$, ideoque erit vel 1, vel 2, vel 3 etc.

Exempli gratia, sit formula proposita $3xx+2yy$ et sumatur $x=7$, $y=2$, ac prodit numerus 155, cujus divisor sumatur $=i=31$, ut jam proxime fiat $\frac{i}{x} = \frac{31}{7} = \frac{\alpha}{\beta} = \frac{9}{2}$, sive $\alpha=9$ et $\beta=2$; tum enim fit

(*) Tomus I p. 1 ad 346, ab A. 1766 ad med. Apr. 1775; tomus II p. 1 ad 246, inde usque ad Junium 1779; tomus III p. 1 ad 184, inde usque ad mortem Euleri, 1783.

$\alpha x-\beta i=63-62=+1$, et altera radix $\alpha y-\gamma i=18-31=-13=r$,

unde fiat sequens tabula:

$$\begin{array}{cccccccc} x & 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ y & 13, & 5, & 8, & 10, & 3, & 15, & 2. \end{array}$$

Minima formula hinc nascens erit secunda: $3.2^2+2.5^2=62$, quod per 31 divisum dat quotum minimum 2.

A. m. T. I. p. 94. 95.

3.

THEOREMA. Si fuerit numerus primus formae $p=8n+5$, constat semper dari formam $aa+1$ per illum numerum p divisibilem, tum vero nulla hujusmodi forma $xx\pm ayy$ unquam erit per p divisibilis. Contra autem pro numeris primis formae $p=8n+1$ datur etiam forma $aa+1$ per p divisibilis, tum vero dabuntur formulae $xx\pm ayy$ per p divisibiles.

Demonstratio eo nititur fundamento, quod priori casu numerus a semper sit non-residuum, in posteriori vero residuum; illud autem inde ostenditur, quod numerus residuorum sit $4n+2$, inter quos quilibet numerus utroque signo $+$ et $-$ occurrit, unde multitudo diversorum residuorum erit $2n+1$, scilicet impar; sin autem numerus ille a inter residua esset, haec multitudo prodiret par, quod esset absurdum.

A. m. T. I. p. 114.

4.

(N. Fuss I.)

THEOREMA. Si formula $naa+bb$ divisibilis sit per numerum p, semper dari poterit formula $n+qq$ divisibilis per eundem numerum p, ita ut $q<\frac{1}{2}p$.

DEMONSTRATIO. Quaeratur primo formula generalis $nxx+yy$ per numerum p divisibilis, quod fit sumendo $x=\alpha a-\beta p$ et $y=\alpha b-\gamma p$; tum enim ista formula erit $\alpha\alpha(naa+bb)-2p(na\alpha\beta+\alpha b\gamma)+pp(n\beta\beta+\gamma\gamma)$, quae ergo per p est divisibilis. Jam semper numeros α et β ita accipere licebit, ut fiat $\alpha a-\beta p=1$, ideoque $x=1$, quaerendo scilicet fractionem $\frac{\alpha}{\beta}$ proxime aequalem ipsi $\frac{p}{a}$. Cum igitur sit $y=\alpha b-\gamma p$, numerum semper ita accipere licebit, ut fiat y non solum minus quam p, sed etiam minus quam $\frac{1}{2}p$.

PROBLEMA. Quando formula $naa+bb$ divisibilis est per numerum p, quotum ex divisione resultantem per formulam integram exprimere.

SOLUTIO. Cum igitur detur numerus q, ut sit $n+qq$ divisibile per p, ponatur $\frac{n+qq}{p}=r$ sumaturque $b=qa+pd$ eritque $naa+bb=naa+qqaa+2pqad+ppdd$, quae ob $n=pr-qq$ abit in $p(raa+2qad+pdd)$, quae ergo per p divisa dat $raa+2qad+pdd$. Quod autem poni possit $b=qa+pd$, sive ut $\frac{b-qa}{p}$ semper sit numerus integer, inde patet, quod etiam detur formula $n+qq$ per p divisibilis, ideoque etiam $naa+aaqq$, quarum differentia $bb-aaqq$ per p divisibilis erit, unde cum p supponatur numerus primus, vel $b+aq$, vel $b-aq$ per p divisibile, utrumvis perinde est. Quia ergo $\frac{b-qa}{p}$ integer sit $=d$, ideoque poni semper poterit $b=qa+pd$.

A. m. T. II. p. 209.

5.

THEOREMA. Omnis numerus primus formae $8n+1$ semper in forma $xx+2yy$ continetur.

DEMONSTRATIO. Sufficiet ostendisse semper exhiberi posse formam A^2+2B^2 per $8n+1$ divisibilem. Demonstratum autem est, hanc formam $a^{8n}-b^{8n}$ semper divisibilem esse per $8n+1$, quicunque numeri primi a et b accipiantur, scilicet primi ad $8n+1$. Ergo $a^{4n}-b^{4n}$, vel $a^{4n}+b^{4n}$ erit divisibilis. Facile autem demonstratur non omnes numeros $a^{4n}-b^{4n}$ divisibiles esse. Dantur ergo casus, quibus forma $a^{4n}+b^{4n}$ est divisibilis. Habebitur ergo summa duorum biquadratorum divisibilis A^4+B^4. Quare cum sit $a^4+b^4=(aa-bb)^2+2aabb$, propositum est demonstratum. — Ita cum 97 in forma $8n+1$ contineatur, reperitur $97=5^2+2.6^2$.

Theorema. Omnis numerus primus formae $8n+3$ simul in forma $xx+2yy$ continetur.

Demonstratio. Iterum sufficiet ostendisse, dari formam A^2+2B^2 per $8n+3$ divisibilem. Cum igitur haec forma $a^{8n+2}-b^{8n+2}$ semper sit divisibilis, quicunque numeri pro a et b accipiantur, erit vel $a.a^{4n}-b.b^{4n}$, vel $a.a^{4n}+b.b^{4n}$ divisibilis. Jam sumatur $a=cc$ et $b=2dd$ ut $a.a^{4n}$ fiat quadratum A^2 et $b.b^{4n}$ duplum quadratum, puta $2B^2$, sicque vel forma A^2-2B^2, vel A^2+2B^2 divisionem admittet per $8n+1$. At vero demonstratum est, formam A^2-2B^2 alios divisores non admittere, nisi vel formae $8n+1$, vel formae $8n-1$, unde sequitur alteram formam A^2+2B^2 divisibilem esse. Ita cum sit $107=8.13+3$, reperitur esse $107=3^2+2.7^2$, hocque semper unico modo, quod ita demonstratur:

Sit $P=aa+2bb$ simulque $P=cc+2dd$, numerus P necessario est compositus. Cum enim sit

$$aa+2bb=cc+2dd,\quad \text{erit}\quad aa-cc=2(dd-bb),$$

unde sequitur $\frac{a+c}{b+d}=\frac{2(d-b)}{a-c}=\frac{p}{q}$. Erit ergo $a+c=\alpha p$ et $d+b=\alpha q$, $d-b=\beta p$ et $a-c=2\beta q$. Hinc $2a=\alpha p+2\beta q$ et $2b=\alpha q-\beta p$; quare cum $4P=4aa+2.4bb$ erit $4P=(\alpha\alpha+2\beta\beta)(pp+2qq)$, sicque $4P$ certe duos habet factores, quorum neuter unquam esse potest neque 1 neque 4; sequitur P ad minimum duos habere factores:

$$\begin{array}{ll} 3=1^2+2.1^2 & 59=3^2+2.5^2 \\ 11=3^2+2.1^2 & 67=7^2+2.3^2 \\ 19=1^2+2.3^2 & 83=9^2+2.1^2 \\ 43=5^2+2.3^2 & 107=3^2+2.7^2. \end{array}$$

Notandum hic, praeter casum primum, in omnibus reliquis alterum quadratum semper per 9 esse divisibile.

A. m. T. III. p. 180. 181.

6.

Theorema. Propositis numeris quibuscunque a, b, c, d, si numerus formae $abpp+cdqq$ multiplicetur per numerum formae $acrr+bdss$, tum productum semper continebitur in hac forma $bcxx+adyy$.

Demonstratio facile patet. Sumto enim $x=apr+dqs$ et $y=bps-cqr$, postrema forma $bcxx+adyy$ reperitur productum binarum praecedentium.

A. m. T. III. p. 182.

(Golovin.)

Theorema. Productum ex duabus hujusmodi formulis $aa+ab+bb$ et $cc+cd+dd$ semper ad similem formam $xx+xy+yy$ reduci potest. Est enim duplici modo

$$\text{vel}\quad x=ac+b(c+d)\quad \text{et}\quad y=ad-bc$$

$$\text{vel etiam}\quad x=ad+b(c+d)\quad \text{et}\quad y=ac-bd.$$

Ita si fuerit $a=3$ et $b=2$, tum vero $c=1$ et $d=5$, erit $aa+ab+bb=19$ et $cc+cd+dd=31$; prior igitur resolutio dat $x=15$ et $y=13$, hincque $xx+xy+yy=589$.

A. m. T. II. p. 204.

(Lexell.)

Theorema. Si formula $\alpha aa+2\beta ab+\gamma bb$ per aliam sui similem $\alpha pp+2\beta pq+\gamma qq$ multiplicetur, productum prodit hujus formae

$$\alpha xx+2\beta xy+\alpha\gamma yy\quad \text{eritque}\quad x=\alpha ap-\gamma bq\quad \text{et}\quad y=aq+bp+\frac{2\beta}{\alpha}bq.$$

Corollarium. Ita si fuerit $\alpha=1$, $2\beta=1$ et $\gamma=1$, erit $(aa+ab+bb)(pp+pq+qq)=xx+xy+yy$ existente $x=ap-bq$ et $y=aq+bp+bq$.

Nota Editorum. Casum specialem, quo $\beta=0$, vide Comment. arithm. T. II, p. 201.

A. m. T. I. p. 26.

Theorema. Si formula $aapp + b\beta qq$ ducatur in formulam $abrr + \alpha\beta ss$, productum erit:

$$ab(aapprr + \beta\beta qqss) + a\beta(\alpha\alpha ppss + bbqqrr) = \alpha b(apr \pm \beta qs)^2 + a\beta(\alpha ps \pm bqr)^2,$$

hujus ergo producti forma est $\alpha bxx + a\beta yy$ existente $x = apr \pm \beta qs$ et $y = \alpha ps \pm bqr$.

(Conf. pro casu $a = b = 1$ Comment. arithm. T. II, p. 201.)

Problema. Formulam $aaxx + b\beta yy$ in aliam ejusdem generis transformare.

Solutio. Ponatur $x = bmp + \beta nq$ et $y = anp - \alpha mq$ et prodibit

$$aabbmmpp + a\alpha\beta\beta nnqq + b\beta aannpp + b\beta\alpha\alpha mmqq = \alpha bmm(abpp + \alpha\beta qq) + a\beta nn(\alpha\beta qq + abpp) =$$
$$(\alpha bmm + a\beta nn)(abpp + \alpha\beta qq).$$

A. m. T. I. p. 130.

7.

(N. Fuss I.)

Theorema. Si numerus formae $xx + nyy$ divisibilis fuerit per numerum $pp + nqq$, quotus semper erit numerus ejusdem formae $A^2 + nB^2$.

Demonstratio. Cum numeri x et y ad $pp + nqq$ debeant esse primi, et p et q quoque sint primi inter se, quicunque fuerint numeri x et y, semper per p et q ita repraesentari possunt, ut sit $x = \alpha p + \beta q$ et $y = \gamma p + \delta q$. Hoc modo formula $xx + nyy$ abit in hanc: $pp(\alpha\alpha + n\gamma\gamma) + qq(\beta\beta + n\delta\delta) + 2pq(\alpha\beta + n\gamma\delta)$, quae per $pp + nqq$ divisa praebeat quotum Δ, ita ut sit

$$pp(\alpha\alpha + n\gamma\gamma) + qq(\beta\beta + n\delta\delta) + 2pq(\alpha\beta + n\gamma\delta) = \Delta pp + n\Delta qq.$$

Hinc igitur patet fore $\Delta = \alpha\alpha + n\gamma\gamma$, $n\Delta = \beta\beta + n\delta\delta$ et $\alpha\beta + n\gamma\delta = 0$, unde jam patet formam ipsius Δ esse $\alpha\alpha + n\gamma\gamma$. Tum etiam erit $n\Delta = \beta\beta + n\delta\delta$ et $\alpha\beta + n\gamma\delta = 0$. Ex ultima fit $\frac{\beta}{\delta} = \frac{-n\gamma}{\alpha}$. Ponatur ergo

$$\delta = \alpha f \text{ et } \beta = -n\gamma f, \quad \text{erit} \quad \beta\beta + n\delta\delta = nff(\alpha\alpha + n\gamma\gamma) = n\Delta, \quad \text{unde} \quad \Delta = ff(\alpha\alpha + n\gamma\gamma).$$

Cum igitur sit $\Delta = \alpha\alpha + n\gamma\gamma$, sequitur fore $f = \pm 1$. His valoribus fit

$$x = \alpha p \mp n\gamma q \quad \text{et} \quad y = \gamma p \pm \alpha q.$$

Hinc fit $$xx + nyy = pp(\alpha\alpha + n\gamma\gamma) + nqq(\alpha\alpha + n\gamma\gamma) = (pp + nqq)(\alpha\alpha + n\gamma\gamma),$$

sicque quotus, uti jam vidimus, $\Delta = \alpha\alpha + n\gamma\gamma$.

A. m. T. III. p. 184.

8.

Theoremata demonstranda. I. Si fuerit $4na + bb$ numerus primus, erit semper hujus formae $xx \mp ayy$

II. Si fuerit $4na - bb$ numerus primus, erit semper hujus formae $ayy -$

A. m. T. II. p. 154.

9.

Theorema. Si numerus $mnff + gg$ divisorem habeat primum $p = \frac{maa + nbb}{\Delta}$, tum etiam quotus q, ex hac divisione ortus, erit quoque ejusdem formae scilicet $q = \frac{mcc + ndd}{\Delta}$.

Explicatio. Quaerantur primo duo numeri λ et μ, ut sit $\lambda a - \mu p = \pm 1$; deinde ut formula $mnff + gg$ divisorem admittat p, alteram litteram f pro lubitu accipere licet, tum vero altera g ita esse debet comparata ut sit $g = n\lambda bf - \nu p$, quibus notatis cum sit $mnff + gg = pq$ existente $q = \frac{mcc + ndd}{\Delta}$, litterae c et d sequenti modo determinantur

$$c = n\mu bf - \nu a \quad \text{et} \quad d = m\mu af + \nu b - \lambda\Delta f.$$

A. m. T. II. p. 211.

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b) *De divisoribus numerorum formae* $fa^n + gb^n$.

10.

(*Lexell.*)

Problema. Si formula $fa^n + gb^n$ divisorem habeat d, invenire infinitas alias similes formas $fx^n + gy^n$ per eundem numerum d divisibiles.

Solutio. Capiatur $x = ma \pm \mu d$, et $y = mb \pm \nu d$, et quaesito satisfiet; si enim μ et $\nu = 0$, res est manifesta; sin autem multipla ipsius d accedant, omnes termini post primos ex evolutione nati, per se sunt divisibiles per d.

Problema. Invenire omnes divisores primos formulae $x^4 + y^4$. Cum haec formula sit factor hujus $x^8 - y^8$, demonstratum est, omnes ejus divisores contineri in forma $8n + 1$, quod etiam hoc modo ostenditur: Cum formae $a^2 + b^2$ omnes divisores sint formae $4n + 1$, ponamus formulae $aa + bb$ divisorem primum esse $4n + 1 = d$; tum ergo etiam omnes formulae $xx + yy$ per eundem numerum erunt divisibiles sumendo $x^2 = ma \pm \mu d$, $y^2 = mb \pm \nu d$. Pro nostro ergo casu hi ambo numeri debent esse quadrati. Pro priore sumto $\mu = 0$, hoc fiet si $m = app$, ut fiat $x = ap$. Superest ergo, ut et haec forma $y^2 = abpp \pm \nu d$ fiat quadratum, idque sive positivum sive negativum. Ponatur ergo $abpp \pm \nu d = \pm qq$ et res huc redit, ut $abpp \pm qq$ divisibile fiat per d, et quia statui potest $a^2 + b^2 = d$, quaeritur ergo quibus casibus formula $abpp \pm qq$ divisibilis fieri possit per d. Varios ergo casus evolvamus:

I. Sit $d = 5$, erit $a = 2$ et $b = 1$, unde formula $2pp \pm qq$ divisorem habere deberet 5, id quod fieri nequit, neque vero 5 continetur in forma $8n + 1$, atque hinc vicissim concludere possumus, neque $2pp + qq$, nec $2pp - qq$ unquam divisibile esse per 5.

II. Sit $d = 13$, erit $a = 2$ et $b = 3$, et nunc quaeritur an formula $6pp \pm qq$ divisibilis esse possit per 13, id quod negari debet, quia 13 non est formae $8n + 1$.

III. Sit $d = 17$, erit $a = 1$ et $b = 4$, nunc quaeritur an $4pp \pm qq$ divisibilis esse possit per 17, quod utique affirmandum, verum est etiam $17 = 8n + 1$.

IV. Sit $d = 29$ erit $a = 2$ et $b = 5$, et quaeritur an $10pp \pm qq$ divisibilis esse possit per 29, quod quia 29 non est $8n + 1$, negari debet.

Corollarium 1. Hic ergo distingui oportet duos casus, prouti existente b numero impari, numerus a fuerit vel impariter par, vel pariter par. Priori casu divisor d non erit formae $8n + 1$, sed formae $8n + 5$, ideoque hic casus est excludendus. Sit igitur

$$a = 4\alpha \pm 2, \text{ et } b = 4\beta \pm 1, \text{ eritque } aa + bb = 16(\alpha^2 + \beta^2) \pm 16\alpha \pm 8\beta + 5.$$

Ergo per talem divisorem nunquam divisibilis erit haec forma $(16\alpha\beta \pm 4(\alpha \pm 2\beta) \pm 2)pp \pm qq$. Per numerum ergo primum $16(\alpha^2 + \beta^2) + 16\alpha + 8\beta + 5$ talis formula $16\alpha\beta + 4(2\beta + \alpha) + 2$ nunquam est divisibilis.

Corollarium 2. Sin autem manente $b = 4\beta + 1$ (ubi β etiam negative capere licet) sit $a = 4\alpha$, erit $aa + bb = 16(\alpha\alpha + \beta\beta) + 8\beta + 1$, et nunc certi sumus, dari formulas $4\alpha(4\beta + 1)pp \pm qq$, quae divisorem habeant $16(\alpha\alpha + \beta\beta) + 8\beta + 1$.

Corollarium 3. Si igitur verum est, omnes numeros primos formae $8n + 1$ divisores esse posse formulae $x^4 + y^4$, sequitur nostram formulam $16(\alpha^2 + \beta^2) + 8\beta + 1$ omnes plane numeros $8n + 1$ in se continere siquidem fuerint primi. Aequemus ergo has formas et reperimus

$$n = 2(\alpha^2 + \beta^2) \pm \beta,$$

ubi n denotat omnes plane numeros saltem eos, qui faciunt $8n + 1$ primos:

$$0, \; 2, \; 8, \; 18, \; 32, \; 50, \; 72, \; 98$$

$$1, \; 4, \; 11, \; 22, \; 37, \; 56, \; 79, \; 106$$

$$1, \; 2, \; 7, \; 16, \; 29, \; 46, \; 67, \; 92$$

sive $2(\alpha^2+\beta^2)\pm\beta = 0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 38,\ 36,\ 45,\ 55,\ 66,\ 78,\ 91,\ 105$

$2,\ 3,\ 5,\ 8,\ 12,\ 17,\ 23,\ 30,\ 38,\ 47,\ 57,\ 68,\ 80,\ 93,\ 107.$

$8,\ 9,\ 11,\ 14,\ 18,\ 23,\ 29,\ 36,\ 44,\ 53,\ 63,\ 74,\ 86,\ 99$

$18,\ 19,\ 21,\ 24,\ 28,\ 33,\ 39,\ 46,\ 54,\ 63,\ 73,\ 84,\ 96,\ 109$

$32,\ 33,\ 35,\ 38,\ 42,\ 47,\ 53,\ 60,\ 68,\ 77,\ 87,\ 98\ \ldots$

$50,\ 51,\ 53,\ 56,\ 60,\ 65,\ 71,\ 78,\ 86,\ 95\ \ldots$

$71,\ 72,\ 74,\ 77,\ 81,\ 86,\ 92,\ 99\ \ldots$

$98,\ 99\ \ldots$

Hic omnes numeri non occurrunt, sed excluduntur 4, 7, 13, 16, 20, 22, 25, 26, 27, etc. at vero ex his omnibus $8n+1$ non fit primus.

Si igitur A denotet numerum impariter parem $4n+2$ et B numerum pariter parem sive $4n$, et C numerum imparem $2n+1$, tum haec duo habentur theoremata:

I. Per numerum primum A^2+C^2 neutra formula $ACpp\pm qq$ unquam dividi potest; neque etiam summa duorum biquadratorum, unde sequitur, si singula quadrata per A^2+C^2 dividantur, tum in residuis neque $+AC$, neque $-AC$ occurrere, sed certo esse non-residua.

II. Sin autem divisor primus fuerit B^2+C^2, tum semper datur formula $BCpp\pm qq$ per eum divisibilis, ac propterea etiam summa duorum biquadratorum, atque in residuis quadratorum, per eundem numerum primum B^2+C^2 divisorum, tam $+BC$, quam $-BC$ reperientur.

Problema. Invenire omnes divisores primos formulae fx^4+gy^4.

Cum omnes constent divisores formulae $faa+gbb$, qui sive in formula $f\alpha\alpha+g\beta\beta$, sive in hac $\alpha\alpha+fg\beta\beta$ continentur, sit quilibet eorum $=d$, per quem formula $faa+gbb$ sit divisibilis; tum sumto $X=ma\pm\alpha d$ et $Y=mb\pm\beta d$, ut formula fX^2+gY^2 etiam per d fiat divisibilis, jam reddatur primo X quadratum, quod fit si $m=app$; tum vero erit $Y=abpp\pm\beta d$, quod etiam quadratum reddi debet, quod sit $\pm qq$, et nunc oportet ut $abpp\pm qq$ divisibile fiat per d, eritque $Y=\pm qq$ et $X=aapp$, quare sumto $x=ap$ et $y=q$ fiet fx^4+gy^4 per d divisibile. Huc ergo redit quaestio: quibus casibus formula $abpp\pm qq$ dividi queat per memoratum divisorem, qui est vel $f\alpha\alpha+g\beta\beta$, vel $\alpha\alpha+fg\beta\beta$.

Exemplum 1. Sit $f=1$ et $g=2$, ideoque $d=\alpha\alpha+2\beta\beta$, qui numeri sunt vel $8n+1$, vel $8n+3$, quos valores percurramus. Sit

I. $d=3$, per quem formula $aa+2bb$ divisibilis fit; si $a=1$ et $b=1$, unde quaeritur an formula $pp\pm qq$ divisibilis fieri queat per 3, quod cum eveniat, etiam 3 erit divisor formulae x^4+2y^4.

II. Sit $d=11$, erit $a=3$ et $b=1$, hinc nostra formula $3pp\pm qq$ divisibilis per 11, at ipsius $3pp+qq$ divisores sunt formae $12n+1$, $12n+7$, formulae autem $3pp-qq$ divisores sunt vel $12n+1$, vel $12n-1$, ideoque postremus casus quaestioni satisfacit, ergo datur formula x^4+2y^4 per 11 divisibilis.

III. Sit $d=17$, $a=3$, $b=2$, ergo formula nostra per 17 divisibilis erit $6pp\pm qq$, at prior $6pp+qq$ non est divisibilis, neque etiam posterior, unde sequitur nullam formam x^4+2y^4 dividi posse per 17.

IV. Sit $d=19$, erit $a=1$ et $b=3$ et formula per 19 divisibilis erit $3pp\pm qq$, id quod fieri potest ponendo ex. causa $p=1$ et $q=4$, hinc $x=1$ et $y=4$, atque formula x^4+2y^4 erit divisibilis per 19.

V. Sit $d=41$, erit $a=3$ et $b=4$, et haec formula nostra per 41 divisibilis reddenda fit $12pp\pm qq$, sive haec $3pp\pm qq$,* at 41 in nulla harum formularum $12n\pm1$, $12n+7$ continetur. Ergo non datur x^4+2y^4 per 41 divisibilis.

VI. Sit $d=43$, erit $a=5$ et $b=3$, hinc formula per 43 divisibilis $15pp\pm qq$, sive etiam $5pp\pm 3qq$, id quod succedit, cum sit $43=3.4^2-5.1^2$, ergo datur forma x^4+2y^4 per 43 divisibilis. Si $x=ap=20$, $y=$ sive $x=4$, $y=1$.

VII. Sit $d = 59$, erit $a = 3$ et $b = 5$, hinc formula $15pp \pm qq$, sive $5pp \pm 3qq$, ubi manifesto $15.2^2 - 1$, ergo $x = 6$, $y = 1$, et formula $x^4 + 2y^4$ per 59 divisibilis.

Corollarium 1. Videtur ergo, quoties fuerit $d = 8n + 3$, tum fore divisorem formae $x^4 + 2y^4$, nec non et hujus $abpp \pm qq$, at vero tum fiunt ambo numeri a et b impares; quoties ergo $aa + 2bb$ fuerit numerus primus, semper datur formula $abpp \pm qq$ per eum divisibilis, sive inter residua quadratorum reperietur vel $+ab$, vel $-ab$.

Corollarium 2. Contra autem non omnes numeri $8n + 1$ excluduntur, quia numerus $113 = 3^4 + 2.2^4$.

VIII. Sit $d = 67$, $a = 7$, $b = 3$, formula $21pp \pm qq$, vel $7pp \pm 3qq$, $p = 5$, $q = 6$, vel $x = 35$, $y = 18$.

Exemplum 2. Sumatur $f = 1$ et $g = 3$, ut quaerantur divisores formulae $x^4 + 3y^4$ et divisor d erit $aa + 3bb$, erit ergo vel formae $12n + 1$, vel $12n + 7$.

I. Sit $d = 7$, erit $a = 2$ et $b = 1$, et formula $\frac{2pp \pm qq}{7}$, quod succedit quia $7 = 2.2^2 - 1$, unde $p = 2$, $q = 1$, $x = 4$, $y = 1$.

II. Sit $d = 13$, erit $a = 1$ et $b = 2$ et formula $\frac{2pp \pm qq}{13}$, quae est impossibilis.

III. Sit $d = 19$, erit $a = 4$ et $b = 1$ et formula $\frac{4pp \pm qq}{19}$, quae succedit: $p = 9$, $q = 1$, $x = 36$ et $y = 1$.

IV. Sit $d = 31$, erit $a = 2$ et $b = 3$ et formula $\frac{6pp \pm qq}{31}$, vel $\frac{3pp \pm 2qq}{31}$, $x = 18$, $y = 5$.

V. Sit $d = 37$, erit $a = 5$ et $b = 2$ et formula $\frac{10pp \pm qq}{37}$, vel $\frac{5pp \pm 2qq}{37}$, $p = 1$, $q = 8$, $x = 5$, $y = 8$.

VI. Sit $d = 43$, erit $a = 4$ et $b = 3$ et formula $\frac{12pp \pm qq}{43}$, vel $\frac{3pp \pm qq}{43}$, $x = 12$, $y = 8$, $x = 3$, $y = 2$.

Hic igitur maxime est mirandum, quod solus numerus 13 hic sit exclusus.

Problema superius de divisoribus $fx^4 + gy^4$ ita concinnius resolvitur:

Sit d divisor hujus formulae, qui necessario erit divisor talis formulae $fa^2 + gb^2$. Cum igitur hae duae formulae $faa + gbb$ et $fx^4 + gy^4$ habere debeant communem divisorem d, multiplicetur prior per x^4 et posterior per aa, horumque productorum differentia, quae est $gbbx^4 - gaay^4 = g(bx^2 - ay^2)(bx^2 + ay^2)$ etiam nunc erit divisibilis per d; unde si d sit numerus primus, per quem neque f, neque g divisibilis esse potest, ob

$$bbx^4 - aay^4 = (bx^2 + ay^2)(bx^2 - ay^2),$$

necesse est, ut horum factorum alter $bx^2 \pm ay^2$ sit divisibilis per d. Quare proposito numero primo d, qui dividat formulam $faa + gbb$, quoties assignari poterit formula $bxx \pm ayy$ per d divisibilis, tunc etiam formula $fx^4 + gy^4$ per eundem numerum d divisibilis erit.

Corollarium. Si datur formula $bxx \pm ayy$ per d divisibilis, etiam haec formula $zz \pm abyy$ divisibilis erit sumto $z = bx$; hoc autem eveniet, si inter residua quadratorum per d divisorum, occurrat numerus $\pm ab$.

Theorema. Quoties divisor primus d fuerit formae $4n - 1$, isque dividat formulam $faa + gbb$, tum semper dabitur formula $fx^4 + gy^4$ per d divisibilis.

Demonstratio. Cum divisor d sit formae $4n - 1$, sive $4n + 3$, si quadrata singula per eum dividantur, inter residua omnes plane numeri occurrent, sive signo plus, sive minus affecti, ergo etiam occurret numerus vel $+ab$, vel $-ab$, dabitur ergo formula $zz \pm abyy$, ideoque etiam $bxx \pm ayy$ per d divisibilis.

Corollarium. At si d fuerit formae $4n + 1$, quia in residuis quadratorum non omnes numeri occurrunt, sed semissis adeo penitus excludatur, sive positive, sive negative capiantur, utique fieri potest, ut $\pm ab$ inter eas non occurrat et tum nulla dabitur formula $fx^4 + gy^4$ per d divisibilis. Observatum autem est (nondum vero demonstratum) omnes divisores formulae $axx \pm byy$ contineri in tali forma $4abn + kk$.

*

Hic jam duo occurrunt casus considerandi, prout vel ambo numeri a et b sunt impares, vel unus par et alter impar. Priori casu, semper possibile videtur, ut divisor d in hac forma contineatur; at vero si a fuerit numerus par, puta $2c$, forma divisorum erit $8bcn+kk$, quae reducitur ad formam $8n+1$. Quoties ergo hoc casu divisor d formam habet $8n+5$, tum casus est impossibilis, unde sequitur haec conclusio:

Quoties ergo $d=8n+5$ *fuerit divisor formulae* $faa+gbb$, *insuperque alteruter numerorum* a *et* b *par, tum nulla dabitur formula* fx^4+gy^4 *per* d *divisibilis.*

Theorema. Si numerus primus formae $4n+3$ dividat formulam $faa+gbb$, sive $aa+fgbb$, tum nulla dabitur formula $faa-gbb$, sive $aa-fgbb$ per d divisibilis.

Demonstratio. Si enim formula $aa+fgbb$ divisibilis sit per d, tum inter residua quadratorum reperietur $-fg$, at fg erit non-residuum, unde etiam nulla formula $aa-fgbb$ divisibilis erit per d.

Theorema. Si numerus primus formae $4n+1$ dividat formulam $faa+gbb$, sive $aa+fgbb$, tum etiam semper dabitur formula $faa-gbb$, sive $aa-fgbb$ divisibilis per d.

Demonstratio. Quia d dividit formulam $aa+fgbb$, in residuis quadratorum occurret $-fg$; ideoque ob formam $4n+1$, ibidem quoque occurret $+fg$, ergo dabitur formula $faa-gbb$, sive $aa-fgbb$ itidem per d divisibilis.

Corollarium. Quoties ergo evenit, ut formulae $faa+gbb$ divisor $d=4n+1$, non simul dividat formulam fx^4+gy^4, tum quia idem divisor est quoque formulae $faa-gbb$, forte erit divisor formulae fx^4-gy^4. Hoc autem secus evenit casu $f=1$, $g=2$ et $d=17$. Etsi enim $17=3^2+2.2^2$ et simul $17=2.3^2-1$, tamen neutra harum formularum x^4+2y^4 et x^4-2y^4 per 17 est divisibilis. Quo hoc accuratius scrutemur, consideremus residua ex divisione biquadratorum nata pro divisoribus $4n+1$, quae semper tantum numero n.

Divisor	Residua
5	1
13	1, 3, 9
17	$+1, +4$; $-1, -4$
29	$+1, +7, +20$; $-4, -5, -6, -13$
37	$+1, +7, +9, +10, +12, +16$; $-3, -4, -11$
41	$+1, +4, +10, +16, +18$; $-1, -4, -10, -16, -18$

Hinc ergo discimus, si divisor fuerit formae $8n+5$, tum numerum residuorum esse $2n+1$, ideoque imparem unde nullum utroque signo occurrit, unde, si formula fx^4+gy^4 fuerit divisibilis, altera fx^4-gy^4 certe non erit divisibilis, quod autem vicissim non valet, quia numerus non-residuorum triplo major est, quam residuorum. Pro tali ergo divisoris forma vel neutra formularum $fx^4\pm gy^4$, vel unica saltem est divisibilis.

At si divisor fuerit formae $8n+1$, quodvis residuum utroque signo affectum occurrit, unde si una harum formularum fuerit divisibilis, etiam altera erit divisibilis, sive vel utraque, vel neutra divisibilis erit. Hinc sequitur *primo* si divisor primus $=8n+5$ dividat formulam $faa+gbb$, quo casu etiam dividet formulam $fa'a'-gb'b'$, illinc autem pro biquadratis formula $axx\pm byy$ per d fuerit divisibilis, tum certe formula $a'x^2\pm b'y^2$ non erit divisibilis. *Deinde* si fuerit $d=8n+1$ et dividat tam formulam $faa+gbb$ quam $fa'a'-gb'b'$, tum si formula $axx\pm byy$ fuerit divisibilis, certe etiam altera $a'xx\pm b'yy$ erit divisibilis, et si illa non erit, etiam haec non erit.

A. m. T. I. p. 218—223.

11.

(*N. Fuss I.*)

Problema. Invenire omnes summas binorum biquadratorum x^4+y^4, quae sint divisibiles per datum numerum primum formae $8m+1=\Delta$.

Solutio. Cum haec formula x^n+y^n alios divisores non admittat nisi formae $2in+1$, sequitur formulam x^4+y^4 alios divisores habere non posse nisi formae $8i+1$. Tales autem numeri sunt

17, 41, 73, 89, 97, 113, 137, 193, 233, 241, 257, 281, 313, 337, 353, 401, etc.

qui numeri cum omnes sint summae duorum quadratorum, sit $\Delta=aa+bb$. Deinde cum alter numerorum x et y pro lubitu accipi queat, sumatur $x=a$, et pro y inveniendo quaeratur numerus quadratus formae $i\Delta\pm ab$, qui sit pp atque sumi poterit $y=p$, vel in genere $y=\alpha\Delta\pm p$. Cum enim sit $pp=i\Delta\pm ab$, neglecto multiplo ipsius Δ, quippe quod semper adjici potest, erit $y^4=p^4=aabb$, hinc ergo erit $x^4+y^4=aa(aa+bb)=aa\Delta$, ideoque x^4+y^4 divisorem habebit Δ. Idem valor $y=p$ valet quoque pro $x=b$; tum enim erit

$$x^4+y^4=bb(aa+bb)=bb\Delta.$$

Praeter p autem dabitur alius valor q, ut sit $p:q=a:b$, ideoque $q=\frac{bp}{a}$, sive $q=\frac{bp+i\Delta}{a}$; unde valor ipsius q semper erit integer. Sumto enim

$$x=a \text{ et } y=q=\frac{bp}{a}, \text{ erit } x^4+y^4=a^4+\frac{b^4p^4}{a^4} \text{ et ob } p^4=aabb, \text{ erit } x^4+y^4=\frac{a^6+b^6}{aa}.$$

At vero a^6+b^6 semper habet factorem $aa+bb=\Delta$. Eodem modo patet, sumto $x=b$ et $y=q$, etiam x^4+y^4 factorem Δ esse habiturum. Sumto igitur sive a sive b pro x, tum pro y sumi poterit sive p sive q; unde patet, si pro x capiatur vel na vel nb, tum pro y sumi debere vel np vel nq, qui valores, cum semper multiplum ipsius Δ auferre liceat, omnes hos valores infra $\frac{1}{2}\Delta$ deprimere licebit. Praeterea vero ad singulos hos valores quaevis multipla ipsius Δ addi possunt. Hoc modo pro quovis divisore Δ tabula construi poterit duabus constans columnis, quarum prior binos valores ipsius x, altera vero binos ipsius y exhibebit, id quod exemplis illustremus.

I. Sit $\Delta=17=4^2+1^2$, erit $a=1$ et $b=4$. Nunc igitur erit $pp=17n\pm4$, unde statim sumi potest $n=0$ et $p=2$ et ob $1:4=p:q$ erit $q=8$. Hinc

x	y
1, 4	2, 8
2, 8	4, 1
3, 5	6, 7

ubi, quia x et y sunt permutabiles, secundi valores, utpote in primis jam contenti, omitti possunt, ita ut tabula duos tantum casus involvit, scilicet pro x, 1, 4 et 3, 5, et pro y, 2, 8 et 6, 7. Ita v. gr. sumto $x=5$, sumi poterit $y=6$; quia igitur $5^4=625$ et $6^4=1296$, erit $x^4+y^4=1921=17.113$.

II. Sit $\Delta=41=4^2+5^2$, eritque $a=4$ et $b=5$, ideoque $pp=41n\pm20$, ideoque $n=4$ et $p=12$. Jam $4:5=12:q$, ergo $q=15$. Hinc pro divisore 41 nostra tabula erit:

x	y
1, 9	3, 14
2, 18	6, 13
4, 5	12, 15
7, 19	16, 20
8, 10	11, 17

Ita sumto $x=1$ et $y=3$, erit $x^4+y^4=82=41.2$.

Simili modo tabulam construximus pro sequentibus

$\Delta=73$		$\Delta=89$		$\Delta=97$	
x	y	x	y	x	y
1, 27	10, 22	1, 34	12, 37	1, 22	33, 47
5, 11	23, 36	2, 21	15, 24	2, 44	3, 31
2, 19	20, 29	3, 13	22, 36	4, 9	6, 35
3, 8	30, 7	4, 42	30, 41	5, 13	29, 41
4, 35	33, 15	5, 8	7, 29	7, 40	37, 38
6, 16	13, 14	6, 26	17, 44	8, 18	12, 27
9, 24	17, 21	9, 39	19, 23	10, 26	15, 39
12, 32	26, 28	10, 16	14, 31	11, 48	25, 32
18, 25	34, 31	11, 18	38, 43	14, 17	21, 23
		20, 32	27, 28	16, 36	24, 43
		25, 40	33, 35	19, 30	20, 45
				28, 34	42, 46

Sit $\Delta=89$ sumto $x=5$ et $y=7$, erit $x^4+y^4=3026=89.34$.

Cum hae tabulae facillime ex positione litterarum a, b, et p, q construantur, istam positionem pro singulis divisoribus Δ hic apponamus:

Δ	$a,\ b$	$p,\ q$	Δ	$a,\ b$	$p,\ q$	Δ	$a,\ b$	$p,\ q$	Δ	$a,\ b$	$p,\ q$
17	1, 4	2, 8	193	7, 12	63, 85	353	8, 17	131, 146	569	13, 20	150, 187
41	4, 5	12, 15	233	8, 13	77, 96	401	1, 20	45, 98	577	1, 24	152, 186
73	3, 8	7, 30	241	4, 15	32, 120	409	3, 20	39, 198	593	8, 23	121, 171
89	5, 8	7, 29	257	1, 16	4, 64	433	12, 17	44, 82	601	5, 24	214, 295
97	4, 9	6, 35	281	5, 16	19, 117	449	7, 20	44, 195	617	16, 19	173, 244
113	7, 8	13, 31	313	12, 13	16, 65	457	4, 21	86, 223	641	4, 25	10, 258
137	4, 11	27, 40	337	9, 16	12, 91	521	11, 20	48, 182	673	12, 23	95, 126

Hic igitur praecipuum negotium in inventione quadrati $pp=n\Delta\pm ab$ consistit, quod autem sequenti modo haud difficulter praestabitur. Cum enim semper dentur numeri p et q, minores quam $\frac{1}{2}\Delta$, eorum complementa etiam erunt $<\Delta$, semper ergo dantur quatuor tales numeri minores quam Δ, quorum duo erunt pares et duo impares, atque cognito uno, reliqui tres facile inveniuntur.

Quaeramus igitur numerum imparem pro p et cum sit $pp=n\Delta\pm ab$, tum vero $pp<\Delta\Delta$, singulos numeros n tentando non ultra $n=\Delta$ progredi opus est. Deinde, quia $\Delta=aa+bb=8m+1$, numerorum a et b alter erit pariter par, alter vero impar, unde productum ab habebit vel formam $8i+4$, vel $8i$. Pro priore casu, quo $ab=8i+4$, quia Δ est $8\alpha+1$, quadrata autem imparia formam habent $8i+1$, ut talis forma oriatur, sumi debet $n=5$, vel $n=8\alpha+5$, sicque casuum examinandorum numerus octies erit minor. Pro altero casu, quo $ab=8m$ numeri pro n sumendi erunt $1, 9, 17 \ldots 8i+1$. Inter hos autem numeri etiam statim excludi possunt ii, qui desinunt in 3 vel 7, tum etiam ii, qui sunt formae $3i-1$. Praeterea vero etiam ipsam formam $pp=n\Delta\pm ab$ in alias similes transformare licet. Si enim fuerit $ab+\alpha\Delta=ffA$, erit $pp=ff(n\Delta\pm A)$; tum vero si fuerit $\alpha\Delta+A=ggB$, erit etiam $pp=ffgg(n\Delta\pm B)$ et ita porro. Inter quas plurimas formas plerumque casus sponte se produnt, quibus quadrata emergunt. Ex his autem egregia theoremata deduci possunt:

I. Si fuerit $\Delta=aa+bb=8m+1$, haec formula $n\Delta\pm ab$ semper quadratum reddi potest.

II. Si fuerit $\Delta=aa+bb=8m+5$, tum ista formula $n\Delta\pm ab$ nunquam quadratum fieri potest. Ita si $\Delta=5$, ob $a=2$ et $b=1$, haec forma $5n\pm 2$ nunquam esse potest quadratum, quod per se constat.

Deinde sumto $a=2$, $b=3$ et $\Delta=13$, haec forma $13n\pm 6$ nunquam quadratum esse potest. Item si $\Delta=29$, ob $a=2$, $b=5$, haec forma $29m\pm 10$ nunquam fit quadratum.

Alia Solutio problematis praecedentis. Sit $8m+1=aa+bb=\Delta$ esseque oportet

$$x^4+y^4=(aa+bb)(pp+qq).$$

Jam sit proxime $\frac{a}{b}=\frac{\alpha}{\beta}$, ita ut sit $a\beta-b\alpha=\pm 1$. Sit nunc $x=c$ et sumatur $p=bf\Delta+\beta cc$ et $q=af\Delta+\alpha cc$, et cum sit $xx=ap-bq$ et $yy=aq+bp$, erit $x^4+y^4=(aa+bb)(pp+qq)$, erit itaque $xx=(a\beta-b\alpha)cc=cc$ at $yy=(aa+bb)f\Delta+(a\alpha+b\beta)cc$, quod ergo esse debet quadratum. Sit nunc $cc=n\Delta+d$, fiet $yy=i\Delta\pm(a\alpha+b\beta)d$.

Exemplum 1. Sit $aa+bb=41=\Delta$, erit $a=5$ et $b=4$, hinc $\frac{5}{4}=\frac{\alpha}{\beta}$, proxime hinc $\alpha=1$ et $\beta=1$. Sumatur porro $c=1$, eritque $d=1$, ergo $yy=41i\pm 9=\square$, unde sumto $i=0$, erit $y=3$ et $x=1$, eritque $x^4+y^4=82=2.41$.

Exemplum 2. Sit $\Delta=601$, erit $a=24$ et $b=5$, tum vero $\alpha=5$ et $\beta=1$. Sumto ergo $x=1$, erit $d=1$ et $yy=601i\pm 125$, hinc sumto $i=6$, erit $y=59$.

Jam x pro lubitu sumi potest, verbi gr. $x=c$, erit $y=59c\pm 601i$, unde omnes valores redigi possunt infra 300.

A. m. T. III. p. 171—174.

12.

De divisoribus primis formae a^4+2b^4.

Primo patet hanc formam alios divisores habere non posse, nisi qui dividant formam a^2+2b^2, qui omnes continentur vel in hac forma $8n+1$, vel in hac $8n+3$. Ac primo quidem omnes numeri primi hujus formae $8n+3$ possunt esse divisores cujuspiam numeri formae a^4+2b^4. Longe secus autem res se habet de altera forma $8n+1$. Non enim omnes numeri primi in hac forma contenti divisores esse possunt formae a^4+2b^4, sed tantum sequentes: 73, 89, 113, 233, 257, 281, 337, 353, 577, etc. Hinc ergo excluduntur hi numeri ejusdem formae: 17, 41, 97, 137, 193, 241, 313, 401, 409, 433, 449, 457, 569, etc., neque tamen ulla ratio patet, qua has duas species numerorum formae $8n+1$ a se invicem distinguere liceat.

Ad divisores formae a^4+2b^4 supra allatos et in formula $8n+1$ contentos insuper accedunt 601 et 617. Est enim 601 divisor ipsius $14^4+2.5^4$ et 617 divisor ipsius $16^4+2.7^4$.

A. m. T. III. p. 181. 182.

13.

Problema. Invenire exponentem e, ut formula a^e-b^e per datum numerum Δ fiat divisibilis, si quidem numeri a et b sint primi ad Δ.

Solutio. Sint p, q, r, s numeri primi, et considerentur sequentes casus

si $\Delta=p$,	erit $e=p-1$
» $\Delta=p^2$	» $e=p(p-1)$
» $\Delta=p^3$	» $e=p^2(p-1)$
. . . . .	. . . . .
» $\Delta=p^n$	» $e=p^{n-1}(p-1)$
» $\Delta=pq$	» $e=(p-1)(q-1)$
» $\Delta=pqr$	» $e=(p-1)(q-1)(r-1)$
» $\Delta=p^\lambda q^\mu r^\nu$	» $e=p^{\lambda-1}(p-1)q^{\mu-1}(q-1)r^{\nu-1}(r-1)$.

Corollarium 1. Hinc si loco a scribatur a^α et b^β loco b, etiam haec formula $a^{\alpha e}-b^{\beta e}$ erit per Δ divisibilis.

Corollarium 2. Hinc si exponens e divisorem habeat n, ut sit $e = dn$, tum semper dari poterit forma $x^n - y^n$ per Δ divisibilis. Cum enim $a^{dn} - b^{dn}$ sit divisibilis, sumatur $x = a^d$ et $y = b^d$, vel etiam $x = a^d \pm \alpha\Delta$ et $y = b^d \pm \beta\Delta$, vel adhuc generalius $x = fa^d \pm \alpha\Delta$ et $y = fb^d \pm \beta\Delta$.

NB. In his formulis, ubi productum $(p-1)(q-1)(r-1)$ occurrit, sufficit ejus loco minimum commune dividuum numerorum $p-1$, $q-1$, $r-1$, scribere.

Quoniam formula $x^n - y^n$ praeter $x - y$ nullos habet divisores, nisi in forma $\lambda n + 1$ contentos, sic casu $n = 5$ formae $x^5 - y^5$, praeter $x - y$, divisores sunt $5\lambda + 1$ hoc est: 11, 31, 41, 61, 71, 101, 131, etc. Si ergo proponatur formula $x^5 - 1$, eaque casu $x = a$ divisorem habeat $5\lambda + 1$, eundem divisorem habebit casibus $x = a^2$, $x = a^3$, $x = a^4$, etc., sicque ex uno casu reliqui omnes deduci possunt, cum sit

$$x = a^\mu \pm M(5\lambda + 1),$$

unde sequens tabula est confecta:

Div. pr. p.	Valores x	generatim
11	$1 - 2 + 4 + 3 + 5 +$ etc.	$(-2)^\mu \pm 11M$
31	$1 + 2 + 4 + 8 + 16 +$ etc.	$(+2)^\mu \pm 31M$
41	$1 - 4 + 16 + 18 + 10 +$ etc.	$(-4)^\mu \pm 41M$
61	$1 - 3 + 9 - 27 + 20 +$ etc.	$(-3)^\mu \pm 61M$
71	$1 + 5 + 25 - 17 - 14 +$ etc.	$(+5)^\mu \pm 71M$
101	$1 - 6 + 36 - 14 - 17 +$ etc.	$(-6)^\mu \pm 101M$
131	$1 + 53 + 58 + 61 - 42 +$ etc.	$(-42)^\mu \pm 131M$
(11^2) 121	$1 + 3 + 9 + 27 + 81 +$ etc.	$(+3)^\mu \pm 121M$
(11^3) 1331	$1 - 161 + 632 - 596 + 124 +$ etc.	$(+124)^\mu \pm 1331M$

minimus autem valor ipsius x ex proprietate supra allata reperitur. Ita si divisor $= 31$, quia $a^{30} - 1$ divisorem habet 31, sumatur $x = a^6$, fiet $x^5 - 1$. Sumatur $a = 2$, erit $x = 64 \pm 2.31$, unde minimus $= 2$. Ita si $p = 101$, quia $a^{100} - 1$ divisibile per 101, sumatur $x^5 = a^{100}$, sive $x = a^{20} \pm 101M$.

Ut formula $x^6 + y^6$ divisibilis fiat per 37, numeri x et y ex sequenti schemate:

$$x \begin{cases} 1, & 10, & 11 \\ 2, & 17, & 15 \\ 3, & 7, & 4 \end{cases} \qquad y \begin{cases} 8, & 6, & 14 \\ 16, & 12, & 9 \\ 13, & 18, & 5 \end{cases}$$

scilicet ex eadem linea horizontali sumi debent.

At ut $x^6 + y^6$ divisibile fiat per 61, x et y ex sequenti schemate sumuntur

$$x \begin{cases} 1, & 13, & 14 \\ 2, & 26, & 28 \\ 4, & 9, & 5 \\ 7, & 30, & 24 \\ 8, & 18, & 10 \end{cases} \qquad y \begin{cases} 11, & 21, & 32 \\ 22, & 19, & 3 \\ 17, & 23, & 6 \\ 16, & 25, & 20 \\ 27, & 15, & 12 \end{cases}$$

singulis autem his numeris adjici intelligenda est $\pm 61M$. Hinc casus simplicissimus est $2^6 + 3^6$. Singuli autem hi terniones in unica forma comprehendi possunt, quae simplicissima est $4n$, $5n$, $9n$, vel in hac $1n$, $13n$, $14n$.

Problema. Ut formula $x^6 - 1$ divisibilis fiat per divisorem idoneum Δ, valores ipsius x definire.

Solutio. Divisor Δ necessario debet contineri in hac formula $\Delta = \frac{a^3 \pm 1}{a \pm 1}$, cujus factor quicunque dabit valorem idoneum pro Δ; tum autem tres habebuntur valores principales pro x, qui sunt 1, $\pm a$, $\pm aa$, quibus adjici potest $\pm M\Delta$. Ita si sumatur $a = 2$, erit $\Delta = \frac{8 \pm 1}{2 \pm 1}$, ideoque vel $\Delta = 3$, vel $\Delta = 7$, et tum erit $x = 1, 2, 4$. Si $a = 3$, erit $\Delta = \frac{27 \pm 1}{3 \pm 1}$, ideoque vel $\Delta = 7$, vel $\Delta = 13$, eritque $x = 1, 3, 9$. Si $a = 4$, erit $\Delta = \frac{64 \pm 1}{4 \pm 1}$

ideoque vel $\varDelta = 13$, vel $\varDelta = 21 = 3.7$, tum $x = 1, 4, 16$. Si $a = 5$, erit $\varDelta = \frac{125 \pm 1}{5 \pm 1}$; ideoque $\varDelta = 21$, vel $\varDelta = 31$, $x = 1, 5, 25$, etc.

Problema. Ut formula $x^{10} - 1$ divisibilis fiat per $\varDelta$, valores ipsius x assignare.

Solutio. Hic debet esse $\varDelta = \frac{a^5 \pm 1}{a \pm 1}$, ac tum quinque habentur valores principales pro x, scilicet $1, a, aa, a^3, a^4$, quibus adjici potest $M\varDelta$. Sic sumto $a = 2$, erit $\varDelta = \frac{32 \pm 1}{2 \pm 1}$, vel $\varDelta = 11$, vel $\varDelta = 31$, eritque $x = 1, 2, 4, 8, 16$. Si $a = 3$, erit $\varDelta = \frac{243 \pm 1}{3 \pm 1}$, ideoque vel $\varDelta = 61$, vel $\varDelta = 121$, hinc $x = 1, 3, 9, 27, 81$. Si $a = 4$, erit $\varDelta = \frac{1024 \pm 1}{4 \pm 1}$, vel $\varDelta = 205$, vel $\varDelta = 341 = 11.31$ et $x = 1, 4, 16, 64, 256$. Si $a = 5$, erit $\varDelta = \frac{3125 \pm 1}{5 \pm 1}$, ideoque vel $\varDelta = 521$, vel $\varDelta = 781 = 11.71$, $x = 1, 5, 25, 125, 625$, etc.

NB. Omnes divisores primi hic sunt formae $10n + 1$. Dato ergo tali divisore, veluti 131, quaeri debet numerus a, ut $a^5 \pm 1$ divisionem admittat per 131, quod hoc casu non evenit, nisi sumatur vel $a = 42$, vel $a = 53$, vel $a = 58$, vel $a = 70$; tum enim habebitur $x = 1, 42, 70, 58, 53$.

Quando autem divisor $\varDelta$ datur, in forma $10n + 1$ contentus, valor litterae a hoc modo eruetur. Cum $\varDelta$ debeat esse divisor formae $a^5 - 1$, capiatur $a = b^n$, erit $a^5 = b^{5n}$, semper autem est $b^{10n} - 1$ divisibile per $10n + 1$, ideoque vel $b^{5n} + 1$, vel $b^{5n} - 1$, quocirca sumi debet $a = b^n$. Ita pro casu $\varDelta = 131$ est $n = 13$, ideoque $a = b^{13}$; sumto ergo $b = 2$, erit $b^{13} = 8192$, quod divisum per 131 relinquit 61, et valores ipsius x erunt $1, 61, 61^2, 61^3, 61^4$. Est vero $61^2 = 3721$, quod dat 53, et 61.53 dat 42, et 61.42 dat 58. Sicque $x = 1, 61, 53, 42, 58$. Eodem modo si proponatur $\varDelta = 151$, erit $n = 15$ et $a = 19$; $a^2 = 59$, $a^3 = 64$, $a^4 = 8$.

Ut formula $x^8 + y^8$ divisibilis fiat per 97, numeri x et y ex sequenti tabula desumantur

x				y			
1,	33,	22,	47	8,	27,	18,	12
2,	31,	44,	3	16,	43,	36,	24
4,	35,	9,	6	32,	11,	25,	48
5,	29,	13,	41	40,	38,	7,	37
10,	39,	26,	15	17,	21,	14,	23
46,	13,	42,	28	29,	19,	45,	30

ita casus simplicissimus est $5^8 + 7^8$.

Ut formula $x^{10} - 1$ dividi queat per 11^3, valores ipsius x erunt

$$1, \quad 124, \quad 596, \quad 699, \quad 161.$$

Cum enim $3^5 - 1 = 2.11^2$, ponatur $z = 3 + 11^2 y$, eritque $z^5 - 1 = 2.11^2 + 5.3^4.11^2 y +$ etc. quod divisum per 11^2 dat $\frac{z^5 - 1}{11^2} = 2 + 5.3^4 y +$ etc. Tantum ergo y ita sumatur, ut $2 + 5.81y$ divisibile sit per 11, sive $2 - 2y$, vel $1 - y$. Sumatur $y = -10$, erit $z = 1207 = 124$.

A. m. T. II. p. 162—164.

c) *De numeris formae* $x^p \pm 1$.

14.

(*Lexell.*)

Problema. Invenire numerum formae $2^n + 1$, qui habeat datum divisorem.

Solutio. Divisor repraesentetur per simplices potestates binarii, et quotus quaeratur sequenti modo per partes; ubi tenendum est, quoniam tandem omnes minores potestates binarii in producto excludi debent, si ex aliquot partibus quoti prodierit productum $1 + 2^\alpha +$ etc. tum sequentem quoti partem esse debere 2^α, deinde tantum notetur esse $2^\alpha + 2^\alpha = 2^{\alpha+1}$.

Exemplum I. Sit divisor $=1+2^7+2^9$, ac prima pars quoti erit 1, et operatio sequenti modo instituetur

Partes quoti	Productum
1	$1+{}_{*}2^{7}+{}_{*}2^{9}$
2^{7}	${}_{*}2^{7}+{}_{*}2^{14}+{}_{*}2^{16}+{}_{*}2^{8}$
2^{8}	${}_{*}2^{8}+{}_{*}2^{15}+{}_{*}2^{17}+{}_{*}2^{10}$
2^{10}	${}_{*}2^{10}+{}_{*}2^{17}+{}_{*}2^{19}+{}_{*}2^{11}+{}_{*}2^{15}$
2^{11}	${}_{*}2^{11}+{}_{*}2^{15}+{}_{*}2^{20}+{}_{*}2^{12}+{}_{*}2^{21}$
2^{12}	${}_{*}2^{12}+{}_{*}2^{19}+{}_{*}2^{21}+{}_{*}2^{13}+{}_{*}2^{22}$
2^{13}	${}_{*}2^{13}+{}_{*}2^{20}+{}_{*}2^{22}+{}_{*}2^{17}+{}_{*}2^{23}$
2^{17}	${}_{*}2^{17}+{}_{*}2^{24}+{}_{*}2^{26}+{}_{*}2^{18}$
2^{18}	${}_{*}2^{18}+{}_{*}2^{25}+{}_{*}2^{27}+{}_{*}2^{21}$
2^{21}	${}_{*}2^{21}+{}_{*}2^{28}+{}_{*}2^{30}+{}_{*}2^{22}$
2^{22}	${}_{*}2^{22}+{}_{*}2^{29}+{}_{*}2^{31}+2^{32}$

ergo forma est $2^{32}+1$, cujus divisor est $1+2^7+2^9=641$ et

quotus $=1+2^7+2^8+2^{10}+2^{11}+2^{12}+2^{13}+2^{17}+2^{18}+2^{21}+2^{22}$.

Exemplum II. Sit divisor $73=1+2^3+2^6$.

Partes quoti	Productum
1	$1+{}_{*}2^{3}+{}_{*}2^{6}$
2^{3}	${}_{*}2^{3}+{}_{*}2^{6}+{}_{*}2^{9}+{}_{*}2^{4}+{}_{*}2^{7}$
2^{4}	${}_{*}2^{4}+{}_{*}2^{7}+{}_{*}2^{10}+{}_{*}2^{5}+{}_{*}2^{8}$
2^{5}	${}_{*}2^{5}+{}_{*}2^{8}+{}_{*}2^{11}+{}_{*}2^{6}+{}_{*}2^{9}+{}_{*}2^{10}+{}_{*}2^{11}+{}_{*}2^{12}$
2^{6}	${}_{*}2^{6}+{}_{*}2^{9}+{}_{*}2^{12}+{}_{*}2^{7}+{}_{*}2^{13}$
2^{7}	${}_{*}2^{7}+{}_{*}2^{10}+{}_{*}2^{13}+{}_{*}2^{8}+{}_{*}2^{14}$
2^{8}	${}_{*}2^{8}+{}_{*}2^{11}+{}_{*}2^{14}+{}_{*}2^{9}+{}_{*}2^{10}+{}_{*}2^{11}+{}_{*}2^{12}+{}_{*}2^{15}$
2^{12}	${}_{*}2^{12}+{}_{*}2^{15}+{}_{*}2^{18}+{}_{*}2^{13}+{}_{*}2^{16}$
2^{13}	${}_{*}2^{13}+{}_{*}2^{16}+{}_{*}2^{19}+{}_{*}2^{14}+{}_{*}2^{17}$
2^{14}	${}_{*}2^{14}+{}_{*}2^{17}+{}_{*}2^{20}+{}_{*}2^{15}+{}_{*}2^{18}+{}_{*}2^{19}+{}_{*}2^{20}+{}_{*}2^{21}$
2^{15}	${}_{*}2^{15}+{}_{*}2^{18}+{}_{*}2^{21}+{}_{*}2^{16}+{}_{*}2^{22}$
2^{16}	${}_{*}2^{16}+{}_{*}2^{19}+{}_{*}2^{22}+{}_{*}2^{17}+{}_{*}2^{23}$
2^{17}	${}_{*}2^{17}+{}_{*}2^{20}+{}_{*}2^{23}+{}_{*}2^{18}+{}_{*}2^{19}+{}_{*}2^{20}+{}_{*}2^{21}+{}_{*}2^{24}$
2^{21}	${}_{*}2^{21}+{}_{*}2^{24}+{}_{*}2^{27}+{}_{*}2^{22}+{}_{*}2^{25}$
2^{22}	${}_{*}2^{22}+{}_{*}2^{25}+{}_{*}2^{28}+{}_{*}2^{23}+{}_{*}2^{26}$
2^{23}	${}_{*}2^{23}+{}_{*}2^{26}+{}_{*}2^{29}+{}_{*}2^{24}+{}_{*}2^{27}+{}_{*}2^{28}+{}_{*}2^{29}+{}_{*}2^{30}$
2^{24}	${}_{*}2^{24}+{}_{*}2^{27}+{}_{*}2^{30}+{}_{*}2^{25}+{}_{*}2^{31}$
2^{25}	${}_{*}2^{25}+{}_{*}2^{28}+{}_{*}2^{31}+{}_{*}2^{26}+{}_{*}2^{32}$
2^{26}	${}_{*}2^{26}+{}_{*}2^{29}+{}_{*}2^{32}+{}_{*}2^{27}+{}_{*}2^{28}+{}_{*}2^{29}+{}_{*}2^{30}+{}_{*}2^{33}$
2^{30}	${}_{*}2^{30}+{}_{*}2^{33}+2^{36}+2^{31}+2^{34}$

Plane non datur talis forma per 73 divisibilis.

Exemplum III. Sit divisor $41=1+2^3+2^5$, erunt

partes quoti	productum
1	$1+{}_{*}2^{3}+{}_{*}2^{5}$
2^{3}	${}_{*}2^{3}+{}_{*}2^{6}+{}_{*}2^{8}+{}_{*}2^{4}$
2^{4}	${}_{*}2^{4}+{}_{*}2^{7}+{}_{*}2^{9}+{}_{*}2^{5}+{}_{*}2^{6}+{}_{*}2^{7}+{}_{*}2^{8}+{}_{*}2^{9}+2^{10}$

ergo forma $1+2^{10}$ divisibilis est per 41 et quotus erit $1+2^3+2^4=25$.

Exemplum IV. Sit divisor $11 = 1 + 2^1 + 2^3$, erunt

partes quoti	productum
1	$1 + {}_*2^1 + {}_*2^3$
2^1	${}_*2^1 + {}_*2^2 + {}_*2^4 + {}_*2^2 + {}_*2^3 + {}_*2^4 + 2^5$

unde $1 + 2^5 = 11\,(1 + 2)$.

Exemplum V. Sit divisor $13 = 1 + 2^2 + 2^3$, erunt

partes quoti	productum
1	$1 + {}_*2^2 + {}_*2^3$
2^2	${}_*2^2 + {}_*2^4 + {}_*2^5 + {}_*2^3 + {}_*2^4 + {}_*2^5 + 2^6$

unde $2^6 + 1 = 13\,(1 + 2^2)$.

Exemplum VI. Sit divisor $7 = 1 + 2 + 2^2$, erunt

partes quoti	productum
1	$1 + {}_*2 + {}_*2^2$
2^1	${}_*2 + {}_*2^2 + {}_*2^3 + {}_*2^2 + {}_*2^3 + {}_*2^4$
2^2	${}_*2^2 + {}_*2^3 + {}_*2^4 + {}_*2^3 + {}_*2^4 + {}_*2^5$
2^4	${}_*2^4 + {}_*2^5 + {}_*2^6 + {}_*2^5 + {}_*2^6 + {}_*2^7$
2^5	${}_*2^5 + {}_*2^6 + {}_*2^7 + {}_*2^6 + {}_*2^7 + {}_*2^8$
2^7	${}_*2^7 + {}_*2^8 + {}_*2^9 + {}_*2^8 + {}_*2^9 + {}_*2^{10}$
2^8	${}_*2^8 + {}_*2^9 + {}_*2^{10} + {}_*2^9 + 2^{10} + 2^{11}$
2^{10}	etc.

Pro hoc ergo divisore non datur forma binomialis $1 + 2^n$; dantur autem trinomiales:

$$1 + 2 + 2^2,\ \ 1 + 2^2 + 2^4,\ \ 1 + 2^4 + 2^5,\ \ 1 + 2^5 + 2^7,\ \ 1 + 2^7 + 2^8,\ \ 1 + 2^8 + 2^{10},\ \ 1 + 2^{10} + 2^{11}.$$

(*Krafft.*)

Problema. Invenire numerum formae $2^n - 1$, qui habeat datum divisorem.

Solutio. Primo notetur esse

$$2^n - 1 = 1 + 2 + 2^2 + 2^3 + 2^4 + \ldots . 2^{n-1},$$

sicque omnes potestates ab unitate usque ad maximam occurrere debent. Si igitur, ut ante, quotus per partes quaeratur; in producto ex aliquot partibus orto notetur minima potestas, quae adhuc deficit, eaque ipsa erit nova pars quoti.

Exemplum I. Sit divisor $23 = 1 + 2 + 2^2 + 2^4$, erunt

partes quoti	productum
1	$1 + 2 + 2^2 + {}_*2^4$
2^3	$2^3 + {}_*2^4 + {}_*2^5 + {}_*2^7 + {}_*2^5 + {}_*2^6$
2^4	$2^4 + 2^5 + {}_*2^6 + {}_*2^8 + {}_*2^7 + {}_*2^8 + 2^9$
2^6	$2^6 + 2^7 + 2^8 + 2^{10}$

unde erit $n = 11$ sicque $2^{11} - 1$ divisibile est per 23 quoto existente

$$1 + 2^3 + 2^4 + 2^6 = 89.$$

Nota. Forma numerorum perfectorum est $2^{n-1}(2^n - 1)$, quoties fuerit factor posterior $2^n - 1$ numerus primus.

Exemplum II. Sit divisor $47 = 1 + 2 + 2^2 + 2^3 + 2^5$, erunt

partes quoti	productum
1	$1 + 2 + 2^2 + {}_*2^3 + {}_*2^5$
2^4	$2^4 + {}_*2^5 + {}_*2^6 + {}_*2^7 + {}_*2^9 + {}_*2^6 + {}_*2^7 + {}_*2^8$
2^5	$2^5 + 2^6 + 2^7 + {}_*2^8 + {}_*2^{10} + {}_*2^9 + {}_*2^{10} + {}_*2^{11}$
2^8	$2^8 + 2^9 + {}_*2^{10} + {}_*2^{11} + {}_*2^{13} + {}_*2^{12}$
2^{11}	$2^{11} + {}_*2^{12} + {}_*2^{13} + {}_*2^{14} + {}_*2^{16} + {}_*2^{13} + {}_*2^{14} + {}_*2^{15}$
2^{12}	$2^{12} + {}_*2^{13} + {}_*2^{14} + {}_*2^{15} + {}_*2^{17} + {}_*2^{14} + {}_*2^{15} + {}_*2^{16} + {}_*2^{17} + {}_*2^{18}$
2^{13}	$2^{13} + 2^{14} + {}_*2^{15} + {}_*2^{16} + {}_*2^{18} + {}_*2^{16} + {}_*2^{17} + {}_*2^{19}$
2^{15}	$2^{15} + 2^{16} + {}_*2^{17} + {}_*2^{18} + {}_*2^{20} + {}_*2^{18} + {}_*2^{19} + {}_*2^{20} + 2^{21}$
2^{17}	$2^{17} + 2^{18} + 2^{19} + 2^{20} + 2^{22}$

ergo $n = 23$ et $2^{23} - 1$ divisibile est per 47, quoto existente

$$2^{17} + 2^{15} + 2^{13} + 2^{12} + 2^{11} + 2^8 + 2^5 + 2^4 + 1 = 178481.$$

(Lexell.)

Verum haec omnia multo facilius atque adeo multo generalius per sequentem methodum expediri possunt.

Problema. Invenire exponentem x, ut formula $2^x - a$ datum habeat divisorem $= p$.

Solutio. Quaeritur ergo potestas binarii 2^x, quae per numerum p divisa relinquat residuum $= a$; notetur autem pro residuo a in genere scribi posse $a + \lambda p$, loco a igitur sumatur $a \pm p$, qui numerus cum sit par ac fortasse per majorem binarii potestatem divisibilis, ponatur $a \pm p = 2^\alpha b$, atque potestas $2^{x-\alpha}$ dabit residuum b, cujus loco sumatur iterum $b \pm p$, quod sit $= 2^\beta c$, sicque potestas $2^{x-\alpha-\beta}$ residuum dabit c, sive $c \pm p$ quod sit $= 2^\gamma d$, sicque potestas $2^{x-\alpha-\beta-\gamma}$ residuum dabit d, atque hoc modo eo usque procedatur, donec ad residuum perveniatur $= 1$, quod cum sit residuum potestatis 2^0, evidens est ultimum exponentem

$$x - \alpha - \beta - \gamma - \delta - \text{etc. esse debere} = 0,$$

consequenter habebitur
$$x = \alpha + \beta + \gamma + \delta + \text{etc.}$$

Tota haec operatio sequenti modo commode disponetur: Pro divisore $= p$:

potestates	residua	sive
2^x	a	$a \pm p = 2^\alpha b$
$2^{x-\alpha}$	b	$b \pm p = 2^\beta c$
$2^{x-\alpha-\beta}$	c	$c \pm p = 2^\gamma d$
$2^{x-\alpha-\beta-\gamma}$	d	$d \pm p = 2^\delta e$
$\vdots$	$\vdots$	$\vdots$
$2^{x-\alpha-\beta-\gamma-\delta-\cdots}$	$+1$	$x = \alpha + \beta + \gamma + \delta + \text{etc.}$

Exemplum I. Quaeratur formula $2^x + 1$, quae divisorem habeat 641. Pro hoc divisore

potestates	residua	sive
2^x	$a = -1$	$640 = 128.5 = 2^7.5$
2^{x-7}	5	$-636 = -2^2.159$
2^{x-9}	-159	$-800 = -2^5.25$
2^{x-14}	-25	$+616 = +2^3.77$
2^{x-17}	$+77$	$-564 = -2^2.141$
2^{x-19}	-141	$+500 = +2^2.125$
2^{x-21}	$+125$	$-516 = -2^2.129$
2^{x-23}	-129	$512 = +2^9.1$
2^{x-32}	1	ergo $x = 32$

Exemplum II. Quaerere formulam 2^x+1, quae divisorem habeat 29. Pro hoc divisore

potestates	residua	sive
2^x	-1	$-1+29=28=2^2.7$
2^{x-2}	7	$+36=2^2.9$
2^{x-4}	9	$-20=-2^2.5$
2^{x-6}	-5	$24=2^3.3$
2^{x-9}	3	$32=2^5.1$
2^{x-14}	1	$x=14.$

Exemplum III. Quaerere formulam 2^x+1, quae divisorem habeat 73. Pro divisore 73

potestates	residua	sive
2^x	-1	$2^3.9$
2^{x-3}	$+9$	$-64=-2^6.1$
2^{x-9}	-1	$72=2^3.9$
2^{x-12}	9	$-64=-2^6.1$
2^{x-18}	-1	

unde apparet hanc quaestionem esse impossibilem.

Exemplum IV. Quaerere formulam 2^x-1, quae habeat divisorem 23:

potestates	residua	sive
2^x	1	$24=2^3.3$
2^{x-3}	3	$-20=-2^2.5$
2^{x-5}	-5	$-28=-2^2.7$
2^{x-7}	-7	$16=2^4.1$
2^{x-11}	1	ergo $x=11.$

Exemplum V. Quaerere formam 2^x-3, quae habeat divisorem 19:

potestates	residua	sive
2^x	3	$-16=-2^4.1$
2^{x-4}	-1	$-20=-2^2.5$
2^{x-6}	-5	$-24=-2^3.3$
2^{x-9}	-3	$16=2^4.1$
2^{x-13}	1	$x=13.$

Problema generalius. Invenire exponentem x, ut formula AR^x-a datum habeat divisorem $=p$.

Solutio. Numerus ergo AR^x residuum dare debet $=a$, cui aequivalet $a\pm\lambda p=R^\alpha b$, unde numerus $AR^{x-\alpha}$ residuum dare debet b, sive $b\pm\lambda p=R^\beta c$, sicque numerus $AR^{x-\alpha-\beta}$ producet numerum c, sicque hoc modo procedendo, donec perveniatur ad numerum A, quod quia nascitur ex AR^0, manifestum est esse debere

$$x=\alpha+\beta+\gamma+\delta+\text{etc.}$$

Exemplum. Quaerere formulam 5^x-1, quae divisorem habeat 17.

potestates	residua	sive	potestates	residua	sive
5^x	1	$35=5.7$	5^{x-5}	-6	$-40=-5.8$
5^{x-1}	7	$-10=-5.2$	5^{x-6}	-8	$-25=-5^2$
5^{x-2}	-2	$15=5.3$	5^{x-8}	-1	
5^{x-3}	3	$20=5.4$	5^{x-16}	1	$x=16$
5^{x-4}	4	$-30=-5.6$			

Problema. Invenire exponentem x, ut formula $2^{2x}+2^x+1$ datum habeat divisorem p.

Solutio. Cum ergo formula $2^{2x}+2^x$ residuum habere debeat -1, sive $-1+\lambda p$, ponamus potestatem 2^x habere residuum r, atque ejus quadratum 2^{2x} residuum habebit rr, ideoque illius formae residuum erit $rr+r$. Quaeratur ergo r, ut fiat

$$rr+r=-1+\lambda p,\quad \text{sive}\quad 4rr+4r+1=(2r+1)^2=4\lambda p-3,\quad \text{unde}\quad 2r+1=\sqrt{(4\lambda p-3)};$$

λ igitur ita sumi debet, ut $4\lambda p-3$ sit quadratum. Invento autem r quaeratur potestas 2^x residuum habens r, quod est problema superius.

Sit verbi gratia divisor $p=19$ et quadratum esse debet $76\lambda-3$, quod fit si $\lambda=3$, ergo $2r+1=\pm 15$, consequenter vel $r=+7$, vel $r=-8$.

I. Pro $r=+7$	2^x resid.	$+7$	$-12=-2^2.3$
	2^{x-2}	-3	$16=2^4$
	2^{x-6}	1	hinc $x=6$

ideoque $2^{12}+2^6+1$ divisibile per 19.

II. Pro $r=-8$	2^x resid.	-8	$-2^3.1$
	2^{x-3}	-1	$-20=-2^2.5$
	2^{x-5}	-5	$-24=-2^3.3$
	2^{x-8}	-3	$16=2^4$
	2^{x-12}	1	$x=12$

ideoque $2^{24}+2^{12}+1$ divisibile per 19.

A. m. T. I. p. 143—149.

15.

(*J. A. Euler.*)

Cum sit $a^{2p}-1$ divisibile per $2p+1$, si $2p+1$ fuerit numerus primus, tum vel a^p-1, vel a^p+1 per eum dividi poterit. Duplicis ergo generis sunt potestates a^p, prouti vel formula a^p-1, vel a^p+1 fuerit divisibilis per $2p+1$.

Theorema. Cujus generis fuerit potestas a^p ejusdem generis quoque erunt omnes istae

$$a^{2a+p},\quad a^{4a+p},\quad a^{6a+p}\quad \text{et in genere}\quad a^{2na+p},$$

ubi a debet esse primus ad $2p+1$ et n quoque potest esse numerus negativus.

Praeterea vero etiam ejusdem generis erunt hae potestates

$$a^{2a-p-1},\quad a^{4a-p-1},\quad a^{6a-p-1}\quad \text{et in genere}\quad a^{2na-p-1}$$

hoc autem posterius tantum valet, si a fuerit numerus positivus; si enim sit negativus, hae posteriores potestates ad alterum genus pertinent. Ratio hujus exceptionis manifesta est: si enim p fuerit numerus par, perinde est sive capiatur $+a$, sive $-a$; sin autem p sit impar, loco a sumendo $-a$, ipsa potestas fit negativa. Sicque si formula $(+a)^p\pm 1$ fuerit per $2p+1$ divisibilis, tum $(-a)^p\mp 1$ divisibilis erit.

Exemplum. Quia 2^1+1 per $2.1+1=3$ est divisibile, ubi $a=2$ et $p=1$, ad idem genus pertinebunt hae potestates

$$2^1,\ 2^5,\ 2^9,\ 2^{13},\ 2^{17},\ 2^{21}\ \ldots\ 2^{4n+1}$$

deinde etiam istae $\quad 2^2,\ 2^6,\ 2^{10},\ 2^{14},\ 2^{18},\ 2^{22}\ \ldots\ 2^{4n+2}$.

Examinemus casum 2^{21}, an $2^{21}+1$ divisibile sit per 43, sive an 2^{21} per 43 divisum relinquat -1, quod ope methodi supra expositae ita fiet:

divisor: 43	residua
2^{21}	$-1-43=-44=-2^2.11$
2^{21-2}	$-11+43=32=2^5.1$
2^{21-7}	1. Capiatur cubus
$2^{3.21-21}$	1. At
$2^{2.21}$	1. Dividatur
2^{21-21}	1, vel
2^0	1

quod cum sit verum, etiam prima formula est vera.

Examinetur jam potestas 2^{18}, num per 37 divisa relinquat -1. Calculus ita fiet

divisor: 37	residua
2^{18}	$-1+37=2^2.9$
2^{18-2}	$+9-37=-2^2.7$
2^{18-4}	$-7-37=-2^2.11$
2^{18-6}	-11
$2^{2.18}$	$-1331=+1$, quod etiam est verum.

Exemplum. Sit $a=3$ et $p=2$, erit 3^2+1 divisibile per 5; hujus ergo generis erunt omnes hae potestates:

$$3^2,\ 3^8,\ 3^{14},\ 3^{20},\ 3^{26}\ldots.3^{6n+2},\ \text{item hae}$$

$$3^3,\ 3^9,\ 3^{15},\ 3^{21},\ 3^{27}\ldots.3^{6n+3}.$$

Examinetur 3^{26} an per 53 divisa relinquat -1:

divisor: 53	residua
3^{26}	$+1+53=3^3.2$
3^{23}	$+2$
3^{46-26} vel 3^{20}	$+4$
3^{43-26} vel 3^{17}	$+8$
3^{14}	$+16$
3^5	$+128$ vel 22
3^2	44 vel -9,

quod cum sit falsum, residuum non erit $+1$ vel -1.

Examinetur 3^{33} an per 67 divisa relinquat $+1$.

divisor: 67	residua
3^{33}	$1+134=3^3.5$
3^{30}	5
3^{27}	25
3^{24}	125 vel -9
3^{18}	-225 vel -24
3^9	$+216$ vel $+15$
3^{33} vel 3^0	-135 vel -1

quod quia est falsum, nostra regula confirmatur.

Exemplum. Sit $a=6$ et fieri nequit $p=1$, quia neque 6^1+1, neque 6^1-1 per 3 est divisibile, ideoque excluduntur exponentes 1, 13, 25, 37, 49, etc., tum etiam

10, 22, 34, 46, 58, etc.

At 6^2-1 est per 5 divisibile, sive 6^2 per 5 divisum dat residuum $+1$, ergo $p=2$, et idem dabunt hae potestates

$$6^{14},\ 6^{26},\ 6^{38},\ 6^{50},\ 6^{62},\ \text{etc., tum etiam}$$
$$6^{9},\ 6^{21},\ 6^{33},\ 6^{45},\ 6^{57},\ \text{etc.}$$

Examinemus potestatem 6^{50} num per 101 divisa relinquat $+1$:

divisor: 101	residua
6^{50}	$+1+101=102=6.17$
6^{49}	$17-101=-6.14$
6^{48}	$-14-202=-216=-6^3$
6^{45}	-1
6^{5}	$-1-101=-6.17$
6^{4}	-17
6^{3}	$+14$
6^{1}	$-196+202=+6$

quod quia est verum, patet regula.

Examinetur potestas 6^{33} an per 67 divisa relinquat $+1$:

divisor: 67	residua
6^{33}	$+1-67=-6.11$
6^{32}	$-11-67=-6.13$
6^{31}	$-13+67=+6.9$
6^{30}	$+9$
6^{27}	$+81$ vel $+14$
6^{21}	$+196$ vel -5
6^{9}	$+25$
6^{3}	$+350$ vel 15
6^{0}	$+135-134$ vel $+1$

Utra formula $a^p\pm1$ per numerum primum $2p+1$ sit divisibilis sequens tabella ostendit:

$2p+1$		$2p+1$		$2p+1$		$2p+1$	
pro $a=2$		*pro* $a=3$		*pro* $a=5$		*pro* $a=6$	
$8n\pm1$	2^p-1	$12n\pm1$	3^p-1	$20n\pm1$	5^p-1	$24n\pm1$	6^p-1
$8n\pm3$	2^p+1	$12n\pm5$	3^p+1	$20n\pm3$	5^p+1	$24n\pm5$	6^p-1
				$20n\pm7$	5^p+1	$24n\pm7$	6^p+1
				$20n\pm9$	5^p-1	$24n\pm11$	6^p+1
pro $a=7$		*pro* $a=8$		*pro* $a=10$		*pro* $a=11$	
$28n\pm1$	7^p-1	$32n\pm1$	8^p-1	$40n\pm1$	10^p-1	$44n\pm1$	11^p-1
$28n\pm3$	7^p-1	$32n\pm3$	8^p+1	$40n\pm3$	10^p-1	$44n\pm3$	11^p+1
$28n\pm5$	7^p+1	$32n\pm5$	8^p+1	$40n\pm7$	10^p+1	$44n\pm5$	11^p-1
$28n\pm9$	7^p-1	$32n\pm7$	8^p-1	$40n\pm9$	10^p-1	$44n\pm7$	11^p-1
$28n\pm11$	7^p+1	$32n\pm9$	8^p-1	$40n\pm11$	10^p+1	$44n\pm9$	11^p-1
$28n\pm13$	7^p+1	$32n\pm11$	8^p+1	$40n\pm13$	10^p-1	$44n\pm13$	11^p+1
		$32n\pm13$	8^p+1	$40n\pm17$	10^p+1	$44n\pm15$	11^p+1
		$32n\pm15$	8^p-1	$40n\pm19$	10^p+1	$44n\pm17$	11^p+1
						$44n\pm19$	11^p-1
						$44n\pm21$	11^p+1

pro a = 12		pro a = 13		pro a = 14		pro a = 15	
$48n \pm 1$	$12^p - 1$	$52n \pm 1$	$13^p - 1$	$56n \pm 1$	$14^p - 1$	$60n \pm 1$	$15^p - 1$
$48n \pm 5$	$12^p + 1$	$52n \pm 3$	$13^p - 1$	$56n \pm 3$	$14^p + 1$	$60n \pm 7$	$15^p - 1$
$48n \pm 7$	$12^p + 1$	$52n \pm 5$	$13^p + 1$	$56n \pm 5$	$14^p - 1$	$60n \pm 11$	$15^p - 1$
$48n \pm 11$	$12^p - 1$	$52n \pm 7$	$13^p + 1$	$56n \pm 9$	$14^p - 1$	$60n \pm 13$	$15^p + 1$
$48n \pm 13$	$12^p - 1$	$52n \pm 9$	$13^p - 1$	$56n \pm 11$	$14^p - 1$	$60n \pm 17$	$15^p - 1$
$48n \pm 17$	$12^p + 1$	$52n \pm 11$	$13^p + 1$	$56n \pm 13$	$14^p - 1$	$60n \pm 19$	$15^p + 1$
$48n \pm 19$	$12^p + 1$	$52n \pm 15$	$13^p + 1$	$56n \pm 15$	$14^p + 1$	$60n \pm 23$	$15^p + 1$
$48n \pm 23$	$12^p - 1$	$52n \pm 17$	$13^p - 1$	$56n \pm 17$	$14^p + 1$	$60n \pm 29$	$15^p + 1$
		$52n \pm 19$	$13^p + 1$	$56n \pm 19$	$14^p + 1$		
		$52n \pm 21$	$13^p + 1$	$56n \pm 23$	$14^p + 1$		
		$52n \pm 23$	$13^p - 1$	$56n \pm 25$	$14^p - 1$		
		$52n \pm 25$	$13^p - 1$	$56n \pm 27$	$14^p + 1$		

A. m. T. I. p. 211—213. 215. 216.

16.

Theorema. Si potestas a^p per N divisa relinquat r, at potestas a^q residuum s, tum formula $s^p - r^q$ per N erit divisibilis.

Demonstratio. Cum $a^p - r$ sit divisibilis per N, tum etiam $a^{np} - r^n$ erit divisibilis, ergo etiam $a^{pq} - r^q$. Simili modo cum $a^q - s$ sit divisibilis per N, etiam $a^{pq} - s^p$ erit divisibilis; unde sequitur etiam $s^p - r^q$ fore divisibile per N. Hinc si $r = 1$, tum $s^p - 1$ erit divisibile.

Theorema. Si fuerit $r + \lambda N = a^\alpha s$, tum $a^{p-\alpha} - s$ est divisibile per N. Hic ergo est $r = a^\alpha s - \lambda N$ et $q = p - \alpha$; erit ergo

$$s^p - (a^\alpha s - \lambda N)^{p-\alpha} \text{ per } N \text{ divisibile.}$$

A. m. T. I. p. 214.

17.

(*Krafft.*)

Si fuerit p numerus impar, tum $\frac{2^p+1}{3}$ semper est numerus integer, qui quoties p est numerus primus, videtur etiam esse numerus primus, id quod examinetur.

Ponatur $\frac{2^p+1}{3} = y$; erit sequens $\frac{2^{p+2}+1}{3} = \frac{4.2^p+1}{3}$. At ex priore est $2^p = 3y - 1$; unde sequens erit $\frac{12y-3}{3} = 4y - 1$, unde formetur sequens series

$$\begin{array}{lcccccccccc} p \ldots 1, & 3, & 5, & 7, & (9), & 11, & 13, & (15), & 17, & 19, & \text{etc.} \\ y \ldots 1, & 3, & 11, & 43, & (171), & 683, & 2731, & (10923), & 43691, & 174763, & \text{etc.} \end{array}$$

Hinc suspicio confirmatur usque ad ultimum 174763, qui sit $= a$, ita, ut sit $3a = 2^{19} + 1$; at hic numerus continetur in forma $2f^2 + g^2$, quae alios divisores non habet, nisi in eadem forma contentos; necesse ergo est, sit $174763 = 2r^2 + s^2$ idque unico modo. Si ergo hic numerus unico modo in forma $2r^2 + s^2$ contineatur, certo erit primus; sin autem pluribus modis contineatur, tum demum erit compositus; id quod non adeo difficile est explorare. Est autem

$$174763 = 2.295^2 + 713 = 2.294^2 + 1891 = 2.293^2 + 3065 \ldots (= 2.171^2 + 341^2).$$

(*Lexell.*)

At sine tanto calculo demonstrari potest hunc numerum esse primum. Si enim haberet divisorem, is *primo* minor esset, quam radix quadrata hujus numeri, quae est 418, sive < 419. *Secundo* divisor iste continebitur in forma vel $8n+1$, vel $8n+3$. *Tertio* divisor etiam formam habebit $19\lambda+1$, ubi λ primo esse debet par, erit ergo vel $\lambda = 8n$, vel $8n+2$, vel $8n+4$, vel $8n+6$. Prima dat formam $8n+1$, quae congruit cum priore, at $\lambda = 8n+2$ dat $8n+39$, ideoque $\lambda = 8n+2$ excluditur; similiter $\lambda = 8n+4$ dat $8n+5$, unde $\lambda = 8n+4$ excluditur; at $\lambda = 8n+6$ dat $8n+3$, quae valet. Duae ergo formae relinquuntur pro λ, $8n$ et $8n+6$, ergo ex priori $19\lambda+1$ habentur: 1, 153, 305, 457, et ex posteriori $19\lambda+1$: 115, 267, 419. Hi autem numeri, minores quam 419, omnes sunt compositi.

Neque vero propositio supra memorata est vera, plures enim casus assignari possunt, quibus fallit. Cum enim numerus primus $2p+1$ quoties fuerit formae $8n+3$, sit divisor formulae 2^p+1, ob $p = 4n+1$ utique fieri potest, ut p sit numerus primus; iis ergo casibus etiam formula $\frac{2^p+1}{3}$ divisorem habebit $8n+3$, hoc itaque evenit, quoties tam $4n+1$ quam $8n+3$ fuerint numeri primi, cujusmodi casus sunt:

$$\begin{array}{ccccc} 5, & 29, & 41, & 53, & 89 \\ 11, & 59, & 83, & 107, & 179. \end{array}$$

A. m. T. I. p. 217.

18.

(*J. A. Euler.*)

Ut formulae x^4+1 divisor sit 17, erit . . . . $x = 2$, vel 8, vel 9, vel 15
Ut ejusdem formulae divisor sit 41, erit . . . . $x = 3$, vel 14, vel 27, vel 38
" " " 73, erit . . . . $x = 10$, vel 22, vel 51, vel 63
" " " 89, erit . . . . $x = 12$, vel 37, vel 52, vel 77

In omnibus scilicet casibus, si fuerit $x = a$, erit etiam

$$x = a^3 \quad \text{et} \quad = a^5 \quad \text{et} \quad = a^7 \quad \text{etc.}$$

A. m. T. I. p. 201.

d) *De divisoribus et residuis numerorum quadratorum.*

19.

Theorema, cujus demonstratio desideratur.

Si pro divisore d inter residua quadratorum occurrat $\pm r$, tum etiam pro divisore $4nr+d$, si fuerit numerus primus, inter residua quadratorum idem quoque residuum $\pm r$ occurret. Ita si sit $d = 7$, inter residua quadratorum occurrit 2; ideoque quoties $8n+7$ fuerit numerus primus, (eo divisore) inter residua quadratorum reperiatur 2 necesse est.

Ratio in eo quaerenda videtur, quod si $8n+7$ est numerus primus, tum numerus residuorum semper est $4n+3$, dum si non fuerit primus, multitudo residuorum multo est minor: scilicet pro $8n+7 = 15$, multitudo residuorum non est 7, sed tantum 5.

A. m. T. I. p. 210.

20.

Theoremata demonstranda.

I. Si per numerum primum $4n+1$ omnia quadrata dividantur, inter residua occurret non solum ipse numerus n, sed etiam omnes ejus divisores et quidem singuli utroque signo affecti.

II. Si per numerum primum $4n-1$ omnia quadrata dividantur, inter residua non solum occurret ipse numerus n, sed etiam omnes ejus divisores signo $+$ affecti; iidem enim signo $-$ affecti erunt non-residua.

Haec duo theoremata ita generalius proponi possunt: Denotante i numerum imparem quemcunque,

I. si per numerum primum $4n+ii$ quadrata dividantur, inter residua occurrent omnes divisores numeri n tam signo $+$ quam signo $-$ affecti.

II. Corollarium. Hinc si $4n+ii$ est numerus primus et d aliquis divisor numeri n, semper dari poterit formula $xx+dyy$ per illum numerum $4n+ii$ divisibilis.

III. Si per numerum primum $4n-ii$ quadrata dividantur, inter residua occurrent omnes divisores numeri n positive sumti, iidem vero negative sumti erunt non-residua.

IV. Corollarium. Ergo si d fuerit divisor quicunque numeri n, semper dabuntur hujusmodi formulae $xx+dyy$ per numerum primum $4n-ii$ divisibiles: contra vero nulla dabitur talis formula $xx+dyy$ per hunc numerum primum divisibilis. A. m. T. I. p. 224.

21.

(Krafft.)

Problema. Invenire omnes numeros primos formae $4n+1$, per quos si quadrata dividantur, inter residua occurrat datus numerus $\pm a$.

Solutio. Ante vidimus, si divisor primus fuerit $4ap+ii$, inter residua certo occurrere $\pm a$. Statuatur ergo $4n+1 = 4ap+ii$, et quia i est impar, ponatur $i = 2c+1$, ut prodeat

$$4n+1 = 4ap+4c^2+4c+1, \text{ seu } n = ap+c^2+c.$$

Quoties ergo fuerit $n = ap+c^2+c$, quicunque numeri pro c et p statuantur, tum numerus $4n+1$ satisfaciet, siquidem fuerit primus.

Corollarium. Simili modo patebit, ut divisori primo $4n-1$ conveniat in residuis numerus $+a$, tum sumi debere $n = ap-c^2-c$.

Formula autem c^2+c hos praebet numeros: $0, 2, 6, 12, 20, 30, 42, 56$, etc., quibus per a divisis sit quodvis residuum $= r$, quodvis autem non-residuum sit ϱ, atque sequentia theoremata obtinebuntur:

I. Si fuerit $4n+1$ primus et $n = ap+r$, tum in residuis quadratorum per $4n+1$ divisorum occurrunt numeri $+a$ et $-a$, ideoque dabuntur formulae x^2+ay^2 et x^2-ay^2 per $4n+1$ divisibiles; tum vero etiam formula $a^{2n}-1$ quoque erit divisibilis.

II. Existente $4n+1$ numero primo, si fuerit $n = ap+\varrho$, tum in residuis quadratorum neque $+a$ neque $-a$ occurret, et neutra formula x^2+ay^2 et x^2-ay^2, neque etiam haec $a^{2n}-1$ erit divisibilis per $2n+1$; cum ergo $a^{4n}-1$ sit divisibilis, sequitur, formulam $a^{2n}+1$ fore divisibilem per $4n+1$.

III. Si divisor primus $= 4n-1$ atque $n = ap-r$, tum in residuis quadratorum occurret $+a$, non vero $-a$, ideoque dabitur formula $xx-ayy$ per $4n-1$ divisibilis, non vero $xx+ayy$; tum vero formula $a^{2n-1}-1$ divisibilis erit per $4n-1$.

IV. Si divisor primus $4n-1$ at $n = ap-\varrho$, inter residua quadratorum non occurret $+a$, sed $-a$, ideoque dabitur formula $xx+ayy$ divisibilis per $4n-1$, et jam formula $(-a)^{2n-1}-1$, sive $a^{2n-1}+1$ divisibilis erit per $4n-1$.

In his autem theorematibus praecedentia fere omnia continentur, id quod sequentibus ostendamus exemplis.

1) Sit $a = 2$, erit $r = 0$ et $\varrho = 1$, unde sequitur

pro I. $n = 2p$, ideoque divisor $4n+1 = 8p+1$; sequientes igitur dabuntur formulae per $8p+1$ divisibiles: x^2+2y^2, x^2-2y^2 et $2^{4p}-1$;

pro II. $n = 2p+1$, ergo $4n+1 = 8p+5$, per quem numerum scilicet primum neutra formularum x^2+2y^2 et x^2-2y^2, at vero $2^{4p+2}+1$ erit divisibilis;

pro III. $n = 2p$ et divisor primus $4n-1 = 8p-1$, per quem divisibilis erit formula x^2-2y^2, tum vero etiam $2^{4p-1}-1$;

pro IV. $n = 2p-1$ et divisor $4n-1 = 8p-5$, sive $8p+3$, per quem divisibiles erunt formulae x^2+2y^2 et $a^{4p-3}+1$, sive $a^{4p-2}+1$.

2) Sit $a=3$, ubi $r=0$, 2 et $\varrho=1$, ergo

pro I. $n=3p+0$, vel $3p+2$, ideoque $4n+1=12p+1$, $12p+9$, ubi casus posterior est rejiciendus;

ut sit $4n+1=12p+1$, per quem divisibiles sunt $x^2\pm 3y^2$ et $3^{2n}-1$;

pro II. $n=3p+1$ et divisor $4n+1=12p+5$, per quem divisibilis est formula $3^{2n}+1$;

pro III. $n=3p+0$, 2, et divisor $4n-1=12p-1$, sive $12p-9$, quod sponte excidit, per quem formulae divisibiles x^2-3y^2 et $3^{2n-1}-1$;

pro IV. $n=3p-1$, hinc $4n-1=12p-5$; formulae divisibiles x^2+3y^2 et $3^{2n-1}+1$.

3) Sit $a=5$, ubi $r=0$, 2, 1 et $\varrho=3$, 4, sive $=-1$, -2.

Pro I. $n=5p+0$, 1, 2; $4n+1=20p+1$, (5), 9; formulae divisibiles $x^2\pm 5y^2$ et $5^{2n}-1$;

pro II. $n=5p-1$, -2; $4n+1=20p-3$, -7; formula divisibilis $5^{2n}+1$;

pro III. $n=5p-0$, 1, 2; $4n-1=20p-1$, (-5), -9; formulae divisibiles x^2-5y^2, $5^{2n-1}-1$;

pro IV. $n=5p+1$, 2; $4n-1=20p+3$, 7, formulae divisibiles x^2+5y^2 et $5^{2n-1}+1$;

4) Sit $a=6$, $r=0$, 2 et $\varrho=1$, 3, 4, 5, sive $=-2$, -1.

Pro I. $n=6p+0$, 2; $4n+1=24p+1$; formulae divisibiles $x^2\pm 6y^2$ et $6^{2n}-1$;

pro II. $n=6p+1$, 3, -2, -1; $4n+1=24p+5$, 13, -7; formula divisibilis $6^{2n}+1$;

pro III. $n=6p-0$, 2 et $4n-1=24p-1$, 9; formulae divisibiles x^2-6y^2 et $6^{2n-1}-1$;

pro IV. $n=6p-1$, -3, $+2$, $+1$; $4n-1=24p-5$, -13, $+7$; formulae divisib. x^2+6y^2 et 6^{2n-1}...

Ubi notandum est, unitatem hic perperam referri ad ϱ, valores enim literarum r et ϱ inter se aequales esse debent et oporteret 1 ad r referre, ita ut pro 1 sit etiam $4n+1=24p+5$, quod etiam confirmatur per residua; si enim $n=0$, pro divisore 5 utique occurrit residuum 6, utpote $1+5$.

Idem inconveniens occurret, quoties n est numerus par; id vero incongruum ita diluendum videtur: Cum per divisorem 6 dividi debeant numeri 0, 2, 6, 12, 20, etc. utrinque diviso per 2, habebuntur numeri 0, 1, 3, 6, 10, etc. per 3 dividendi; unde manifesto oritur residuum 1 praeter praecedentia, quod ergo ex ϱ expungi debet: Ita si $a=10$, primo pro r reperimus hos valores 0, 2, 6; per binarium autem dividendo insuper accedunt ad r 1, 3, ita, ut valores ipsius r jam sint 0, 1, 2, 3, 6, ergo ipsius ϱ:

4, 5, 7, 8, 9; $4r+1=1$, (5), 9, 13, (25); $4\varrho+1=17$, 21, 29, 33, 37, sive 17, -19, -11, -7, ...

hic ergo etiam numerus 9 ab ϱ ad r est transferendus.

(*Lexell.*)

Vera autem solutio hujus difficultatis in indole numeri a est quaerenda, qui si fuerit primus, valores ... r et ϱ supra assignati recte se habent; sin autem est compositus, valores quidem pro r oriundi recte se habent, sed non omnes per regulam supra datam reperiuntur, sed aliunde insuper alii accedunt. Ut enim formula $(ab)^x-1$ divisibilis sit per numerum primum $2x+1$, id duplici modo contingere potest: priori quando a^x-1 et b^x-1 divisionem admittunt; si enim a^x-1 est divisibile, erit etiam $(ab)^x-b^x$; addatur formula divisibilis b^x-1, prodit formula divisibilis $(ab)^x-1$, atque hos casus regula nostra suppeditat. Praeterea vero formula $(ab)^x-1$ erit divisibilis, si istae a^x+1 et b^x+1 fuerint divisibiles; cum enim ex priori sequatur $(ab)^x+b^x$ divisibilis, auferendo hinc b^x+1 remanet $(ab)^x-1$ divisibilis. Hinc igitur novi valores ad r accedunt, qui supra ad ϱ perperam erant relati. Totum igitur hoc argumentum accuratius sequenti modo simulque concinnius pertractatur.

Denotet $2m+1$ semper numerum primum, et supra affirmavimus, si fuerit $2m+1=4ab+ii$ (denotantibus ... numeros impares), tum in residuis quadratorum tam $+a$ quam $-a$ reperiri; sin autem fuerit $2m+1=4ab-ii$, tum tantum $+a$ in residuis occurrere; utroque autem casu, hoc est si $2m+1=4ab\pm ii$, formulam a^m ... divisibilem esse per $2m+1$. Hujus quidem demonstratio nondum perfecta habetur, sed tamen non longe abesse [illegible]emur, cum enim quadrata per numerum $2m+1$ dividi debeant, ut residua eruantur, per $2m+1=4ab+ii$ dividatur ipsum quadratum ii, et residuum erit $-4ab$, ideoque etiam $-ab$, et quia divisor est formae $4n+1$, etiam $+ab$ erit residuum. Superest igitur tantum, ut demonstretur, tam $+a$ quam $+b$ seorsim inter residua occurrere; si enim ambo essent non-residua, nihilominus productum ab foret residuum. Ad hoc dilucidandum, proponatur divisor primus $2m+1=4ab+(2c+1)^2$, ita ut ab certe sit residuum; quoniam hic numerus pluribus aliis modis similiter exhiberi potest. Statuamus $2m+1=4p+(2q+1)^2$, et nunc etiam p certe erit residuum. Aequentur hae duae formulae inter se, et reperiemus $p=ab+cc+c-qq-q$, ubi q pro lubitu assumere licet, sicque plura alia residua prodibunt, inter quae si occurrat alteruter numerus a vel b, etiam alter certe erit residuum. Ut hoc uberius explicetur, notasse juvabit, inter residua primum omnia occurrere quadrata, deinde si occurrant numeri α, β, γ, etc., etiam producta ex binis vel pluribus occurrent. Et si occurrant numeri α et $\alpha\gamma$, etiam γ occurret, et si occurrat $\alpha\gamma^2$, etiam α occurret; hoc igitur exemplis illustremus.

Exemplum I. Sit $a=2$, $b=2$, ideoque $2m+1=16+(2c+1)^2$.

1) Sit $c=0$ eritque $p=4-qq-q=4-(0,\,2,\,6,\,12)$, hinc capiatur $p=4-2=2$, ergo 2 certe est residuum.

2) Sit $2c+1=5$, erit $p=4+6-qq-q=10-(0,\,2,\,6,\,12)$ et sumto $q=1$, erit $p=8=2.4$, ergo 2 residuum.

3) Sit $c=4$ sive $2m+1=97$, unde $p=24-qq-q$; sumatur $q=2$, erit $p=18$, ideoque 2 residuum.

Exemplum II. Sit $a=2$ et $b=3$ et $2m+1=24+(2c+1)^2$. Sit $c=3$, ut fiat $2m+1=73$, ergo $p=6+12-qq-q=18-qq-q$; sumatur $q=0$ fit $p=2.9$, ergo et 2 et 3 residua.

Exemplum III. Sit $a=3$ et $b=3$ et $2m+1=36+(2c+1)^2$. Sit $c=0$, ut fiat $2m+1=37$, ergo $p=9-qq-q=9-(0,\,2,\,6)$; sumto $q=2$, $p=3$. Sit deinde $c=2$, unde $2m+1=61$, hinc

$$p=9+6-qq-q=15-(0,\,2,\,6),\quad \text{ergo}\quad p=15-12=3.$$

Exemplum IV. Sit $ab=2.3.5$, ideoque $2m+1=8.3.5+(2c+1)^2$. Sumto $c=5$, ut sit $2m+1=241$, erit

$$p=2.3.5+30-qq-q=60-(0,\,2,\,6,\,12,\,20,\,30,\,42,\,56).$$

At $60-6$ dat $54=6.9$, ergo 6 est residuum, ergo et 5; deinde $p=60-12$ dat $48=3.16$, unde 3 est residuum et 2, sicque singuli factores 2, 3, 5 sunt residua.

Exemplum V. Sit $ab=3.5.7=105$, ideoque $2m+1=420+(2c+1)^2$ et sumto $c=0$, $2m+1=421$, unde

$$p=105-q(q+1)=105-(0,\,2,\,6,\,12,\,20,\,30,\,42,\,56,\,72,\,90,\,110).$$

Hinc $105-30=75=3.25$, ergo 3 est residuum, ideoque et 35. Deinde $p=105-42=63=7.9$, ideoque 7 residuum ut et 5; sicque singuli factores sunt residua.

Hinc ergo tuto concludi posse videtur, quotcunque etiam factores habeat productum ab, singulos semper quoque inter residua occurrere, quod idem simili modo de altera forma $4ab-(2c+1)^2$ ostenditur; posito enim

$$4ab-(2c+1)^2=4p-(2q+1)^2,\quad \text{erit}\quad p=ab-cc-c+qq+q,$$

ubi p certo est residuum.

Exemplum I. Sit $ab=2.2$, $2m+1=16-(2c+1)^2$, sumto $c=1$, $2m+1=7$, ergo

$$p=4-2+qq+q=2+qq+q,$$

unde, si $q=0$, patet 2 esse residuum.

Exemplum II. Sit $ab=2.3=6$, erit $2m+1=24-(2c+1)^2$; posito $c=0$, $2m+1=23$, ergo

$$p=6+qq+q=6+(0,\,2,\,6,\,12,\,20),\quad \text{unde}\quad p=6+2=8=2.4,$$

ergo 2 residuum, ideoque et 3, sive $p=6+6=12=3.4$, ergo 3 residuum.

Exemplum III. Sit $ab=2.2.3.5=60$ et $2m+1=240-(2c+1)^2$; posito

$$c=0,\quad 2m+1=239,\quad \text{unde}\quad p=60+(0,\,2,\,6,\,12,\,20,\,30,\,42,\,\text{etc.}).$$

Hinc $p = 60 + 12 = 72 = 2.36$, ergo 2 est residuum. Porro $p = 60 + 20 = 5.16$, ergo 5 residuum, ideoque etiam 3. Sive sumto $p = 60 + 30 = 90 = 10.9$, ergo 10 residuum, hinc etiam 5. Sumto autem $a = 2$, fit $2m + 1 = 191$ primus, ergo

$p = 60 + 12 + qq + q = 48 + qq + q = 48 + (0, 2, 6, 12, 20, 30, 42)$.

Hinc statim $p = 48 = 3.16$ dat 3 pro residuo, deinde $p = 50 = 2.25$ dat 2 pro residuo. Porro $p = 48 + 42 = 90 = 10.9$, ergo 10 residuum, ideoque et 5.

Quamquam haec prorsus certa videntur, tamen demonstratio desideratur.

Uberior consideratio formulae $2m + 1 = 4ab \pm ii$,

ubi primo inquirendum, quibusnam casibus a inter residua reperiatur. Quia ii semper est numerus formae $4r+1$, nostra formula ita referetur $4ab \pm (4r+1)$, at formula $4r+1$ continet primo omnia quadrata imparia, quae quidem cum $4ab$ numeros primos dare possunt, majora autem infra $4a$ deprimi possunt, dum ab iis subtrahitur $4a$ quoties fieri possit, hocque modo pro quovis casu numeri a, formula $4r+1$ certos sortietur valores minores, quam $4a$, ac si a fuerit numerus primus, hoc modo omnes prodeunt idonei valores pro $4r+1$, qui autem numeri hujus formae non occurrunt, eos formula $4\varrho+1$ indicemus, atque his numeris utriusque generis $4r+1$ et $4\varrho+1$ pro quovis numero primo a definitis, sequentia habebimus theoremata.

I. Si fuerit $2m+1 = 4ab \pm (4r+1)$, tum formula $a^m \mp 1$ semper erit divisibilis per $2m+1$, ac casu signi superioris tam $+a$ quam $-a$ inter residua quadratorum reperientur, casu autem signi inferioris, tantum $+a$ erit residuum, et $-a$ non-residuum.

II. Si fuerit $2m+1 = 4ab \pm (4\varrho+1)$, tum semper formula $a^m \pm 1$ dividi poterit per $2m+1$, tum vero pro signo superiore $+$ neque a nec $-a$ erit residuum, sive neque $xx + ayy$ nec $xx - ayy$ unquam per $2m+1$ dividi poterit. Pro signo autem inferiore $-$, inter residua erit $-a$, sive formula $xx + ayy$ divisibilis erit per $2m+1$; probe autem hic notetur, haec tantum valere, si a fuerit numerus primus, numeri enim compositi aliam requirunt evolutionem. Nunc igitur pro singulis numeris primis a exhibeamus numeros illos duplicis generis in formulis $4r+1$ et $4\varrho+1$ contentos.

$$a = 2\left\{\begin{array}{l|l} 4r+1 = 1, & 9,\ 17,\ 25,\ 33,\ 41,\ 49,\ 57,\ \text{etc.} \\ 4\varrho+1 = 5, & 13,\ 21,\ 29,\ 37,\ 45,\ 53,\ 61,\ \text{etc.} \end{array}\right.$$

$$a = 3\left\{\begin{array}{l|l} 4r+1 = 1, & 13,\ 25,\ 37,\ 49,\ 61,\ 73,\ \text{etc.} \\ 4\varrho+1 = 5, & 17,\ 29,\ 41,\ 53,\ 65,\ 77,\ \text{etc.} \end{array}\right.$$

$$a = 5\left\{\begin{array}{l|l} 4r+1 = 1,\ \ 9, & 21,\ 29,\ 41,\ 49,\ 61,\ 69,\ 81,\ 89,\ \text{etc.} \\ 4\varrho+1 = 13,\ 17, & 33,\ 37,\ 53,\ 57,\ 73,\ 77,\ 93,\ 97,\ \text{etc.} \end{array}\right.$$

$$a = 7\left\{\begin{array}{l|l} 4r+1 = 1,\ \ 9,\ 25, & 29,\ 37,\ 53,\ 57,\ 65,\ 81,\ \text{etc.} \\ 4\varrho+1 = 5,\ 13,\ 17, & 33,\ 41,\ 45,\ 61,\ 69,\ 73,\ \text{etc.} \end{array}\right.$$

$$a = 11\left\{\begin{array}{l|l|l} 4r+1 = 1,\ \ 5,\ \ 9,\ 25,\ 37, & 45,\ 49,\ 53,\ 69,\ 81, & 89,\ \ 93,\ \ 97,\ \text{etc.} \\ 4\varrho+1 = 13,\ 17,\ 21,\ 29,\ 41, & 57,\ 61,\ 65,\ 73,\ 85, & 101,\ 105,\ \text{etc.} \end{array}\right.$$

$$a = 13\left\{\begin{array}{l|l} 4r+1 = 1,\ \ 9,\ 17,\ 25,\ 29,\ 49, & 53,\ 61,\ 69,\ 77,\ \ 81,\ 101,\ \text{etc.} \\ 4\varrho+1 = 5,\ 21,\ 33,\ 37,\ 41,\ 45, & 57,\ 73,\ 85,\ 89,\ \ 93,\ \ 97,\ \text{etc.} \end{array}\right.$$

$$a = 17\left\{\begin{array}{l|l} 4r+1 = 1,\ \ 9,\ 13,\ 21,\ 25,\ 33,\ 49,\ 53, & 69,\ 77,\ 81,\ 89,\ 93,\ 101,\ \text{etc.} \\ 4\varrho+1 = 5,\ 29,\ 37,\ 41,\ 45,\ 57,\ 61,\ 65, & 73,\ 97,\ 105,\ \text{etc.} \end{array}\right.$$

$$a = 19\left\{\begin{array}{l|l} 4r+1 = 1,\ \ 5,\ \ 9,\ 17,\ 25,\ 45,\ 49,\ 61,\ 73, & 77,\ 81,\ 85,\ \text{etc.} \\ 4\varrho+1 = 13,\ 21,\ 29,\ 33,\ 37,\ 41,\ 53,\ 65,\ 69, & \text{etc.} \end{array}\right.$$

$$a = 23\left\{\begin{array}{l} 4r+1 = 1,\ \ 9,\ 13,\ 25,\ 29,\ 41,\ 49,\ 73,\ 77,\ 81,\ 85,\ \text{etc.} \\ 4\varrho+1 = 5,\ 17,\ 21,\ 33,\ 37,\ 45,\ 53,\ 57,\ 61,\ 65,\ 89,\ \text{etc.} \end{array}\right.$$

Geminas has series pro quovis numero primo a facile in infinitum continuare licet, eas autem in periodos distinximus, quarum prima continet numeros formae $4n+1$, minores quam $4a$, secunda periodus continet eosdem numeros $+4a$. Tertia continet numeros secundae periodi $+4a$ et ita porro.

Hinc igitur pro casibus, quibus a est primus, judicare licet, utrum formula a^m-1 an a^m+1 per numerum primum $2m+1$ sit divisibilis; prius scilicet evenit, quoties fuerit $2m+1=4ab\pm(4r+1)$, posterius vero quoties fuerit $2m+1=4ab\pm(4\varrho+1)$. Circa has series notari oportet, in qualibet periodo contineri $\frac{a-1}{2}$ terminos, ita ut in ordine $4\varrho+1$ totidem sint termini quot in $4r+1$; deinde omnes termini ordinis $4r+1$ vel ipsi sunt quadrata, vel tales, ut $4r+1+4an$ fieri possit quadratum. Contra vero numeri $4\varrho+1$ omnes ita sunt comparati, ut formula $4\varrho+1+4an$ nunquam fieri possit quadratum, quicunque numerus pro n capiatur.

PROBLEMA. Nunc videamus, quomodo judicium institui debeat, quando numerus a habet factores, scilicet tum etiam investigemus tam terminos $4r+1$ quam $4\varrho+1$ tali numero a convenientes.

SOLUTIO. Sit $a=fg$ et f et g numeri primi. Quaerantur primo pro f numeri tam formae $4r+1$ quam $4\varrho+1$, qui ita designentur ${}^{f}(4r+1)$ et ${}^{f}(4\varrho+1)$, eodemque modo pro numero g habeantur formulae ${}^{g}(4r+1)$ et ${}^{g}(4\varrho+1)$, quo facto excerpantur omnes numeri binis formulis ${}^{f}(4r+1)$ et ${}^{g}(4r+1)$ communes, cujusmodi sit P, et ex praecedentibus patet, si divisor fuerit $4fp\pm P=2m+1$, tum formulam f^m-1 fore divisibilem per $2m+1$. Simili modo pro divisore $2m+1=4gq\pm P$ formulam g^m-1 esse divisibilem. Fiat nunc $p=gn$ et $q=fn$, ut prodeat communis divisor $4fgn\pm P$, per quem ambae formulae f^m-1 et g^m-1 erunt divisibiles, unde sequitur, quoque formulam $(fg)^m-1=a^m-1$ fore divisibilem. Praeterea cum a^m-1 quoque sit divisibile, si tam f^m+1 quam g^m+1 dividi queant, id quod evenit, si ex ordinibus ${}^{f}(4\varrho+1)$ et ${}^{g}(4\varrho+1)$ termini communes excerpantur, quam ob rem pro numero proposito $a=fg$ ordo $4r+1$ primo continebit omnes terminos communes ordinum ${}^{f}(4r+1)$ et ${}^{g}(4r+1)$, praeterea vero etiam terminos communes ordinibus ${}^{f}(4\varrho+1)$ et ${}^{g}(4\varrho+1)$. Reliqui vero numeri formae $4n+1$ hic non occurrentes ad ordinem $4\varrho+1$ sunt referendi, ubi ergo occurrent primo termini communes ordinibus ${}^{f}(4r+1)$ et ${}^{g}(4\varrho+1)$, tum vero etiam communes ordinibus ${}^{g}(4r+1)$ et ${}^{f}(4\varrho+1)$; hoc igitur modo pro numero $a=fg$ facile colligentur numeri ordinis $4r+1$ et $4\varrho+1$.

COROLLARIUM 1. Si fuerit $g=f$, ita ut a fiat quadratum $=ff$, tum pro ordine $4r+1$ omnes plane numeri ordinis $4n+1$ occurrent, alter vero ordo $4\varrho+1$ plane manebit vacuus, id quod etiam inde manifestum est, quod si a fuerit quadratum $=ff$, semper formulam $a^m-1=f^{2m}-1$ esse divisibilem per numerum $2m+1$.

COROLLARIUM 2. Sin autem factores f et g fuerint dispares, ex praecedentibus ordinibus serierum facile pro quovis numero $a=fg$ termini utriusque ordinis colligentur, quemadmodum ex sequentibus exemplis patebit.

EXEMPLUM I. Sit $a=2.3$, ideoque $4a=24$, et terminus communis ordinum ${}^{2}(4r+1)$ et ${}^{3}(4r+1)$ est 1 cum sequentibus 25, 49, 73, 97; at vero terminus ordinibus ${}^{2}(4\varrho+1)$ et ${}^{3}(4\varrho+1)$ communis est 5, unde in primo ordine tantum occurrunt 1, 5, at pro ordine $(4\varrho+1)$ terminus communis ordinibus ${}^{2}(4r+1)$ et ${}^{3}(4\varrho+1)$ est 17, ordinibus autem ${}^{3}(4r+1)$ et ${}^{2}(4\varrho+1)$ communis est 13. Qui ordines ita referantur

$$a=6,\quad 4a=24\left\{\begin{array}{l|l} 4r+1=\ 1,\ \ 5, & 25,\ 29,\ 49,\ 53,\ 73,\ 77,\ 97,\ 101 \\ 4\varrho+1=13,\ 17, & 37,\ 41,\ 61,\ 65,\ 85,\ 89 \end{array}\right.$$

EXEMPLUM 2. Sit $a=2.5=10$ et $4a=40$. Hic termini communes ordinum ${}^{2}(4r+1)$ et ${}^{5}(4r+1)$ sunt 1, 9, at termini communes ordinum ${}^{2}(4\varrho+1)$ et ${}^{5}(4\varrho+1)$ sunt 13, 37. At pro ordine $4\varrho+1$ sunt termini communes ${}^{2}(4r+1)$ et ${}^{5}(4\varrho+1)$, 17, 33, at ordines ${}^{5}(4r+1)$ et ${}^{2}(4\varrho+1)$ communes habent 21, 29, unde fit

$$a=10,\quad 4a=40\left\{\begin{array}{l|l} 4r+1=\ 1,\ 9,\ 13,\ 37, & 41,\ 49,\ 53,\ 77,\ 81,\ 89,\ 93 \\ 4\varrho+1=17,\ 21,\ 29,\ 33, & 57,\ 61,\ 69,\ 73,\ 97 \end{array}\right.$$

Exemplum 3. Si $a = 2\cdot 7 = 14$ et $4a = 56$

$$\begin{array}{l|l} 4r+1 = 1,\ 5,\ 9,\ 13,\ 25,\ 45, & 57,\ 61,\ 65,\ 69,\ 81,\ 101 \\ 4\varrho+1 = 17,\ 29,\ 33,\ 37,\ 41,\ 53, & 73,\ 85,\ 89,\ 93,\ 97,\ 109 \end{array}$$

Exemplum 4. Sit $a = 3\cdot 5 = 15$ et $4a = 60$

$$\begin{array}{l|l} 4r+1 = 1,\ 17,\ 49,\ 53, & 61,\ 77,\ 109 \\ 4\varrho+1 = 13,\ 29,\ 37,\ 41, & 73,\ 89,\ 97 \end{array}$$

Corollarium. Si fuerit $a = ffg$, quia factor ff in ordine r continet numeros, in ordine ϱ autem nullos, termini communes ordinibus r sunt omnes, qui pro g habentur, at termini ordinibus ϱ communes sunt nulli; ergo pro hoc casu ordo $4r+1$ congruit cum numero g, similique modo congruit ordo $4\varrho+1$. Id quod his casibus apparet:

$$a = 8,\quad 4a = 32 \begin{cases} 4r+1 = 1,\ 9,\ 17,\ 25, \mid 33,\ 41,\ 49,\ 57, \mid 65,\ 73,\ 81,\ 89 \\ 4\varrho+1 = 5,\ 13,\ 21,\ 29, \mid 37,\ 45,\ 53,\ 61, \mid 69,\ 77,\ 85,\ 93 \end{cases}$$

$$a = 4\cdot 3 = 12,\quad 4a = 48 \begin{cases} 4r+1 = 1,\ 13,\ 25,\ 37, \mid 49,\ 61,\ 73,\ 85, \mid 97 \\ 4\varrho+1 = 5,\ 17,\ 29,\ 41, \mid 53,\ 65,\ 77,\ 89, \end{cases}$$

A. m. T. I. p. 226—235.

22.

(Krafft.)

Si fuerit $4n+1$ numerus primus, in residuis quadratorum non solum hi numeri $n - qq - q$, sed etiam omnes eorum divisores occurrent.

Quia numerus residuorum diversorum est $= 2n$, omnia prodibunt, si loco q substituantur numeri $0, 1, 2, 3$ usque $(2n-1)$. Reliqui ergo haec residua erunt, ut sequens tabella indicat, ubi ultima columna ostendit ea quadrata, unde haec residua nascuntur:

$n - 0$	$4n^2$
$n - 2$	$(2n-1)^2$
$n - 6$	$(2n-2)^2$
$n - 12$	$(2n-3)^2$
$n - 20$	$(2n-4)^2$
$-4n^2+3n$	1

Nunc autem demonstrandum restat, etiam omnes factores horum residuorum esse residua, quod eo magis est mirandum, quod cum etiam hae formulae

$$\lambda(4n+1) + n - qq - q$$

pariter residua exhibeant, tamen non omnes factores etiam futuri sint residua.

A. m. T. I. p. 255.

23.

(N. Fuss I.)

Observatio. Formula $xx+1$ divisibilis erit per sequentes numeros quadratos:

I.	per	5^2	si fuerit	$x = 5^2 t \pm 7$
II.	„	13^2	„	$x = 13^2 t \pm 70$
III.	„	17^2	„	$x = 17^2 t \pm 38$

IV.	per	25^2	si fuerit	$x = 25^2 t \pm 57$
V.	„	29^2	„	$x = 29^2 t \pm 41$
VI.	„	37^2	„	$x = 37^2 t \pm 117$
VII.	„	41^2	„	$x = 41^2 t \pm 378$
VIII.	„	53^2	„	$x = 53^2 t \pm 500$
IX.	„	61^2	„	$x = 61^2 t \pm 682$
X.	„	65^2	„	$x = 65^2 t \pm 268$
XI.	„	73^2	„	$x = 73^2 t \pm 74$

Hinc etiam valores ipsius x assignari poterunt, ut haec formula $xx + aa$ per eosdem numeros fiat divisibilis; sicque $xx + aa$ divisibilis erit per 41^2, si fuerit $x = 41^2 t + 378a$; ita hoc problema resolvi potest, quo quaeruntur valores ipsius x, ut haec formula $xx + aa$ divisibilis fiat per $(ff + gg)^2$.

Problema. Invenire numerum x, ut $xx + 1$ dividi queat per $aa + bb$.

Solutio. Primo patet, si satisfaciat $x = \alpha$, etiam satisfacturum esse $x = m(a^2 + b^2) \pm \alpha$. Deinde sumto $x = \frac{a}{b}$ satisfacit, quia fit $xx + 1 = \frac{aa + bb}{bb}$. Ponatur ergo $x = \frac{m(aa + bb) \pm a}{b}$, qui ergo numerus debet esse integer. Quia vero est $x = mb + \frac{a(ma \pm 1)}{b}$, debet esse $ma \pm 1$ divisibile per b. Quaeratur fractio $\frac{\alpha}{\beta}$ fractioni $\frac{a}{b}$ proxime aequalis, quod fit si $\alpha b - \beta a = \pm 1$. Sumatur ergo $m = \beta$ eritque $x = \beta b + \frac{a(\beta a \pm 1)}{b}$. Cum igitur sit $\beta a \pm 1 = \alpha b$, erit $x = \beta b + \alpha a$. In genere ergo $x = m(aa + bb) \pm (\alpha a + \beta b)$.

Problema. Invenire numerum x, ut formula $x^4 + 1$ divisibilis fiat per $a^4 + b^4$.

Solutio. Primo patet hoc fieri, si $x = \frac{a}{b}$. Ponatur ergo $x = \frac{m(a^4 + b^4) \pm a}{b} = mb^3 + \frac{a(ma^3 \pm 1)}{b}$. Quaeratur nunc fractio $\frac{\alpha}{\beta}$ proxime aequalis huic $\frac{a^3}{b}$, ita ut sit $\beta a^3 - \alpha b = \pm 1$, et sumatur $m = \beta$ eritque $x = \beta b^3 + \alpha a$, generaliter ergo

$$x = m(a^4 + b^4) \pm (\beta b^3 + \alpha a).$$

Potuissemus etiam ponere

$$x = \frac{aa}{bb}, \text{ fiat igitur } x = \frac{m(a^4 + b^4) \pm aa}{bb} = mb^2 + \frac{a^2(ma^2 \pm 1)}{bb}.$$

Quaeratur nunc fractio $\frac{\gamma}{\delta}$ proxime aequalis ipsi $\frac{aa}{bb}$, ut sit $\gamma bb - \delta aa = \pm 1$, sumaturque $m = \delta$ eritque

$$x = \delta bb + \gamma aa \quad \text{et generaliter} \quad x = m(a^4 + b^4) \pm (\delta bb + \gamma aa),$$

quod ergo debet esse quadratum, cujus radix jam ante est assignata, unde patet hanc formulam

$$m(a^4 + b^4) \pm (\gamma aa + \delta bb)$$

semper ad quadratum reduci posse, sive si omnia quadrata dividantur per divisorem $a^4 + b^4$, inter residua certe occurret tam $\gamma aa + \delta bb$ quam $-\gamma aa - \delta bb$. Sit $a = 3$ et $b = 2$ et quaeratur fractio $\frac{\alpha}{\beta}$ proxime aequalis ipsi $\frac{27}{2}$, erit $\alpha = 13$ et $\beta = 1$, hinc ergo erit $x = 97m \pm 47$. Potuissemus quoque facere $a = 2$ et $b = 3$, et fractio $\frac{\alpha}{\beta}$ proxime $= \frac{8}{3}$, quod fit sumendo $\alpha = 3$ et $\beta = 1$, tum erit $x = 97m \pm 33$. Patet ergo tam $47^4 + 1$ quam $33^4 + 1$ divisibile esse per 97.

A. m. T. II. p. 147.

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e) *Diversa.*

24.

(*Krafft.*)

Formulae in producendis numeris primis foecundae:

I. $x^2 + x + 17$ dat: 17, 19, 23, 29, 37, 47, 59, 73, 89, 107, 127, 149, 173, 199, 227, 257, 289.

II. $x^2 + x + 41$ dat: 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281.

Jam autem demonstratum est, nullam dari hujusmodi formulam algebraicam, cujus omnes plane termini sint numeri primi.

A. m. T. I. p. 234.

25.

(*N. Fuss I.*)

Theorema. Haec formula $x^{2n} + x^n + 1$ semper est divisibilis per $xx + x + 1$, dummodo n non sit multiplum ternarii.

Demonstratio. Si enim illa formula multiplicetur per $x^n - 1$, productum $x^{3n} - 1$ semper est divisibile per $x^3 - 1$, ideoque etiam per $xx + x + 1$; quia ergo multiplicator $x^n - 1$ non est divisibilis, necesse est ipsam formulam esse divisibilem. Q. e. d.

Theorema. Haec formula $x^{4n} + x^{3n} + x^{2n} + x^n + 1$ semper est divisibilis per $x^4 + x^3 + x^2 + x + 1$, dummodo exponens n non fuerit multiplum ipsius 5.

Demonstratio similis praecedenti.

Theorema. Si capiatur angulus $\vartheta = \left(\frac{m}{n \pm 1}\right) 360^0$, haec formula $x^{2n} - 2x^n \cos\vartheta + 1$ semper est divisibilis per hanc $xx - 2x\cos\vartheta + 1$. A. m. T. I. p. 285.

26.

Theorema, cujus demonstratio etiamnunc desideratur. Si haec formula $4mnk + maa + nbb$ fuerit numerus primus, puta P, tum semper assignari possunt numeri x et y, ut fiat $mxx + nyy = P$.

Sit $m = 3$, $n = 2$, $a = 1$ et $b = 1$, erit $maa + nbb = 5$ et $4mnk + 5 = 24k + 5$. Sumatur $k = 2$, erit $P = 53$ et esse debebit $3xx + 2yy = 53$, sit $x = 1$ et $y = 5$. Plerumque quidem tales numeri pro x et y dantur integri, interdum tamen non nisi fractos assignare licet, veluti si fuerit $m = 7$ et $n = 2$, praeterea vero $a = 1$ et $b = 1$, ita ut sit $P = 56k + 9$, unde sumto $k = 4$ fit $P = 233$, qui numerus in integris esse nequit $= 7xx + 2yy$. At si capiatur $x = \frac{5}{3}$, erit $233 = \frac{175}{9} + 2yy$, ergo $2yy = \frac{1922}{9}$, ergo $y^2 = \frac{961}{9}$ et $y = \frac{31}{3}$. A. m. T. I. p. 300.

27.

Theorema. Non dantur tria biquadrata, quorum summa esset divisibilis vel per 5, vel per 29, quae sola excipiuntur. A. m. T. II. p. 161.

28.

Observatio. Proposito quocunque numero primo $p = 2n + 1$, omnes numeri eo minores, qui sunt $1, 2, 3, 4 \ldots 2n$, semper tali ordine disponi possunt, ut certis multiplis ipsius p aucti, progressionem geometricam constituant, sive tales assignari possunt numeri x, ut progressionis geometricae $1, x, x^2, x^3, x^4 \ldots$ si singuli termini per p divisi deprimantur, omnes numeri ipso p minores prodeant, uti ex sequentibus exemplis patebit. Notetur autem potestatem x^{2n} hoc modo semper dare unitatem, propterea quod $x^{2n} - 1$ semper per p dividi potest, unde sequentes potestates x^{2n+1}, x^{2n+2}, x^{2n+3}, etc. eosdem reproducunt numeros, ut ab initio.

I. Sit $p = 3$ et $n = 1$ et progressio geometrica erit $1, x, xx$. Sumto ergo $x = 2$, progressio geometrica erit 1, 2, 1, 2, 1, 2, 1, 2, etc.

II. Sit $p = 5$ et $n = 2$ et progressio geometrica $1, x, x^2, x^3$, etc. Hinc sumto $x = 2$ habetur

1, 2, 4, 3, 1, 2, 4, 3, 1, etc.

sumto autem $x = 3$, erit ea 1, 3, 4, 2, 1, 3, 4, 2, 1, etc.

III. Sit $p = 7$ et $n = 3$, erit progressio $1, x, x^2, x^3, x^4$, etc. Hinc sumto $x = 2$, erit ea 1, 2, 4, unde patet, hinc tantum terminos pares oriri, unde x ita sumi debet, ut fiat $xx - 2 = 7m$, ideoque $x = 3$ et progressio geometrica erit 1, 3, 2, 6, 4, 5, 1, 3, 2, etc. Loco x autem etiam sumi posset alia potestas si modo λ ad 6 fuerit primus, ita sumto $\lambda = 5$, capi poterit $x = 5$, unde oritur 1, 5, 4, 6, 2, 3, 1, quae est prioris retrograda. Semper autem series retrograda aeque satisfacit.

IV. Sit $p = 11$ et $n = 5$, at sumto $x = 2$ erit progressio

$$\begin{array}{cccccccccc} 1, & 2, & 4, & 8, & 5, & 10, & 9, & 7, & 3, & 6, \; 1 \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

Hic autem primo etiam retrograda valet:

$$1,\; 6,\; 3,\; 7,\; 9,\; 10,\; 5,\; 8,\; 4,\; 2,\; 1.$$

Praeterea posito $x = x^3$, x^7, x^9, qui numeri sunt 8, 7 et 6, tum erit progressio:

$$1,\; 8,\; 9,\; 6,\; 4,\; 10,\; 3,\; 2,\; 5,\; 7,\; 1,$$

cujus retrograda oritur sumto $x = 7$.

V. Sit $p = 13$ et $n = 6$, at sumto $x = 2$, erit progressio

$$\begin{array}{ccccccccccccc} 1, & 2, & 4, & 8, & 3, & 6, & 12, & 11, & 9, & 5, & 10, & 7, & 1 \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \end{array}$$

Dein pro x sumi possunt numeri 6, 11, 7. Sumto igitur $x = 6$, ea erit

$$1,\; 6,\; 10,\; 8,\; 9,\; 2,\; 12,\; 7,\; 3,\; 5,\; 4,\; 11,\; 1.$$

Reflexiones generales. 1. Perpetuo hic potestati x^n conveniet numerus $2n$. Cum enim ejus quadratum x^{2n} det 1, erit $x^n = \sqrt{1}$, ergo $x^n = -1 = p - 1 = 2n$.

2. Si potestati x^λ respondeat numerus a, tum potestati $x^{n+\lambda}$ respondebit numerus $p - a = 2n + 1 - a$. Cum enim sit

$$x^\lambda = +a \;\text{ et }\; x^n = -1, \;\text{ erit }\; x^{\lambda+n} = -a = p - a = 2n + 1 - a.$$

Sufficit ergo seriem usque ad medium $2n$ continuare, quia sequentes sunt complementa priorum.

3. Posito $x = a$, ejus reciprocum vocemus $\frac{1}{a}$, sive $\frac{mp+1}{a}$, ut prodeat numerus integer, quem designemus per α, ut sit $\alpha = \frac{1}{a}$, eodemque modo $\beta = \frac{1}{b}$, $\gamma = \frac{1}{c}$ etc. Ita casu $p = 13$, si fuerit

$$a = 2,\; 3,\; 4,\; 5,\; 6,\; 7, \;\text{ etc.}$$
$$\text{erit}\quad \alpha = 7,\; 9,\; 10,\; 8,\; 11,\; 2.$$

Notetur enim complementorum reciproca etiam esse complementa.

4. Constitutis his reciprocis, si fuerit $x^\lambda = a$, tum erit $x^{2n-\lambda} = \alpha$, propterea quod productum potestatum est $x^{2n} = 1$, ideoque $a\alpha = 1$. Deinde vidimus esse $x^{n+\lambda} = p - a$, erit igitur $x^{n-\lambda} = p - \alpha$, ita ut, cognito uno termino, simul quatuor innotescant, quod exemplis illustretur.

Sit $p = 19$, $n = 9$.

potestates	numeri	potestates	numeri
1	1	x^9	18
x^1	a	x^{10}	$19 - a$
x^2	b	x^{11}	$19 - b$
x^3	c	x^{12}	$19 - c$
x^4	d	x^{13}	$19 - d$
x^5	$p - \delta$	x^{14}	δ
x^6	$p - \gamma$	x^{15}	γ
x^7	$p - \beta$	x^{16}	β
x^8	$p - \alpha$	x^{17}	α

Hic igitur notetur esse debere $b = a^2$, $c = a^3$, $d = a^4$, etc. Si ergo sumatur $a = 2$, erit $b = 4$, $c = 8$, $d = 16$, tum vero $\alpha = 10$, $\beta = 5$, $\gamma = 12$, $\delta = 6$, unde formatur haec progressio geometrica

*

$$\begin{array}{cccccccccccccccccc} 1, & 2, & 4, & 8, & 16, & 13, & 7, & 14, & 9, & 18, & 17, & 15, & 11, & 3, & 6, & 12, & 5, & 10 \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{array}$$

Loco x autem quoque sumi possunt numeri potestati x^λ respondentes, si modo λ ad 18 fuerit primus. Cum autem $18 = 2.3^2$, multitudo numerorum ad 18 primorum est 6 et valores pro λ sunt 1, 5, 7, 11, 13, 17, unde pro x sumi possunt hi numeri 2, 13, 14, 10, 3, 15, unde sex progressiones geometricas formare licet, quarum tres erunt priorum retrogradae.

Exemplum. Sit $p = 41$ et $n = 20$, et sumatur $x = 2$, unde progressio geometrica oritur

$$\begin{array}{ccccccccccc} 1, & 2, & 4, & 8, & 16, & 32, & 23, & 5, & 10, & 20, & 40 \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

unde pro x^{20} prodit $+1$, ita ut sit $2^{20} = +1$, unde patet esse $xx = 2$, ideoque $x = \sqrt{(2 + 41m)} = 17$ (posito $m = 7$). Factum hinc est sequens schema:

41)

0	1							
1	17	11	11	21	23	31	30	
2	2	12	23	22	39	32	18	
3	34	13	22	23	7	33	19	
4	4	14	5	24	37	34	36	
5	27	15	3	25	14	35	38	
6	8	16	10	26	33	36	31	
7	13	17	6	27	28	37	35	
8	16	18	20	28	25	38	21	
9	26	19	12	29	15	39	29	
10	32	20	40	30	9	40	1	

Jam ad 40 valores ipsius λ primi sunt 1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39, unde pro x accipi poterunt sequentes numeri: 17, 34, 13, 26, 11, 22, 6, 12, 24, 7, 28, 15, 30, 19, 35, 29. Si sumsissemus $x = 3$, prodiisset progressio

$$\begin{array}{ccccc} 1, & 3, & 9, & 27, & 40, \\ & 1 & 2 & 3 & 4 \end{array}$$

sequeretur $3^4 = x^{20}$, ergo $3 = x^5$. Supra autem invenimus esse $2 = x^2$, ergo $4 = x^4$, unde oritur $x = \frac{3}{4}$, sive $x = \frac{3 + 41m}{4} = 11$. Cum igitur formula $a^{40} - 1$ semper dividi queat per 41 h. e. si fuerit $a = x^\lambda$, denotante λ numero quocunque, ista formula $b^{20} - 1$ dividi poterit per 41, si fuerit $b = aa$, h. e. si fuerit $b = x^{2\lambda}$. Quoniam igitur $a^{40} - 1 = (a^{20} - 1)(a^{20} + 1)$, prior vero factor $a^{20} - 1$ divisibilis sit casibus $a = x^{2\lambda}$, sequitur reliquis casibus, h. e. casibus $a = x^{2\lambda + 1}$, formulam $a^{20} + 1$ divisibilem esse per 41, h. e. si fuerit

$a = 17,\ 34,\ 27,\ 13,\ 26,\ 11,\ 22,\ 3,\ 6,\ 12,\ 24,\ 7,\ 14,\ 28,\ 15,\ 30,\ 19,\ 38,\ 35,\ 29.$

Porro quia $a^{20} - 1$ divisibile per 41 si $a = x^{2\lambda}$, erit $b^{10} - 1$ divisibile per 41 si $b = x^{4\lambda}$; hinc sequitur formulam $b^{10} + 1$ divisibilem esse per 41 si $b = x^{4\lambda + 2}$. Porro $a^8 - 1$ divisibile per 41 si $a = x^{5\lambda}$. At $a^4 - 1$ divisibile per 41 si $a = x^{10\lambda}$, ergo $a^4 + 1$ divisibile per 41 si $a = x^5$, x^{15}, x^{25}, x^{35}, etc. h. e. si $a = x^{10\lambda + 5}$. Consequenter formula $a^4 + 1$ divisibilis per 41 his casibus: $a = 27$, 3, 14, 38. Porro quia $a^4 - 1$ divisibile per 41, si $a = x^{10\lambda}$, et $a^2 - 1$ per 41, si $a = x^{20\lambda}$, sequitur fore $a^2 + 1$ divisibile per 41, si a fuerit $x^{20\lambda + 10}$, qui casus sunt $a = 32$ et 9, hoc est in genere si $a = 41m \pm 9$.

A. m. T. II. p. 170. 171.

29.

Regula facilis explorandi numeros formae $4m + 1$, qui desinunt vel in 3, vel in 7, utrum sint primi, nec ne? Sit N talis numerus, et a $2N$ subtrahatur quadratum proxime minus, desinens in 5, cujus radix sit $5n$, sitque residuum $= R$. Ad hoc continuo addantur numeri $100(n - 1)$, $100(n - 3)$, $100(n - 5)$, $100(n - 7)$, etc.

ut prodeant sequentes numeri: R, $R+100(n-1)$, $R+200(n-2)$, $R+300(n-3)$, etc. Quodsi jam inter hos numeros unicus occurrat quadratus, tum numerus propositus N certo est primus, vel per hoc quadratum divisibilis; sin autem vel nullus occurrat quadratus, vel duo pluresve, tum numerus N non est primus. Sit $N=637$, erit $2N=1274$. Proximum quadratum in 5 desinens erit $1225=5^2.7^2$, ideoque $n=7$ et numeri addendi numero $R=49$ erunt 600, 400, 200, unde prodit 649, 1049, 1249, inter quos numeros unicum occurrit quadratum 49, unde numerus propositus vel erit primus, vel per 49 divisibilis.

Sit $N=1073$, erit $2N=2146$; proximum quadratum in 5 desinens $=2025=5^2.9^2$, unde $n=9$ et $R=121$. Numeri addendi sunt 800, 600, 400, 200 eritque 921, 1521, 1921, 2121, inter quos sunt quadrata 121 et 1521, ideoque numerus non est primus.

Sit $N=697$, $2N=1394$, proximum quadratum in 5 desinens $1225=5^2.7^2$, $R=169$ et numeri addendi 600, 400, 200; inde prodeunt 769, 1169, 1369. Hic duo occurrunt quadrata $169=13^2$ et $1369=37^2$, unde numerus ille non est primus, est enim $697=17.41$.

Sit $N=1697$, erit $2N=3394$, proximum quadratum $3025=5^2.11^2$, hinc $R=369$ et numeri addendi 1000, 800, 600, 400, 200; hinc prodeunt 1369, 2169, 2769, 3169, 3369, inter quos unicum est quadratum $1369=37^2$, unde numerus est primus, quandoquidem per 1369 non est divisibilis.

A. m. T. II. p. 188.

30.

(*Golovin.*)

Tabula exhibens per intervallum 420 omnes numeros, qui restant, deletis numeris sequentium formarum: $3n+2$, $4n+3$, $5n+1$, $5n+4$, $7n+3$, $7n+5$ et $7n+6$.

0	78	148	232	310	373
18	85	162	238	312	378
22	88	165	240	322	382
25	93	168	252	330	385
28	100	172	253	333	393
30	102	177	268	337	400
37	105	190	270	340	403
42	112	193	273	345	408
57	120	205	277	350	417
58	130	210	280	352	420
60	133	217	282	357	
70	142	225	288	358	
72	145	228	298	372	

A. m. T. II. p. 195.

31.

(*N. Fuss I.*)

Theoremata numerica.

NB. Denotet hoc signum :: divisibile, ita ut $a::p$ denotet, numerum a per p esse divisibilem.

Theorema fundamentale, a me olim demonstratum. Proposito numero quocunque P, atque ab 1 usque ad P reperiantur π numeri ad P primi, qui scilicet cum eo praeter unitatem nullum habeant factorem communem: tum semper $(a^{\pi}-1)::P$. Hinc fluunt sequentia theoremata.

I. Si fuerit p numerus primus, cum semper sit $(a^p-a)::p$, si fuerit $a=b^p$ erit $(a^p-a)::p^2$. At si fuerit $a=b^{p^p}$, erit $(a^p-a)::p^3$. Et in genere si $a=b^{p^n}$ erit $(a^p-a)::p^{n+1}$.

II. Si p, q, r, s etc. fuerint primi inter se diversi, fueritque $a = b^{q-1}$ erit $(a^p - a) :: pq$. Porro si $a = b^{(q-1)(r-1)}$ erit $(a^p - a) :: pqr$ et ita porro.

III. Si fuerit tam $(x^m - y^m) :: P$ quam $(x^n - y^n) :: P$, sitque $m > n$, erit quoque $(x^{m-n} - y^{m-n}) :: P$, si quidem x et y sint numeri inter se primi.

Demonstratio. Posterior formula ducta in x^{m-n} a priore subtrahatur, erit residuum

$$x^{m-n}y^n - y^m = y^n(x^{m-n} - y^{m-n}),$$

quod ergo etiam est divisibile per P, et quia y^n non est divisibile, necesse est ut $(x^{m-n} - y^{m-n}) :: P$.

IV. Si ut ante tam $(x^m - y^m) :: P$ quam $(x^n - y^n) :: P$ atque inter numeros m et n maximus communis divisor fuerit Δ tum etiam $(x^\Delta - y^\Delta) :: P$.

Demonstratio. Ponatur $m = \mu\Delta$ et $n = \nu\Delta$, et quia Δ est maximus communis divisor, erunt μ et ν primi inter se. Dari igitur poterunt numeri α et β, ut sit $\alpha\mu - \beta\nu = 1$. Hinc igitur quoque erit $(x^{\alpha m} - y^{\alpha m}) :: P$, similique modo $(x^{\beta n} - y^{\beta n}) :: P$, unde per praecedens theorema erit $(x^{\alpha m - \beta n} - y^{\alpha m - \beta n}) :: P$. Est vero exponens $\alpha m - \beta n = \alpha\mu\Delta - \beta\nu\Delta = \Delta$, consequenter erit $(x^\Delta - y^\Delta) :: P$.

A. m. T. III. p. 174, 175.

B. Partitio numerorum in summas polygonalium.

32.

(Léonard Euler.)

Caractère général pour juger, si un nombre entier quelconque N *est somme de trois triangles, tous les nombres plus petits étant tels.*

Soit $N - A$ un nombre moindre quelconque qui soit égal à ces trois triangles: $\Delta p + \Delta q + \Delta r$; ensuite prenant pour a et b des nombres quelconques et posant $A = ab$, s'il arrive que $p - q$, ou $p - r$, ou $q - r$ soit égal à $a - b$, alors le nombre proposé N sera somme de trois triangles, et un seul cas de a et b suffit pour cela.

(Lexell.)

Demonstration. Ayant posé $N - ab = \Delta p + \Delta q + \Delta r$, soit $p - q = a - b$, et pour cet effet mettons $p = x + a$ et $q = x + b$, de sorte que
$$N - ab = \Delta(x + a) + \Delta(x + b) + \Delta r.$$
Alors je dis qu'on aura
$$N = \Delta(x + a + b) + \Delta x + \Delta r;$$
car puisque
$$\Delta(x + a + b) = \frac{xx + 2(a + b)x + (a + b)^2 + x + a + b}{2},$$
on aura
$$N = \frac{1}{2}(xx + 2(a + b)x + (a + b)^2 + x + a + b) + \frac{xx + x}{2} + \Delta r.$$
Mais la première formule donne
$$N - ab = \frac{x^2 + 2ax + a^2 + x + a}{2} + \frac{x^2 + 2bx + b^2 + x + b}{2} + \Delta r$$
ce qui étant ôté de celle-là donne $ab = ab$, ce qu'il fallait démontrer.

Coroll. 1. Puisque $p - q = a - b$ et $p = x + a$ et $q = x + b$, on aura
$$x = p - a = q - b, \text{ donc } x + a + b = p + b = q + a;$$
par conséquent, dès qu'on aura $N - ab = \Delta p + \Delta(p - a + b) + \Delta r$, il s'en suit $N = \Delta(p + b) + \Delta(p - a) + \Delta r$.

Coroll. 2. Qu'on prenne $b = a$, et dès lors il arrive que $N - aa = \Delta p + \Delta p + \Delta r$, c'est à dire que si deux de ces triangles sont égaux entre eux, on en déduira
$$N = \Delta(p + a) + \Delta(p - a) + \Delta r.$$

Exemple. Prenons $N = 17$ et successivement $a = 1, 2, 3, 4$, etc. nous aurons

1. $a=1$ donc $17-1=10+3+3$ ou $17=10+6+1$
2. $a=2$, on aura $17-4=13=1+6+6$ donc $17=1+1+15$
3. $a=3$, on aura $17-9=8=6+1+1$ donc $17=6+10+1$
4. $a=4$, on aura $17-16=1=1+0+0$ donc $17=1+1+15$

Caractères semblables pour la résolution des nombres en quatre carrés.

Soit le nombre proposé $=N$ et un nombre plus petit quelconque $N-2ab$ qui soit $=pp+qq+rr+ss$. S'il arrive que $p-q=a-b$, ou bien $p=q+a-b$, $q=p-a+b$, alors on aura $N=(p+b)^2+(q-a)^2+rr+ss$; car celle-là

$$N=2ab+pp+pp-2p(a-b)+(a-b)^2+rr+ss$$

et celle-ci

$$N=pp+pp+2pb-2ap+bb+aa+rr+ss$$

sont évidemment égales.

Coroll. Prenant $b=a$, si parmi les quatre carrés dont la somme donne $N-2aa$, deux se trouvent égaux entre eux, de sorte que $N-2aa=2pp+rr+ss$, alors on aura

$$N=(p+a)^2+(p-a)^2+rr+ss.$$

Exemples. Soit proposé le nombre $N=71$ et soit

1. $a=1$, on aura $71-2=69=4+4+36+25$ d'où l'on conclut $71=9+1+36+25$.
2. Prenant $a=2$, on aura $71-8=63=9+9+9+36$, et partant $71=36+9+1+25$.
3. Soit $a=3$ et puisque $71-18=53=49+4+0+0$, il s'en suit $71=49+4+9+9$.
4. Soit $a=4$; puisque $71-32=39=36+1+1+1$, il y aura $71=36+1+25+9$.
5. Soit $a=5$; puisque $71-50=21=4+4+4+9$, donc $71=9+4+49+9$.

A. m. T. I. p. 92. 93.

33.

(N. Fuss I.)

Problema. Si omnes numeri minores quam N sint resolubiles in tres numeros trigonales, ipsum numerum N etiam in tres trigonales resolvere.

Solutio. Sint x, y et z radices numerorum trigonalium, quorum summa aequetur numero N, ita ut sit

$$N=\frac{xx+x}{2}+\frac{yy+y}{2}+\frac{zz+z}{2}.$$

Jam consideretur numerus minor quicunque $N-p$, pro quo radices trigonalium sint a, b, c, ut sit

$$N-p=\frac{aa+a}{2}+\frac{bb+b}{2}+\frac{cc+c}{2}.$$

Sumamus autem hic esse $b=a+d$; tum vero statuatur $z=c$, ita ut esse debeat

$$\frac{xx+x}{2}+\frac{yy+y}{2}=\frac{aa+a}{2}+\frac{bb+b}{2}+p.$$

Fiat nunc $x=a-n$ et $y=b+n$ eritque

$$xx+x=aa-(2n-1)a+n(n-1) \text{ et } yy+y=bb+(2n+1)b+n(n+1),$$

quibus valoribus substitutis prodit

$$2p=2bn-2an+2nn, \text{ sive } p=(b-a)n+nn.$$

Cum igitur sit $b=a+d$ ideoque $b-a=d$, erit $p=dn+nn$. Hinc pro variis valoribus litterarum d et n littera p sequentes accipiet valores. Sit primo $n=1$ erit $p=d+1$

existente $n=2$ fiet $p=2d+4$

$n=3$ $p=3d+9$

$n=4$ $p=4d+16$

$n=5$ $p=5d+25$.

Inde pro p sequentes oriuntur valores: $d = -2, -1, 0, 1, 2, 3, 4, 5$

$$n = 1: \quad p = 0, 1, 2, 3, 4, 5, 6$$
$$n = 2: \quad p = 2, 4, 6, 8, 10, 12, 14$$
$$n = 3: \quad p = 3, 6, 9, 12, 15, 18, 21, 24$$
$$n = 4: \quad p = 8, 12, 16, 20, 24, 28, 32, 36$$
$$\text{etc.} \qquad \text{etc.}$$

Unde pro diversis valoribus ipsius p resolutio numeri N in tres trigonales succedet, si fuerit pro numeris minoribus

$$\begin{array}{llll}
 & & \text{tum erit} & \text{et} \\
N-1: & b = a & x = a-1 & y = b+1 \\
N-2 & \left\{\begin{array}{l} b = a+1 \\ b = a-1 \end{array}\right. & \begin{array}{l} x = a-1 \\ x = a-2 \end{array} & \begin{array}{l} y = b+1 \\ y = b+2 \end{array} \\
N-3 & \left\{\begin{array}{l} b = a+2 \\ b = a-2 \end{array}\right. & \begin{array}{l} x = a-1 \\ x = a-3 \end{array} & \begin{array}{l} y = b+1 \\ y = b+3 \end{array} \\
N-4 & \left\{\begin{array}{l} b = a+3 \\ b = a \\ b = a-3 \end{array}\right. & \begin{array}{l} x = a-1 \\ x = a-2 \\ x = a-4 \end{array} & \begin{array}{l} y = b+1 \\ y = b+2 \\ y = b+4 \end{array}
\end{array}$$

His positis ambarum x et y inventio succedet, si inter ternas radices a, b, c, primo pro numero $N-1$ fuerit $b = a$, sive si duae fuerint aequales. Secundo si pro numero $N-2$ fuerit vel $b = a+1$, vel $b = a-1$, hoc est si differentia fuerit inter binas $= 1$, tum vero duplex solutio locum habebit. Tertio si pro $N-3$ fuerit vel $b = a+2$, vel $b = a-2$, hoc est si binae radices binario discrepent. Quarto si pro numero $N-4$ fuerit vel $b = a+3$, vel $b = a$, vel $b = a-3$, hoc est si differentia inter binas radices fuerint vel $= 0$, vel $= 3$. Quinto si pro numero $N-5$ fuerit $b = a \pm 4$ h. e. si differentia inter binas radices fuerit $= \pm 4$. Sexto si pro numero $N-6$ fuerit vel $b = a \pm 5$, vel $b = a \pm 1$, hoc est si inter radices binas occurrat differentia vel 1, vel 5. Quamobrem si demonstrari posset, semper unum saltem horum casuum locum habere debere, tum demonstratum foret omnes numeros esse summas trium trigonalium. Quod cum de minoribus numeris certum per se sit, pro majoribus autem continuo plures casus examinandi occurrant, eo minus dubitari potest, quin resolutio semper sit locum habitura, idque plerumque pluribus modis, quo accedit, quod pro majoribus numeris fere omnes numeri $N-p$ pluribus modis in tres trigonales resolvi possint. Quod quo clarius pateat has resolutiones ab ipso initio numerorum secundum ternas radices contemplemur:

numeri	radices a, b, c	vel radices	vel radices
1	0, 0, 1		
2	0, 1, 1		
3	1, 1, 1	0, 0, 2	
4	0, 1, 2		
5	1, 1, 2		
6	0, 2, 2	0, 0, 3	
7	1, 2, 2	0, 1, 3	
8	1, 1, 3		
9	2, 2, 2	0, 2, 3	
10	0, 0, 4	1, 2, 3	
11	0, 1, 4		
12	0, 3, 3	1, 1, 4	2, 2, 3

Hinc ergo examinemus numerum $N = 13$ et habebimus

	a,	b,	c
pro $N-1 = 12$:	0,	3,	3
	1,	1,	4
	2,	2,	3
pro $N-2 = 11$:	0,	1,	4
pro $N-3 = 10$:	0,	0,	4
	1,	2,	3
pro $N-4 = 9$:	2,	2,	2
	0,	2,	3
pro $N-5 = 8$:	1,	1,	3
etc.		etc.	

Problema. Si omnes numeri minores quam N fuerint summae quatuor quadratorum, ipsum numerum N in quatuor quadrata resolvere.

Solutio. Sint pro numero quocunque minore $N-p$ quadratorum radices a, b, f, g, unde pro numero N statuantur radices x, y et f, g, ac ponatur $x=a+\alpha$ et $y=b+\beta$, eritque ab hoc $N-p$ subtracto

$$p=2a\alpha+\alpha\alpha+2b\beta+\beta\beta.$$

Jam sumatur $\alpha=-n$ et $\beta=+n$, ut fiat $p=2n(b-a)+2nn$; quare si fuerit $b=a+d$, habebitur $p=2(nd+nn)$, qui numerus duplo major est quam casu praecedente, pro numeris trigonalibus; unde eadem criteria locum habebunt, quae ante, si modo numerus p duplo major capiatur. Ita resolutio numeri $N-p$ succedet

si pro numero	$N-2$	fuerit $b=a$	tum erit	$x=a-1$,	$y=b+1$
	$N-4$	$b=a+1$		$x=a-1$,	$y=b+1$
	$N-6$	$b=a+2$		$x=a-1$,	$y=b+1$
	$N-8$	$b=a+3$		$x=a-1$,	$y=b+1$
		$b=a$		$x=a-2$,	$y=b+2$
	$N-10$	$b=a+4$		etc.	etc.
	etc.	etc.			

Hic ergo patet, pro hoc casu numerum criteriorum esse duplo majorem quam casu praecedente: Verum quia hic quatuor occurrunt radices, etiam hic multo probabilius est, inter quaternas radices occurrere duas, quarum differentia sit vel 0, vel 1, vel 2, vel 3, vel etc. Quin etiam plerique numeri pluribus modis in quatuor quadrata resolvi poterunt, unde hoc judicium aeque certum esse potest ac praecedens.

Problema. Si omnes numeri minores quam N fuerint resolubiles in quinque numeros pentagonales, ipsum numerum N in tales partes resolvere.

Solutio. Sint pro numero $N-p$ radices quinque pentagonalium a, b, f, g, h, unde pro ipso numero N statuantur quinque radices x, y, f, g, h, ac ponatur $x=a+\alpha$ et $y=b+\beta$, eritque

$$\frac{3xx-x}{2}-\left(\frac{3aa-a}{2}\right)=\frac{6a\alpha+3\alpha\alpha-\alpha}{2} \quad \text{et} \quad \frac{3yy-y}{2}-\left(\frac{3bb-a}{2}\right)=\frac{6b\beta+3\beta\beta-\beta}{2},$$

$$\text{unde fiet} \qquad p=3a\alpha+\frac{3\alpha\alpha-\alpha}{2}+3b\beta+\frac{3\beta\beta-\beta}{2}.$$

Sumatur nunc $\alpha=-n$ et $\beta=+n$ eritque $p=3n(b-a)+3nn$; quare si fuerit $b=a+d$, fiet $p=3(nd+nn)$, ita ut hoc casu p sit triplo majus quam pro trigonalibus, unde eadem criteria locum habebunt, si modo ipso p valor triplo major tribuatur; hinc igitur resolutio semper succedet

$$\begin{array}{lll} \text{si pro numero} & N-3 & \text{fuerit } b=a \\ & N-6 & \qquad b=a+1 \\ & N-9 & \qquad b=a+2 \\ & N-12 & \left\{\begin{array}{l} b=a+3 \\ b=a \end{array}\right. \end{array}$$

ex quibus criteriis, si unicum tantum locum habuerit, resolutio numeri N certe succedit. Hic quidem triplo pauciora habentur criteria. Verum inter quinque radices reperientur binae, quarum differentia sit vel 0, vel 1, vel 2, vel 3, etc. Praeterea vero etiam plerique numeri multo pluribus modis in quinque pentagonales resolvi possunt.

Problema generale circa numeros polygonales quoscunque, quorum laterum numerus sit $=\pi$.

Si omnes numeri minores quam N *resolvi queant in π numeros polygonales, etiam ipsum numerum* N *in tales resolvere.*

Solutio. Sint pro numero quocunque minore, $N-p$, radices polygonalium a, b, f, g, h, i, k, etc. Tum vero pro ipso numero N radices x, y, f, g, h, i, k, etc. Sit autem in genere $x=a+\alpha$, $y=b+\beta$, et quia radicis x numerus polygonalis est

$$\frac{1}{2}(\pi-2)xx-\frac{1}{2}(\pi-4)x,$$

posito $x=a+\alpha$, iste numerus polygonalis erit

$$\frac{1}{2}(\pi-2)aa+(\pi-2)a\alpha+\frac{1}{2}(\pi-2)\alpha\alpha-\frac{1}{2}(\pi-4)a-\frac{1}{2}(\pi-4)\alpha,$$

unde si subtrahatur polygonalis ipsius a, remanet

$$(\pi-2)a\alpha+\frac{1}{2}(\pi-2)\alpha\alpha-\frac{1}{2}(\pi-4)\alpha;$$

hinc ergo si $N-p$ ab N subtrahatur, relinquetur

$$(\pi-2)a\alpha+\frac{1}{2}(\pi-2)\alpha\alpha-\frac{1}{2}(\pi-4)\alpha+(\pi-2)b\beta+\frac{1}{2}(\pi-2)\beta\beta-\frac{1}{2}(\pi-4)\beta.$$

Sumatur nunc $\alpha=-n$ et $\beta=+n$ fietque $p=n(\pi-2)(b-a)+(\pi-2)nn$; quamobrem si fuerit $b=a+d$, erit

$$p=n(\pi-2)d+(\pi-2)nn=(\pi-2)(nd+nn);$$

ideoque $\pi-2$ vicibus major quam pro numeris trigonalibus; quocirca criteria ita se habebunt:

Si pro numero $N-(\pi-2)$, fuerit $b=a$

$N-2(\pi-2)$, $b=a+1$

$N-3(\pi-2)$, $b=a+2$

$N-4(\pi-2)$, $\begin{cases} b=a+3 \\ b=a \end{cases}$

etc. etc.

Nisi ergo omnia haec criteria fallant, numerus N certe in π numeros polygonales resolvi potest. Pro theoremate igitur Fermatii demonstrando requiritur, ut demonstretur, fieri omnino non posse, ut omnia plane haec criteria simul fallant.

Scholion. Quemadmodum haec criteria deducta sunt ex consideratione binarum radicum x et y, cum binis datis a et b collatarum, ita etiam simpliciora criteria exhiberi possunt, si unica radix x cum a comparetur, manente $y=b$. Tum igitur erit

$$p=(\pi-2)a\alpha+\frac{1}{2}(\pi-2)\alpha\alpha-\frac{1}{2}(\pi-4)\alpha;$$

quare si capiamus $a=0$, fiet

$$p=\frac{1}{2}(\pi-2)\alpha\alpha-\frac{1}{2}(\pi-4)\alpha.$$

Quod si ergo pro numero unico $N-p$ occurrat unica radix $=0$. resolutio etiam certe succedet. Quocirca dispiciendum erit, num pro aliquo horum numerorum ipso N minorum:

$$N-1,\quad N-\pi,\quad N-(3\pi-3),\quad N-(6\pi-8),\quad N-(10\pi-15)\text{ etc.}$$

inter ejus radices una occurrat $=0$. Quod si semel tantum evenerit, numerus N certe resolutionem admittet. Sin autem hoc criterium unquam succedat, tum demum superiora criteria examinari poterunt.

Scholion. Talia criteria possunt etiam derivari ex comparatione ternarum radicum x, y, z, ponendo $x=a+\alpha$, $y=b+\beta$ et $z=c+\gamma$, tum enim erit

$$p=(\pi-2)(a\alpha+b\beta+c\gamma)+\frac{1}{2}(\pi-2)(\alpha\alpha+\beta\beta+\gamma\gamma)-\frac{1}{2}(\pi-4)(\alpha+\beta+\gamma),$$

unde si sumatur $\alpha+\beta+\gamma=0$ simulque fuerit $a\alpha+b\beta+c\gamma=0$, obtinebitur

$$p=\frac{1}{2}(\pi-2)(\alpha\alpha+\beta\beta+\gamma\gamma).$$

Cum autem sit $y = -\alpha - \beta$, erit $a\alpha + b\beta - c\alpha - c\beta = 0$, unde fit

$$c = \frac{a\alpha + b\beta}{\alpha + \beta}; \text{ tum igitur erit } p = (\pi - 2)(\alpha\alpha + \alpha\beta + \beta\beta);$$

quam ob rem, si pro numero $N - p$ inter ejus radices, quarum numerus est π, tres reperiantur a, b, c, ita comparatae, ut sit $c = \frac{a\alpha + b\beta}{\alpha + \beta}$, tum resolutio certe succedet. Sumatur ex. gr. $\alpha = n$ et $\beta = n$, eritque $c = \frac{a + b}{2}$, sive $a + b = 2c$, vel $a = 2c - b$, ex qua conditione sequitur, numeros b, c, a esse in progressione arithmetica, quia hinc fiet $c - b = a - c$. Quare si pro numero $N - p = N - 3nn(\pi - 2)$ inter ejus radices ternae sint in progressione arithmetica, numerus N semper erit resolubilis. Similique modo multitudo criteriorum pro lubitu augeri poterit, quorum si unicum successerit, resolutio numeri N locum habebit. Totum ergo negotium huc est reductum, ut demonstretur, nunquam fieri posse, ut omnia ista criteria simul fallant. In quo negotio imprimis erit perpendendum: in omnibus numeris minoribus $N - p$ omnes plane combinationes radicum 0, 1, 2, 3, 4, 5, etc., quarum quidem numeri polygonales simul sumti numerum N non superant, occurrere; unde demonstrandum erit: fieri non posse, ut in omnibus his combinationibus omnia nostra criteria simul fallant. Tum vero etiam hoc erit perpendendum, in numeris minoribus pro N assumtis resolutionem semper locum habere, ita, ut demonstratio tantum pro majoribus numeris sit suscipienda; ubi non solum numerus criteriorum major evadet, sed etiam numerus omnium combinationum. Quodsi enim nostra criteria unquam fallerent, id maxime metuendum foret in numeris minoribus.

Aliud tentamen in theorema Fermatianum inquirendi.

Sit series numerorum polygonalium 0, 1, A, B, C, D, etc., ac posito numero laterum $= n + 2$, erit

$A = n + 2$, $B = 3n + 3$, $C = 6n + 4$, $D = 10n + 5$, $E = 15n + 6$, $F = 21n + 7$, $G = 28n + 8$, etc.

et in genere pro radice x numerus polygonalis

$$= \frac{1}{2} nxx - \frac{1}{2}(n - 2)x.$$

Quibus positis videamus, quot numeris hujus seriei opus sit ad singulos numeros producendos. Ac primo quidem ab 1 usque ad A quilibet numerus N, minor quam A, componitur ex N unitatibus, unde pro numeris ab 1 ad A ad summum opus est A. Nunc ad intervallum ab A ad B progrediamur, et quia $A + 1$ constat ex duobus, $A + 2$ ex tribus, $A + 3$ ex quatuor, usque ad $A + n + 1$ qui est primus qui postulat $n + 2$ partes, praecedentes vero omnes ex paucioribus constant; est vero $A + n + 1 = 2n + 3$, unde videtur sequentem numerum $2n + 4$ requirere $n + 3$, quia autem est $2n + 4 = 2A$, hic numerus tantum duos postulat; sequens igitur $2n + 5$ postulat 3, $2n + 6$ postulat 4, $2n + 7$ postulat 5 etc. et $2A + n$ postulat $n + 2$. Est vero $2A + n = 3n + 4$; at vero hic numerus est $B + 1$, ideoque tantum postulat duos; unde patet usque ad B unicum esse numerum scilicet $2n + 3$, qui $n + 2$ partes postulat, omnes reliqui pauciores. Nunc a B ad C progrediamur, ac manifestum est, hinc omnes numeros minores quam $B + 2n + 3$ ad summum requirere $n + 2$, numerus autem $B + 2n + 3 = 5n + 6$ videtur $n + 3$ partes requirere; est vero $5n + 6 = 3A + 2n = 4A + n - 2$, ubi $4A$ constat quatuor partibus et $n - 2$ ex $n - 2$ partibus, unde ipse numerus $4A + n - 2$ constat ex $n + 2$. Verum hic excipiendus est casus, ubi $n < 2$, quia $n < 2$ foret negativum: hoc autem casu numerus noster $B + 2n + 3$ fiet $= C - n + 2$; unde casu $n = 1$ erit $C + 1$, ideoque duabus tantum constat partibus. Casu autem $n = 2$ fit $5n + 6 = C$, ideoque ipse est numerus polygonalis; reliquis vero casibus, ubi $n > 2$, iste numerus $5n + 6$ secundus est, qui $n + 2$ partes postulat, dum minores omnes praeter $2n + 3$ paucioribus constant. Sequens autem numerus $5n + 7 = 2A + B$, ideoque tribus tantum constat partibus. Hunc sequens, $5n + 8$, constabit quatuor, ac tandem $5n + 7 + n - 1$ constabit ex $n + 2$; est vero $5n + 7 + n - 1 = C + 2$, ideoque constat tantum tribus. Nunc a C ad D progrediamur usque, ubi primum occurrit $C + 2n + 3$, qui dubius videri potest. Est vero

$$C + 2n + 3 = 8n + 7 = 2B + 2n + 1 = 2B + A + n - 1,$$

quarum partium numerus est $n+2$, qui ergo est tertius numerus $n+2$ partes postulans. Quia deinde ab $2n+3$ usque ad $5n+6$ omnes numeri postulant partes pauciores quam $n+2$, numerus sequens dubius erit $C+5n+6=11n+10$, qui autem jam superat D et ad sequens intervallum pertinet. Simili modo progredientes a D versus E, ubi primum numerum dubium reperimus $D+2n+3=12n+8=2C$, qui ergo duabus tantum constat partibus, unde ulterius progredi licet, usque ad $2C+n$, qui constabit ex $n+2$. Sequens est

$$2C+n+1=13n+9,$$

qui videtur $n+3$ partes requirere: est vero $13n+9=D+B+1$, qui ergo in tres partes resolvitur. Hinc progrediemur usque ad $14n+9=2C+2n+1=2C+A+n-1$, sicque partium numerus reducitur ad $n+2$; sequens vero numerus $14n+10=D+4n+5=D+B+A$ sicque tribus constat partibus, unde progredi licet usque ad $14n+10+n-1=15n+9$ quod jam superat . . . .

A. m. T. I. p. 336—340.

34.

(Lexell.)

Demonstratio sequens, ardua Viro Celeb. la Grange *debetur:*

Si fuerit $Aa=pp+qq+rr+ss$, ubi sumere licet $pp+qq$, ut cum a communem non habeat divisorem. Ponatur $pp+qq=t$ et $rr+ss=u$, ut sit $Aa=t+u$, et per t multiplicando $Aat=tt+tu$. Cum nunc sit $tu=(pp+qq)(rr+ss)$, erit summa duorum quadratorum; ergo ponatur $=xx+yy$, eritque $x=pr\pm qs$ et $y=ps\mp qr$, ita ut sit $Aat=tt+xx+yy$. Jam quaerantur numeri $\alpha, \beta, \gamma, \delta$, ut fiat

$$x=t\alpha+a\gamma \quad \text{et} \quad y=t\beta+a\delta,$$

quod cum infinitis modis fieri possit, casu simplicissimo α et β capere licebit minores quam $\frac{1}{2}a$, eritque

$$Aat=tt(1+\alpha\alpha+\beta\beta)+2at(\alpha\gamma+\beta\delta)+aa(\gamma\gamma+\delta\delta).$$

Debet ergo primum membrum $1+\alpha\alpha+\beta\beta$ factorem habere a, quia autem t ad a est primus, necesse est, ut $1+\alpha\alpha+\beta\beta$ divisibile sit per a; ponatur ergo $1+\alpha\alpha+\beta\beta=aa'$, ita ut nunc habeamus

$$At=a'tt+2t(\alpha\gamma+\beta\delta)+a(\gamma\gamma+\delta\delta),$$

quae per a' multiplicata fit $\quad Aa't=a'a'tt+2a't(\alpha\gamma+\beta\delta)+aa'(\gamma\gamma+\delta\delta),$

formula per t divisibilis. In ultimo membro loco aa' restituatur $1+\alpha\alpha+\beta\beta$, ut habeamus

$$Aa't=a'a'tt+2a't(\alpha\gamma+\beta\delta)+(\alpha\alpha+\beta\beta)(\gamma\gamma+\delta\delta)+\gamma\gamma+\delta\delta,$$

cujus formulae ad dextram tria priora membra manifesto reducuntur ad

$$(a't+\alpha\gamma+\beta\delta)^2+(\beta\gamma-\alpha\delta)^2$$

ita ut nunc habeamus $\quad Aa't=(a't+\alpha\gamma+\beta\delta)^2+(\beta\gamma-\alpha\delta)^2+\gamma\gamma+\delta\delta.$

Supra autem vidimus $\gamma\gamma+\delta\delta$ per t esse divisibile, unde etiam summam duorum priorum quadratorum

$$(a't+\alpha\gamma+\beta\delta)^2+(\beta\gamma-\alpha\delta)^2$$

per t divisibilem esse oportet, ita ut uterque quotus fiat summa duorum quadratorum, quare si faciamus

$$\frac{(a't+\alpha\gamma+\beta\delta)^2+(\beta\gamma-\alpha\delta)^2}{t}=p'p'+q'q' \quad \text{et} \quad \frac{\gamma\gamma+\delta\delta}{t}=r'r'+s's'$$

habebimus $Aa'=p'p'+q'q'+r'r'+s's'$ scilicet summae quatuor quadratorum. Hic vero imprimis notandum est fore $a'<a$. Cum enim sit $a'=\frac{1+\alpha\alpha+\beta\beta}{a}$, ac ut vidimus

$$\alpha<\frac{1}{2}a \quad \text{et} \quad \beta<\frac{1}{2}a, \quad \text{erit} \quad 1+\alpha\alpha+\beta\beta<1+\frac{1}{2}aa,$$

unde sequitur fore $a'<\frac{1}{2}a+\frac{1}{a}$, ideoque certe minor quam a, vel $a'<\frac{1}{2}a+1$. Consequenter si productum

Ad fuerit summa quatuor quadratorum, etiam hoc minus productum Aa' erit talis summa, hocque modo continuo ad minora hujusmodi producta Aa'', Aa''' etc. progredi licet, sicque tandem necessario pervenietur ad productum $A.1$, ideoque A summa quatuor quadratorum. Quae est demonstratio insignis illius et demonstratu difficillimi theorematis, quod si quispiam numerus A fuerit divisor summae quatuor quadratorum, quae quidem inter se factorem non habeant communem, tum ipsum numerum A fore quoque summam quatuor quadratorum, seu, quod eodem redit, summam quatuor quadratorum alios non admittere divisores nisi qui ipsi sint summae quatuor quadratorum.

(*Krafft.*)

Ejusdem theorematis demonstratio mea (*scil.* Euleri).

Lemma I. Si N et n fuerint numeri inter se primi, tum quicunque numerus A ita potest repraesentari, ut sit $A = Nx + ny$, et quia hoc infinitis modis fieri potest, dabitur casus, quo $x < \frac{1}{2}n$. Sit enim

$$A = Nf + ng, \quad \text{erit etiam} \quad A = N(f - \lambda n) + n(g + \lambda N),$$

unde, quantumvis magnus fuerit numerus f, ita accipere licebit λ, ut fiat $f - \lambda n < n$; tum vero si f etiamnunc fuerit $> \frac{1}{2}n$, tum $f - n$ certe minus erit, quam $\frac{1}{2}n$: hic enim ipsi numeri spectantur, et perinde est, sive sint positivi, sive negativi.

Lemma II. Productum ex binis numeris, quorum uterque est summa quatuor quadratorum, in quatuor quadrata resolvere.

Sit hujusmodi productum $(a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$,

ac sumatur

$$A = + a\alpha + b\beta + c\gamma + d\delta, \quad B = + a\beta - b\alpha - c\delta + d\gamma$$
$$C = + a\gamma + b\delta - c\alpha - d\beta, \quad D = + a\delta - b\gamma + c\beta - d\alpha,$$

quorum quadrata si invicem addantur, omnia duplicia producta ex binis se mutuo tollent, et quodvis quadratum latinarum literarum multiplicatur per omnia quadrata litterarum graecarum, atque hinc manifesto fiet

$$(a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = A^2 + B^2 + C^2 + D^2.$$

Theorema. Si numerus primus N fuerit divisor summae quatuor quadratorum $P^2 + Q^2 + R^2 + S^2$, tum ille ipse numerus N erit summa quatuor quadratorum.

Demonstratio. Quantumvis magni fuerint numeri P, Q, R, S, eos semper deprimere licebit infra $\frac{1}{2}N$; nam si loco P scribatur $P - \lambda N$, summa illa etiamnunc erit per N divisibilis, quod etiam de reliquis Q, R et S valet, sicque singulae radices infra N deprimentur, ac si P adhuc majus fuerit quam $\frac{1}{2}N$, ejus loco scribatur $N - P$, quod certo erit minus quam $\frac{1}{2}N$. Sit ergo $p^2 + q^2 + r^2 + s^2$ ista quatuor quadratorum summa per N divisibilis, ita, ut singulae radices minores sint quam $\frac{1}{2}N$, ac denotet n quotum resultantem, ut sit

$$Nn = p^2 + q^2 + r^2 + s^2$$

et haec summa minor erit quam N^2, sicque certo erit $n < N$. Jam sequenti modo istae quatuor radices exhibeantur secundum lemma I:

$$p = Na + n\alpha, \quad q = Nb + n\beta, \quad r = Nc + n\gamma, \quad s = Nd + n\delta,$$

ubi litteras a, b, c, d ita assumere licebit, ut sint minores quam $\frac{1}{2}n$, sive hoc fiat negative, sive positive. His jam valoribus substitutis habebimus

$$Nn = N^2(a^2 + b^2 + c^2 + d^2) + 2Nn(a\alpha + b\beta + c\gamma + d\delta) + n^2(\alpha^2 + \beta^2 + \gamma^2 + \delta^2),$$

ubi notetur esse, per lemma II, $a\alpha + b\beta + c\gamma + d\delta = A$. Duo posteriora membra sponte sunt divisibilia per n; ergo necesse est, ut etiam primum per n sit divisibile. At N^2 dividi nequit per n, ergo necesse est, ut $a^2 + b^2 + c^2 + d^2$ sit per n divisibile. Ponatur ergo

$a^2+b^2+c^2+d^2=nn'$, et quia $a<\frac{1}{2}n$, $b<\frac{1}{2}n$, $c<\frac{1}{2}n$, $d<\frac{1}{2}n$, erit summa $a^2+b^2+c^2+d^2<n^2$, ideoque $nn'<n^2$, ergo $n'<n$, nisi forte sit $n=1$. Divisa ergo per n illa aequatione, prodit

$$N=N^2n'+2NA+n(\alpha^2+\beta^2+\gamma^2+\delta^2);$$

quae per n' multiplicetur, ut habeamus

$$Nn'=N^2n'^2+2Nn'A+nn'(\alpha^2+\beta^2+\gamma^2+\delta^2);$$

quia autem $nn'=a^2+b^2+c^2+d^2$, ultimum illud membrum abit in

$$(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)=A^2+B^2+C^2+D^2$$

per lemma II; consequenter

$$Nn'=N^2n'^2+2Nn'A+A^2+B^2+C^2+D^2=(Nn'+A)^2+B^2+C^2+D^2.$$

Sicque formula Nn' etiam erit summa quatuor quadratorum, existente $n'<n$. Eodem modo pervenire licebit ad formas ulteriores Nn'', Nn''' etc. ita, ut sit $n''<n'$, $n'''<n''$ etc. sicque tandem perveniri necesse est ad formam $N.1$, quae ergo etiam est summa quatuor quadratorum. Q. E. D.

Hinc etiam sequens Theorema facilius demonstrari potest, quam hactenus est factum:

Summa duorum quadratorum inter se primorum alios non admittit divisores, nisi qui ipsi sint summae duorum quadratorum.

Lemma. Productum ex duabus summis duorum quadratorum ipsum in duo quadrata resolvere.

Sit productum $(a^2+b^2)(\alpha^2+\beta^2)$, et sumtis $A=a\alpha+b\beta$ et $B=a\beta-b\alpha$, erit

$$(a^2+b^2)(\alpha^2+\beta^2)=A^2+B^2.$$

Si nunc N fuerit divisor formae p^2+q^2, posito quoto $=n$, habetur $Nn=p^2+q^2$. Nunc igitur p et q ita exhibeantur, ut sit $p=Na+n\alpha$ et $q=Nb+n\beta$, ita, ut a et b sint minores quam $\frac{1}{2}n$; hincque $a^2+b^2<\frac{1}{2}n^2$, quo substituto fit

$$Nn=N^2(a^2+b^2)+2Nn(a\alpha+b\beta)+n^2(\alpha^2+\beta^2),$$

quae cum per n divisibilis esse debeat, statuatur $a^2+b^2=nn'$, et diviso per n erit

$$N=N^2n'+2NA+n(\alpha^2+\beta^2).$$

Multiplicetur per n', erit

$$Nn'=N^2n'^2+2Nn'A+nn'(\alpha^2+\beta^2)$$

at $nn'(\alpha^2+\beta^2)=A^2+B^2$, ergo

$$Nn'=N^2n'^2+2Nn'A+A^2+B^2=(Nn'+A)^2+B^2,$$

sicque Nn' est etiam summa duorum quadratorum, ubi $n'<\frac{1}{2}n$. Hocque modo ulterius progrediendo, mox pervenietur ad $N.1$. Consequenter N certo erit summa duorum quadratorum.

Alia demonstratio simplicior ejusdem theorematis.

Si numerus quicunque N fuerit divisor summae quatuor quadratorum $P^2+Q^2+R^2+S^2$, quae singula seorsim per eum non sint divisibilia, ille ipse numerus quoque erit summa quatuor quadratorum.

Demonstratio. I. Illa quadrata semper ad alia reduci possunt minora quam $\frac{1}{4}N^2$. Ponatur enim

$$P=\mathfrak{A}N\mp p,\quad Q=\mathfrak{B}N\mp q,\quad R=\mathfrak{C}N\mp r,\quad S=\mathfrak{D}N\mp s,$$

ubi literae $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ ita assumi possunt, ut numeri p, q, r, s infra semissem numeri N deprimantur, quibus substitutis evidens est, formulam $p^2+q^2+r^2+s^2$, quae utique minor erit quam N^2, divisibilem fore per N et quotum fore minorem quam N.

II. Sit ergo iste quotus $=n$, ut sit $Nn=p^2+q^2+r^2+s^2$, et ratione hujus numeri n radices istorum quadratorum ita exhiberi poterunt

$$p=a+\alpha n,\quad q=b+\beta n,\quad r=c+\gamma n\quad\text{et}\quad s=d+\delta n,$$

si pro a, b, c et d etiam valores negativi admittantur, hos numeros itidem infra $\frac{1}{2}n$ deprimere licebit, ut sit $a^2+b^2+c^2+d^2<n^2$.

III. His autem valoribus substitutis fiet

$$Nn=a^2+b^2+c^2+d^2+2n(a\alpha+b\beta+c\gamma+d\delta)+n^2(\alpha^2+\beta^2+\gamma^2+\delta^2),$$

quae formula per lemma praemissum abit in hanc

$$Nn=a^2+b^2+c^2+d^2+2nA+n^2(\alpha^2+\beta^2+\gamma^2+\delta^2),$$

quae cum sit divisibilis per n et bina posteriora membra jam in se sint per n divisibilia, necesse est, ut etiam pars prima $a^2+b^2+c^2+d^2$ factorem habeat n. Quare ponatur $a^2+b^2+c^2+d^2=nn'$ et dividendo per n habebimus

$$N=n'+2A+n(\alpha^2+\beta^2+\gamma^2+\delta^2).$$

IV. Multiplicemus nunc in n', et in postremo membro loco nn' substituamus valorem $a^2+b^2+c^2+d^2$, ut prodeat

$$Nn'=n'^2+2n'A+(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2).$$

At per lemma praemissum hoc postremum membrum transformatur in $A^2+B^2+C^2+D^2$, ita, ut nunc habeamus

$$Nn'=n'^2+2n'A+A^2+B^2+C^2+D^2,$$

sive $\quad Nn'=(n'+A)^2+B^2+C^2+D^2=$ summae quatuor quadratorum.

V. Cum autem sit $nn'=a^2+b^2+c^2+d^2<n^2$, utique erit $n'<n$. Quemadmodum igitur ex forma Nn, quae erat summa quatuor quadratorum, pervenimus ad hanc minorem Nn', etiam aequalem summae quatuor quadratorum; ita ulterius pervenire licebit ad formulas Nn'', Nn''' etc. itidem quatuor quadratis aequales, ita, ut numeri n', n'', n''', etc. continuo diminuantur. Tandem ergo haec diminutio usque ad unitatem deducetur; ita, ut tum futurum sit $N.1$, hoc est ipse numerus propositus N aequalis summae quatuor quadratorum. Q. E. D.

Corollarium 1. Haec adeo demonstratio locum habet, etiamsi N non fuerit numerus primus; dummodo ergo numerus quicunque N fuerit factor vel divisor summae cujuspiam quatuor quadratorum, tum certe is ipse numerus quoque erit summa quatuor quadratorum.

Corollarium 2. Quodsi ergo demonstrari posset, proposito quocunque numero N, semper exhiberi posse summam quatuor quadratorum per eum divisibilem, tum utique completa haberetur demonstratio theorematis illius Fermatiani, quod omnis numerus sit summa quatuor quadratorum, vel etiam pauciorum.

Theorema. Proposito quocunque numero primo N, semper exhiberi possunt quatuor quadrata, singula minora quam $\frac{1}{4}N^2$, quorum summa per illum numerum sit divisibilis.

Demonstratio. I. Ratione numeri propositi N omnes plane numeri in aliqua sequentium formularum erunt contenti $\quad \lambda N,\ \lambda N+1,\ \lambda N+2,\ \lambda N+3,\ \lambda N+4,\ \ldots\ \lambda N+(N-1)$, quarum numerus est $=N$. Singulae autem hae formae non omnes continent numeros quadratos; dantur scilicet inter illas ejusmodi formulae, quae numeros quadratos involvunt, reliquae vero quadrata prorsus excludunt. Seposita enim prima forma λN, quae ipsa multipla numeri N continet, reliquarum primae $\lambda N+1$ et ultimae $\lambda N+N-1$, vel $(\lambda+1)N-1$ quadrata in eadem formula continebuntur, nempe $\lambda N+1$. Eodem modo quadrata secundae et penultimae formulae continentur in formula $\lambda N+4$. Simili modo quadrata tertiae et antepenultimae continebuntur in formula $\lambda N+9$, quarum formularum multitudo est $\frac{1}{2}(N-1)$, quae scilicet in se complectuntur quadrata. Reliquae formulae omnes ab his diversae quadrata penitus excludunt, quarum numerus itidem est $\frac{1}{2}(N-1)$.

II. Sint formulae illae quadrata admittentes: $\lambda N+a$, $\lambda N+b$, $\lambda N+c$, $\lambda N+d$, etc., quarum numerus est $\frac{1}{2}(N-1)$ et modo vidimus, inter hos numeros a, b, c, d, etc. reperiri quadratos $1, 4, 9, 16$, etc. quamdiu scilicet sunt minores, quam N. Majorum enim residua ex divisione per N relicta sumuntur. Formulae autem quadrata penitus excludentes sint: $\lambda N+\alpha$, $\lambda N+\beta$, $\lambda N+\gamma$, $\lambda N+\delta$, etc., quorum numerus itidem est $\frac{1}{2}(N-1)$.

III. Facile autem demonstrari potest, binas formulas prioris classis in se multiplicatas etiamnunc ad prio-rem classem pertinere, scilicet cum prior classis contineat formas a, b, c, d, etiam continebit producta ex binis vel quotcunque horum numerorum. Scilicet producta ex binis numeris prioris classis etiam in priore classe occurrent, cujusmodi sunt aa, bb, cc, etc. Tum vero productum ex numero prioris classis in numerum poste-rioris classis cadet in classem posteriorem. Denique productum ex binis numeris posterioris classis etiam cadet in classem priorem.

IV. Jam si in prima classe occurreret formula $\lambda N - a$, sive quod eodem redit, $\lambda N + N - a$, darentur qua-drata formae $\lambda N + a$ et $\lambda N - a$, quorum ergo summa foret per N divisibilis. Quare si quis neget, dari summam quatuor quadratorum per N divisibilem, multo magis negare debebit, dari adeo summam duorum quadratorum divisibilem.

V. Quo igitur nostrum theorema demonstremus, sumamus tantisper, non dari summam quatuor vel pau-ciorum quadratorum, quae non esset divisibilis per numerum propositum N, atque ostendemus hinc maxima absurda esse secutura.

VI. Ista igitur opinione quasi adoptata, quia numerus $-a$ vel $N-a$ in priore classe non occurrit, certe occurret in posteriore classe inter numeros α, β, γ, δ; ergo inter numeros α, β, γ, δ occurrent numeri $-a$, $-b$, $-c$, $-d$, ideoque etiam negativa quadrata -1, -4, -9, -16.

VII. Eodem modo ostendi potest, numerum $-a-b$ certe non in priori classe contineri; si enim ibi con-tineretur, darentur tres numeri quadrati formarum $\lambda N + a$, $\lambda N + b$ et $\lambda N - a - b$, quorum summa esset per N divisibilis; quod cum hypothesi repugnet, hic numerus $-a-b$ in posteriori classe reperiatur necesse est.

VIII. Quia autem in posteriori classe reperitur -1, productum ex -1 in $-a-b$, id est $+a+b$ in prima classe continetur; sicque in priori classe jam occurrerent numeri $1, 2, 4, 5, 8, 9, 10, 13$; eorundem autem negativa occurrent in classe posteriori.

IX. Cum ergo formulae $\lambda N + 1$ et $\lambda N + 2$ sint prioris classis, ibidem non continebitur formula $\lambda N - 3$, quia alioquin haberemus tria quadrata harum formularum, quorum summa foret per N divisibilis. Quia ergo -3 non in priori classe continetur, continebitur in posteriori; ejus vero productum in -1, hoc est $+3$, con-tinebitur in priori.

X. Sit autem generalius f numerus quicunque primae classis, atque dico, in priori classe formulam $\lambda N - f - 1$ non contineri, quia darentur tria quadrata, scilicet $\lambda N + 1$, $\lambda N + f$, et $\lambda N - 1 - f$, quorum summa foret divisibilis per N; unde numerus $-f-1$ in classe posteriori reperiatur necesse est; ejus vero negativum $+f+1$ in priorem classem cadet.

XI. Admissa ergo illa hypothesi, si formula quaecunque $\lambda N + f$ in prima classe contineatur, ibidem quoque occurret formula $\lambda N + f + 1$; quocirca in prima classe occurrerent omnes istae formulae:

$$\lambda N + 1, \quad \lambda N + 2, \quad \lambda N + 3, \quad \lambda N + 4, \quad \text{etc.}$$

hoc est omnes plane formulae forent prioris classis, simul vero in classem posteriorem ingrederentur omnes istae formulae:

$$\lambda N - 1, \quad \lambda N - 2, \quad \lambda N - 3, \quad \lambda N - 4, \quad \text{etc.}$$

hoc est omnes plane formulae tam in priore quam in posteriore classe occurrerent. Quare cum ante sit osten-sum, in priore classe tantum occurrere $\frac{1}{2}(N-1)$ formulas et totidem in posteriore, absurdum est manifestum, quod inde ortum est, quod falso supposuimus, non dari summam trium quadratorum per N divisibilem; quam-obrem verum erit, dari summas trium quadratorum per N divisibiles. Multo magis ergo dantur summae qua-tuor quadratorum per N divisibiles. Q. E. D.

Corollarium. Cum ergo, proposito numero primo quocunque N, dentur summae non solum quatuor sed etiam trium quadratorum per illum divisibiles, ipse ille numerus N erit quoque summa quatuor quadra-torum, vel et pauciorum, et cum producta ex binis vel pluribus numeris, quorum singuli sunt summae quatuor quadratorum, sint etiam summae quatuor quadratorum, jam rigorosissime demonstratum est, omnes plane numeros esse summas quatuor quadratorum.

Observatio singularis. Cum productum ex binis numeris, quorum uterque est summa duorum quadratorum, etiam sit summa duorum quadratorum, tum vero etiam productum ex duobus numeris, quorum uterque est summa quatuor quadratorum, quoque sit summa quatuor quadratorum. Hinc concludendum videtur, idem etiam de summis trium quadratorum valere, quod autem longe secus se habet, neque etiam eo modo, quo in lemmate superiore sumus usi, talis forma $(a^2+b^2+c^2)(\alpha^2+\beta^2+\gamma^2)$ ad tria quadrata revocari potest. Fieri enim saepe potest, ut productum ex binis summis trium quadratorum non in pauciora quam quatuor quadrata resolvi possit, veluti $3=1+1+1$ et $21=1+4+16$; horum tamen productum 63 nullo modo in pauciora quam quatuor quadrata potest resolvi, quandoquidem est numerus formae $8n-1$ sive $8n+7$.

A. m. T. I. p. 177—186.

35.

(*N. Fuss I.*)

Theorema. Nulli numeri in sequentibus formulis contenti in duos numeros trigonales resolvi possunt:

I. $9n+$ 5, 8

II. $49n+$ 5, 19, 26, 33, 40, 47

III. $81n+47$, 74

IV. $121n+$ 8, 19, 41, 52, 63, 74, 85, 96, 107, 118

V. $361n+14$, 33, 52, 71, 109, 128, 147, 166, 185, 204, 223, 242, 261, 280, 299, 318, 337, 356.

Specimen Demonstrationis pro formula $49n+19$:

Sit $49n+19=\frac{aa+a}{2}+\frac{bb+b}{2}$, erit multiplicando per 8

$$392n+152=4aa+4a+4bb+4b,$$

ergo $392n+154=(2a+1)^2+(2b+1)^2$, ideoque summa duorum quadratorum. At numerus $392n+154$ factorem habet 7, ideoque duorum quadratorum summa esse nequit.

Problema. Numeros in hac forma contentos $xx+7$ in quatuor quadrata resolvere.

Solutio. Formula $xx+7$ transformatur in has:

$$(x-1)^2+2x+6, \quad \text{vel} \quad (x-2)^2+4x+3, \quad \text{vel} \quad (x-3)^2+6x-2, \quad \text{vel} \quad (x-4)^2+8x-9,$$

vel in genere
$$(x-n)^2+2nx-nn+7,$$

unde si $2nx-nn+7$ in tria vel pauciora quadrata resolvi potest, quaesito satisfiet. Plerumque statim una harum formularum priorum negotium conficit. Verum dantur etiam casus, quibus longe progredi oportet. Veluti si x fuerit 75, usque ad $n=11$ progredi oportet, tum enim fiet

$$75^2+7=64^2+1650-121+7=64^2+1536.$$

Est vero
$$1536=16\,.\,96=16^2.6, \quad \text{at} \quad 6=4+1+1,$$

unde quatuor quadrata erunt
$$64^2+16^2+32^2+16^2.$$

Aliud exemplum multo notabilius est, quo $x=181$; tum enim formulae supra datae frustra tentantur, donec perveniatur ad $n=53$, tum autem fiet

$$181^2+7=128^2+19186-2802=128^2+16384=128^2+128^2.$$

Sicque hic numerus ad duo quadrata est reductus, neque ullo alio modo vel in tria, vel in plura adhuc quadrata resolvi potest.

Hac occasione sequens theorema omnem attentionem meretur.

Theorema. Omnis potestas binarii 2^n semper est numerus in hac formula contentus: $xx+7yy$.

Demonstratio. Sumto enim $x = \frac{1}{2}$ et $y = \frac{1}{2}$ prodit $xx + 7yy = 2$. Notum autem est omnes potestates formulae $xx + 7yy$ in eadem formula contineri, quandoquidem est

$$(aa + 7bb)(cc + 7dd) = (ac \pm 7bd)^2 + 7(ad \mp bc)^2.$$

Hinc igitur per factores imaginarios erit

$$2 = \frac{1+7}{4} = \frac{1+\sqrt{-7}}{2} \cdot \frac{1-\sqrt{-7}}{2}, \text{ erit ergo } 2^n = \left(\frac{1+\sqrt{-7}}{2}\right)^n \cdot \left(\frac{1-\sqrt{-7}}{2}\right)^n.$$

Binomii autem $\frac{1+\sqrt{-7}}{2}$ potestates sequenti modo progrediuntur

$$\frac{-3+\sqrt{-7}}{2} = \left(\frac{1+\sqrt{-7}}{2}\right)^2$$

$$\frac{-5-\sqrt{-7}}{2} = \left(\frac{1+\sqrt{-7}}{2}\right)^3$$

$$\frac{1-3\sqrt{-7}}{2} = \left(\frac{1+\sqrt{-7}}{2}\right)^4$$

etc. etc.

Harum formularum ambae partes seriem recurrentem constituunt, cujus scala relationis est $1, -2$, unde si ex. gr. ponatur $\frac{1+\sqrt{-7}}{2} = A$, et quia omnes hae formulae per 2 dividuntur, istae formulae sequenti modo continuantur:

$$\begin{array}{ll}
2A = 1 + \sqrt{-7} & 2A^8 = -31 - 3\sqrt{-7} \\
2A^2 = -3 + \sqrt{-7} & 2A^9 = -5 - 17\sqrt{-7} \\
2A^3 = -5 - \sqrt{-7} & 2A^{10} = 57 - 11\sqrt{-7} \\
2A^4 = 1 - 3\sqrt{-7} & 2A^{11} = 67 + 23\sqrt{-7} \\
2A^5 = 11 - \sqrt{-7} & 2A^{12} = -47 + 45\sqrt{-7} \\
2A^6 = 9 + 5\sqrt{-7} & 2A^{13} = -181 - \sqrt{-7} \\
2A^7 = -13 + 7\sqrt{-7} & \text{etc.}
\end{array}$$

Cum igitur sit

$$\left(\frac{1+\sqrt{-7}}{2}\right)^{13} = \frac{-181-\sqrt{-7}}{2}, \text{ erit } \left(\frac{1-\sqrt{-7}}{2}\right)^{13} = \frac{-181+\sqrt{-7}}{2}, \text{ indeque } 2^{13} = \frac{181^2+7}{4},$$

$$\text{ergo} \quad 2^{15} = 181^2 + 7.$$

Ratio autem scalae relationis in hoc sita est, quod si ponatur

$$z = \frac{1+\sqrt{-7}}{2}, \quad \text{fit} \quad z - \frac{1}{2} = \frac{\sqrt{-7}}{2}$$

et sumtis quadratis erit $zz = z - 2$, unde nascitur scala relationis $1, -2$. In superiori progressione, ubi omnes termini in forma $a + b\sqrt{-7}$ continentur, ii casus maxime sunt notatu digni, quibus b est vel $+1$, vel -1, quibus casibus pars rationalis fit maxima. Hincque sequens problema omnino peculiarem postulat solutionem.

Problema. Cum sit uti vidimus $\left(\frac{1+\sqrt{-7}}{2}\right)^n = \frac{a+b\sqrt{-7}}{2}$, investigare eos exponentes n, pro quibus fit $b = \pm 1$, id quod fieri observavimus casibus $n = 1, 2, 3, 5, 13$. Quaerantur igitur casus sequentes.

Solutio. Cum esse debeat $b = \pm 1$, reducatur formula $\frac{1+\sqrt{-7}}{2}$ ad hanc formam $p(\cos\varphi + \sqrt{-1}.\sin\varphi)$ eritque

$p\cos\varphi = \frac{1}{2}$ et $p\sin\varphi = \frac{1}{2}\sqrt{7}$, unde fit $\tang\varphi = \sqrt{7}$, hincque $\sin\varphi = \sqrt{\frac{7}{8}}$ et $\cos\varphi = \sqrt{\frac{1}{8}}$, sicque erit $p = \sqrt{2}$.

Invento igitur angulo φ, ut sit $\tang\varphi = \sqrt{7}$ erit primo

$$\frac{1+\sqrt{-7}}{2} = \sqrt{2}(\cos\varphi + \sqrt{-1}.\sin\varphi) \text{ ideoque } \left(\frac{1+\sqrt{-7}}{2}\right)^n = 2^{\frac{n}{2}}(\cos n\varphi + \sqrt{-1}.\sin n\varphi).$$

Quaestio igitur huc redit, ut membrum imaginarium fiat quam minimum, id quod evenit, quando angulus $n\varphi$ quam minime differt ab π, vel 2π, vel etc. vel $i\pi$. Quod si ergo statuamus $n\varphi = i\pi$ erit $\frac{n}{i} = \frac{\pi}{\varphi}$, quamobrem quaerantur fractiones proxime aequales ipsi $\frac{\pi}{\varphi}$, earumque numeratores dabunt valores pro n. Cum igitur sit

$$\operatorname{tang}\varphi = \sqrt{7} \quad \text{erit} \quad l.\operatorname{tang}\varphi = 0{,}4225490, \quad \text{unde} \quad \varphi = 69^0\ 17'\ 43'' = 249463''.$$

At vero
$$\pi = 180^0 = 648000'', \quad \text{unde} \quad \frac{\pi}{\varphi} = \frac{648000}{249463} = \frac{n}{i}.$$

Evolvatur ergo haec fractio per continuam divisionem, eruntque quotientes 2, 1, 1, 2, 16, 7. Ex his quotis formentur sequentes fractiones

$$\frac{1}{0}, \quad \frac{2}{1}, \quad \frac{3}{1}, \quad \frac{5}{2}, \quad \frac{13}{5}, \quad \frac{213}{82},$$

ex quarum numeratoribus statim patet, quaesito satisfieri casibus 1, 2, 3, 5, 13, unde tuto affirmare licet idem evenire casu $n = 213$. Consideremus casum $n = 13$, eritque

$$13\varphi = 900^0\ 50'\ 19'' = 180^0\ 50'\ 19''.$$

Nunc vero est
$$l2^{\frac{13}{2}} = 1{,}9566950,$$

unde fiet

$$\begin{array}{rr} l2^{\frac{13}{2}} = 1{,}9566950 & l2^{\frac{13}{2}} = 1{,}9566950 \\ l\cos 13\varphi = 9{,}9999534 & l\sin 13\varphi = 8{,}1654040 \\ \hline 1{,}9566484 & 0{,}1220990 \end{array}$$

$$2^{\frac{13}{2}}\cos 13\varphi = -90{,}5 = -\frac{181}{2}, \qquad 2^{\frac{13}{2}}\sin 13\varphi = -1{,}32 = -\frac{1}{2}\sqrt{7},$$

eritque
$$2^{\frac{13}{2}}\sin 13\varphi\sqrt{-1} = -\frac{1}{2}\sqrt{-7}, \quad \text{unde patet esse} \quad \left(\frac{1+\sqrt{-7}}{2}\right)^{13} = \frac{-181-\sqrt{-7}}{2}.$$

Cum sit $181^2 + 7 = 2\,(2^7)^2$, erit $181^2 = 2\square - 7$. Consideretur formula $2xx - 7yy$ reddaturque quadratum: Ponatur $x = 2y + z$ eritque $yy + 8yz + 2zz$, cujus radix statuatur

$$y + \frac{p}{q}z, \quad \text{eritque} \quad 8y + 2z = \frac{2p}{q}y + \frac{pp}{qq}z, \quad \text{unde fit} \quad \frac{y}{z} = \frac{pp - 2qq}{8qq - 2pq}.$$

Statuatur ergo $y = pp - 2qq$ et $z = 8qq - 2pq$, eritque radix illa quadrata $8pq - pp - 2qq$, in qua ergo forma contineri debet 181, quod fit si $q = 5$ et $p - 4q = 13$, ideoque $p = 33$, vel $p = 7$, ergo $y = -1$ et $z = 130$ et $x = 128$. Eritque ergo $181^2 = 2.128^2 - 7$, uti habuimus $181^2 + 7 = 2\,(2^7)^2$.

A. m. T. II. p. 110—113.

36.

(*J. A. Euler.*)

Hujus seriei: 1^2, 3^2, 6^2, 10^2, 15^2, etc. ad minimum duodecim termini conjungi debent, ut omnes numeri prodeant. At seriei

$$1^n, \quad 2^n, \quad 3^n, \quad 4^n, \quad 5^n, \quad 6^n, \quad \text{etc.}$$

ad minimum tot termini jungi debent quot indicat haec formula

$$\frac{3^n}{2^n} + 2^n - 2 = T,$$

ubi pro $\frac{3^n}{2^n}$ numerus integer proxime minor capi debet

$$\begin{array}{llllllllll} \text{si} & n = 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, \\ \text{fit} & T = 1, & 4, & 9, & 19, & 37, & 73, & 143, & 279. \end{array}$$

Pro numeris figuratis litera T ita se habet:

*

	T		T		T
1. 2. 3. 4. 5. 6	1	1. 3. 5. 7. 9	2	$1.\ 1+a.\ 1+2a.\ 1+3a$	a
1. 3. 6. 10. 15. 21	3	1. 4. 9. 16. 25	4	$1.\ 2+a.\ 3+3a.\ 4+6a$	$a+2$
1. 4. 10. 20. 35. 56	5	1. 5. 14. 30. 55	6	$1.\ 3+a.\ 6+4a.\ 10+10a$	$a+4$
1. 5. 15. 35. 70	7	1. 6. 20. 50. 105	8	$1.\ 4+a.\ 10+5a.\ 20+15a$	$a+6$
				$1.\ 5+a.\ 15+6a.\ 35+21a$	$a+8$

(*Krafft.*)

Omnes illae superiores series numerorum figuratorum sequenti forma generali comprehendi possunt

$$1;\quad \frac{n+a}{1};\quad \frac{(n+1)(n+2a)}{1.2};\quad \frac{(n+1)(n+2)(n+3a)}{1.2.3};\quad \frac{(n+1)(n+2)(n+3)(n+4a)}{1.2.3.4}\quad \text{etc.}$$

pro qua superior littera T fit $= a + 2n - 2$.

A. m. T. I. p. 234. 235.

C. Analysis Diophantea.

a) *Quaestiones ad resolutionem unius aequationis ducentes.*

37.

(*J. A. Euler.*)

Problema. Si fuerit $x^3 = m$, et proposita sit formula $axx + bx + c$, invenire multiplicatorem $pxx + qx + r$, ut productum

$$(axx + bx + c)(pxx + qx + r)$$

fiat numerus absolutus non amplius involvens x, posito scilicet $x^3 = m$.

Solutio. Productum ergo erit

$$\begin{aligned} & mapx + (aq + bp)m + (ar + bq + cp)xx \\ & + (br + cq)x + cr. \end{aligned}$$

Debet ergo poni $br + cq + map = 0$ et $ar + bq + cp = 0$, tum enim productum erit

$$m(aq + bp) + cr.$$

Fit autem

$$p = \frac{-br - cq}{ma} = \frac{-ar - bq}{c},$$

unde fit $bcr + ccq = maar + mabq$; hinc $mabq - ccq = bcr - maar$,

consequenter $$\frac{q}{r} = \frac{bc - maa}{mab - cc}.$$

Capiatur ergo $q = bc - maa$, $r = mab - cc$, erit $p = ac - bb$, ita ut multiplicator quaesitus sit

$$(ac - bb)xx + (bc - maa)x + mab - cc;$$

ac tum productum erit $$3mabc - m^2a^3 - mb^3 - c^3.$$

A. m. T. I. p. 50. 51.

38.

Problema. Invenire numeros x et y, ut fiat $xy(xx - yy) = \alpha nn$, existente α numero primo: ubi hi casus sunt notandi:

I. Si $\alpha = 7$, sumatur $x = 16$ et $y = 9$, tum enim fit $xy(xx - yy) = 7.\ 16.\ 9.\ 25$.

II. Si $\alpha = 13$, sumatur $x = 325$ et $y = 36$, erit $xy(xx - yy) = 13.\ 25.\ 36.\ 361.\ 289$.

III. Si $\alpha = 23$, sumatur $x = 156^2$ et $y = 133^2$, erit $xy(xx - yy) = 23.\,156^2.\,133^2.\,289.\,42025$.

IV. Ut $\alpha = 41$, capiatur $x = 21^2$ et $y = 20^2$, erit $xy(xx - yy) = 41.\,21^2.\,20^2.\,29^2$.

V. Ut fiat $\alpha = 31$, sumatur $x = 40^2$ et $y = 9^2$, fit enim

$$xy(xx - yy) = 31.\,9^2.\,40^2.\,7^2.\,41^2.$$

VI. In genere si capiatur $x = 4ppqq$ et $y = (pp - qq)^2$, fiet

$$xy(xx - yy) = (pp - qq + 2pq)(2pq - pp + qq).\square.$$

Unde si sit $2pq + pp - qq = aa$, fit $\alpha = 2pq - pp + qq$, at illa formula $2pq + pp - qq$ fit quadratum sumendo $q = 2rs$ et $p = 2rr + ss - 2rs$.

VII. Deinde vero si sumatur $x = (2pp + qq)^2$ et $y = (2pp - qq)^2$, fiet

$$xy(xx - yy) = 8pp.qq(8p^4 + 2q^4)\square = (4p^4 + q^4)\square$$

unde fit

$$\alpha\square = 4p^4 + q^4 = (2pp + 2pq + qq)(2pp - 2pq + qq).$$

Unde si fuerit $2pp + 2pq + qq = \square$, tunc erit $\alpha = 2pp - 2pq + qq$. At illud evenit

si $p = 2rs$ et $q = rr - ss - 2rs$, unde fit $\alpha = pp + (p - q)^2$.

VIII. Ex casu VII, si $p = 5$ et $q = 7$, capiatur $x = 99^2$ et $y = 1$, erit $\alpha = 29$ et $xy(xx - yy) = 29\square$; vel si capiatur $x = 29.13^2$ et $y = 70^2$.

A. m. T. I. p. 21. 22.

39.

(*Lexell.*)

Problema. Invenire numeros x, y, z, ut fiat $\alpha xx + \beta yy = \gamma zz$, siquidem cognitus fuerit casus

$$\alpha ff + \beta gg = \gamma hh.$$

Solutio. Statuatur $\alpha xx + \beta yy = (\alpha ff + \beta gg)(\alpha pp + \beta qq)^2$, tum enim erit

$$\alpha xx + \beta yy = \gamma hh(\alpha pp + \beta qq)^2, \quad \text{sicque erit} \quad z = h(\alpha pp + \beta qq);$$

illud autem hoc modo per factores praestetur. Sit $x\sqrt{\alpha} + y\sqrt{-\beta} = (f\sqrt{\alpha} + g\sqrt{-\beta})(p\sqrt{\alpha} + q\sqrt{-\beta})^2$, tum enim sponte fit $x\sqrt{\alpha} - y\sqrt{-\beta} = (f\sqrt{\alpha} - g\sqrt{-\beta})(p\sqrt{\alpha} - q\sqrt{-\beta})^2$, quarum formularum productum ipsa est aequatio supposita. Prior autem evoluta dat

$$\begin{aligned} x\sqrt{\alpha} + y\sqrt{-\beta} = \alpha fpp\sqrt{\alpha} &- \beta fqq\sqrt{\alpha} - 2\beta gpq\sqrt{\alpha} \\ &+ \alpha gpp\sqrt{-\beta} - \beta gqq\sqrt{-\beta} + 2\alpha fpq\sqrt{-\beta} \end{aligned}$$

$$x = f(\alpha pp - \beta qq) - 2\beta gpq$$

$$y = g(\alpha pp - \beta qq) + 2\alpha fpq$$

ac tum erit

$$z = h(\alpha pp + \beta qq).$$

Verum haec solutio nondum est generalis, eodem modo enim ponere potuissemus

$$\alpha xx + \beta yy = (\alpha ff + \beta gg)(pp + \alpha\beta qq)^2,$$

unde fit $z = h(pp + \alpha\beta qq)$. Pro hoc ergo casu statuatur

$$x\sqrt{\alpha} + y\sqrt{-\beta} = (f\sqrt{\alpha} + g\sqrt{-\beta})(p + q\sqrt{-\alpha\beta})^2,$$

cujus evolutio praebet

$$x = f(pp - \alpha\beta qq) - 2g\beta pq$$

$$y = g(pp - \alpha\beta qq) + 2f\alpha pq.$$

Verum ne hi ambo quidem casus solutionem praebent generalem, cum sine dubio ejusmodi casus dentur, quibus z non per h fit divisibile, quare pro solutione generali statuatur

$$\alpha xx + \beta yy = (\alpha ff + \beta gg)(\alpha npp + \beta nqq)^2, \quad \text{unde fit} \quad z = hn(\alpha pp + \beta qq),$$

ubi forte n potest esse fractio denominatoris h. Statuatur igitur

$$x\sqrt{\alpha} + y\sqrt{-\beta} = (f\sqrt{\alpha} + g\sqrt{-\beta})(p\sqrt{\alpha n} + q\sqrt{-\beta n})^2,$$

cujus evolutio praebet

$$x = f(\alpha npp - \beta nqq) - 2\beta ngpq, \quad y = g(\alpha npp - \beta nqq) + 2\alpha nfpq.$$

Videamus igitur an esse possit $n = \frac{1}{h}$, manentibus x et y integris. Cum igitur sit

$$x = \frac{f(\alpha pp - \beta qq) - 2\beta gpq}{h}, \quad y = \frac{g(\alpha pp - \beta qq) + 2\alpha fpq}{h},$$

quod evenit si p et q ita sumantur, ut $fq + gp$ fiat per h divisibile.

(*W. L. Krafft.*)

Problematis supra propositi solutio facillime sequenti modo absolvetur, siquidem constet unus casus, quo sit $\alpha ff + \beta gg = \gamma hh$, ubi scilicet $x = f$, $y = g$ et $z = h$. Statuamus $x = fp + \beta gq$ et $y = gp - \alpha fq$, tum enim erit

$$\alpha xx + \beta yy = pp(\alpha ff + \beta gg) + \alpha\beta qq(\alpha ff + \beta gg) = \gamma hh(pp + \alpha\beta qq).$$

Sicque aequatio adhuc resolvenda erit $hh(pp + \alpha\beta qq) = zz$, ita ut $pp + \alpha\beta qq$ debeat reddi quadratum, quod fit capiendo $p = rr - \alpha\beta ss$ et $q = 2rs$, tum enim fit

$$pp + \alpha\beta qq = (rr + \alpha\beta ss)^2.$$

Ideoque $z = h(rr + \alpha\beta ss)$. Ipsarum vero x et y valores erunt

$$x = f(rr - \alpha\beta ss) + 2\beta grs, \quad y = g(rr - \alpha\beta ss) - 2\alpha frs,$$

ubi numeri r et s pro lubitu assumi possunt. (Conf. Comment. arithm. T. I. p. 556.)

A. m. T. I. p. 95. 96. 98. 99.

40.

(*J. A. Euler.*)

Theorema. Si fuerint $naa + pbb = \square = cc$ et $nff + qgg = \square = hh$, tum semper assignare licet x et y ut sit $nxx + pqyy = \square = zz$.

Demonstratio. Cum sit $pbb = cc - naa$ et $qgg = hh - nff$, erit productum

$$pqbbgg = (cc - naa)(hh - nff) = (ch + naf)^2 - n(ah + fc)^2,$$

unde manifestum est fore $n(ah + fc)^2 + pqbbgg = (ch + naf)^2$, sicque erit

$$x = ah + fc, \quad y = bg \quad \text{et} \quad z = ch + naf.$$

A. m. T. I. p. 130.

41.

(*N. Fuss I.*)

Problema. Resolvere aequationem $\lambda zz = \mu xx + \nu yy$, ex cognito casu $\lambda cc = \mu aa + \nu bb$.

Solutio. A priore aequatione in cc ducta subtrahatur posterior in zz ducta, eritque

$$0 = \mu(ccxx - aazz) + \nu(ccyy - bbzz),$$

$$\text{sive} \quad \mu(ccxx - aazz) = \nu(bbzz - ccyy), \quad \text{hinc} \quad \frac{\mu(cx + az)}{bz - cy} = \frac{\nu(bz + cy)}{cx - az}.$$

Utraque haec fractio statuatur $= \frac{p}{q}$ et ex priore elicitur

$$z = \frac{\mu cqx + pcy}{bp - \mu aq} \quad \text{et ex altera} \quad z = \frac{cpx - \nu cqy}{\nu bq + ap},$$

qui duo valores inter se aequati dant

$$\frac{y}{x} = \frac{\mu\nu bcqq + 2\mu acpq - bcpp}{\mu\nu acqq - 2\nu bcpq - acpp}.$$

Statuatur ergo $\quad x = \mu\nu aqq - 2\nu bpq - app \quad$ et $\quad y = \mu\nu bqq + 2\mu apq - bpp$,

$$\text{sive} \quad x = a(\mu\nu qq - pp) - 2\nu bpq \quad \text{et} \quad y = b(\mu\nu qq - pp) + 2\mu apq.$$

Cum autem sit
$$\frac{z}{c}(bp-\mu aq)=\mu qx+py=\mu\mu\nu aq^3-\mu\nu bpqq+\mu appq-bp^3$$
$$=\mu aq(\mu\nu qq+pp)-bp(\mu\nu qq+pp)=(\mu\nu qq+pp)(\mu aq-bp)$$
hinc
$$\frac{z}{c}=-(\mu\nu qq+pp) \quad \text{et} \quad z=-c(\mu\nu qq+pp).$$

Alia Solutio. Quia semper f, g, h invenire licet, ut sit $hh=ff+\mu\nu gg$, per hanc aequationem multiplicetur cognita $\lambda cc=\mu aa+\nu bb$ eritque
$$\lambda cchh=\mu aaff+\mu\nu\nu bbgg+\nu bbff+\mu\mu\nu aagg=\mu(af+\nu bg)^2+\nu(bf-\mu ag)^2.$$
Cum igitur esse debeat $\lambda zz=\mu xx+\nu yy$, capi poterit $z=ch$ deinde $x=af+\nu bg$ et $y=bf-\mu ag$, at vero ut fiat $hh=ff+\mu\nu gg$, debet esse
$$f=\mu\nu qq-pp \quad \text{et} \quad g=2pq, \quad \text{eritque} \quad h=\mu\nu qq+pp,$$
consequenter formula proposita ita resolvetur
$$x=a(\mu\nu qq-pp)+2\nu bpq \quad \text{et} \quad y=b(\mu\nu qq-pp)-2\mu apq \quad \text{et} \quad z=c(\mu\nu qq+pp).$$

Corollarium. Haec solutio duobus modis variari potest, prouti aequationes propositae aliter disponuntur, scil. primo $\mu xx=\lambda zz-\nu yy$ et $\mu aa=\lambda cc-\nu bb$. Ad quem casum solutio praecedens revocabitur

si loco	λ,	μ,	ν,	z,	x,	y,	c,	a,	b
ponatur	μ,	λ,	$-\nu$,	x,	z,	y,	a,	c,	b.

Unde si loco p et q scribamus r et s obtinebimus
$$z=c(-\lambda\nu ss-rr)-2\nu brs, \quad y=b(-\lambda\nu ss-rr)-2\lambda crs, \quad x=a(-\lambda\nu ss+rr).$$

Eodem modo si formulae datae ita disponantur $\nu yy=\lambda zz-\mu xx$ et $\nu bb=\lambda cc-\mu aa$, unde

si loco	λ,	μ,	ν,	z,	x,	y,	c,	a,	b
ponatur	ν,	λ,	$-\mu$,	y,	z,	x,	b,	c,	a

tum loco p et q, t et u, obtinebitur
$$z=c(-\lambda\mu uu-tt)-2\mu atu, \quad x=a(-\lambda\mu uu-tt)-2\lambda ctu, \quad y=b(-\lambda\mu uu+tt).$$

Has igitur tres solutiones ita aspectui opponamus:

Solutiones	x	y	z
I	$a(\mu\nu qq-pp)+2\nu bpq$,	$b(\mu\nu qq-pp)-2\mu apq$,	$c(\mu\nu qq+pp)$
II	$a(rr-\lambda\nu ss)$	$b(rr+\lambda\nu ss)+2\lambda crs$,	$c(rr+\lambda\nu ss)+2\nu brs$
III	$a(tt+\lambda\mu uu)+2\lambda ctu$,	$b(tt-\lambda\mu uu)$,	$c(tt+\lambda\mu uu)+2\mu atu$
I	$6qq-pp+6pq$,	$6qq-pp-4pq$,	$6qq+pp$
II	$rr-15ss$,	$rr+15ss+10rs$,	$rr+15ss+6rs$
III	$tt+10uu+10tu$,	$tt-10uu$,	$tt+10uu+4tu$.

Ubi notatu dignum, quod ternae formulae in qualibet columna eosdem numeros praebere queant, dummodo fuerit $\lambda cc=\mu aa+\nu bb$.

Exemplum. Sit proposita haec formula $5zz=2xx+3yy$, ut sit $\lambda=5$, $\mu=2$ et $\nu=3$, tum vero quia $5.1^2=2.1^2+3.1^2$, erit $c=1$, $a=1$ et $b=1$, unde ternae nostrae solutiones in tabella hic supra apponamus. Hinc si $p=1$ et $q=1$, erit $x=11$, $y=1$ et $z=7$; si $p=1$ et $q=-1$, erit $x=\pm 1$, $y=9$ et $z=7$, qui valores satisfaciunt. Sit porro pro secunda solutione $r=1$ et $s=1$ eritque $x=\pm 14$ seu ± 7, $y=26$ seu 13 et $z=22$ seu 11. Unde fit $5zz=2xx+3yy$ sive $605=98+507$. Sit $r=1$ et $s=-1$, ut sit $x=14$ seu 7, $y=6$ seu 3 et $z=10$ seu 5. Pro tertia sit $t=1$ et $u=1$ et erit $x=21$ seu 7, $y=\pm 9$ seu 3 et $z=15$ seu 5. Sit $t=1$ et $u=-1$ fietque $x=1$, $y=\pm 9$ et $z=7$.

A. m. T. I. p. 299. 300.

42.

(J. A. Euler.)

Criterium ad dignoscendum, utrum hujusmodi aequatio $fxx + gyy = hzz$ *sit possibilis, nec ne?*

Si est possibilis, casu $h = a$, tunc etiam erit possibilis casu $h = \frac{a(pp+fg)}{qq}$: hic scilicet pro p ejusmodi numerus sumi debet, ut $pp + fg$ divisorem habeat a, fuerit nempe $pp + fg = ab$ et sumto $q = a$ etiam casu $h = b$ erit possibilis. Tum vero pro b eodem modo operatio instituatur, sicque continuo ad minores numeros pervenietur, donec tandem judicium fiat facile.

Exemplum. I. Sit $7xx + 113yy = 114zz$, quae aequatio an sit possibilis, quaeritur. Hic est $f = 7$, $g = 113$, et quaeritur an sit possibilis casu $h = 114 = 2.3.19$? Statuatur ergo

$$h = \frac{114(pp+791)}{qq} \text{ et sumto } p = 3 \text{ et } q = 40, \text{ prodit casus } h = \frac{114.800}{1600} = 57.$$

II. Nunc iterum fiet $h = \frac{57(pp+791)}{qq}$ et fiat $pp + 791$ divisibile per 19, quod si fieri potest, dabitur casus quo $p < 19$. Ut, ex. gr. $pp + 12$ fiat divisibile per 19, debet esse $p = 8$, unde $h = \frac{855}{3.19} = 15$.

III. Quaestio ergo huc est reducta, an aequatio $7xx + 113yy = 15zz$ sit possibilis? quae hoc modo repraesentetur $15zz - 7xx = 113yy$, ubi $f = 15$, $g = -7$, $fg = -105$ et $h = 113$. Nunc fiat $h = \frac{113(pp-105)}{qq}$ reddatur $pp - 105$ divisibile per 113, quod fit sumendo $p = 52$, tum autem fiat $h = \frac{113.2599}{\square} = \frac{113.113.23}{\square}$ ergo quadrato sublato fit $h = 23$ et quaestio huc est reducta, an aequatio $15zz - 7xx = 23yy$ sit possibilis.

IV. Fiat ergo $h = \frac{23(pp-105)}{\square}$ sitque $pp - 13$ per 23 divisibile, sive $pp = 23n + 13$, quod fit si $n = 1$ et $p = 6$, ergo $h = \frac{-23.69}{\square} = -3$. Habetur igitur haec aequatio

$$15zz - 7xx = -3yy, \text{ sive } 7xx - 3yy = 15zz,$$

ubi $f = 7$, $g = -3$ et $fg = -21$, $h = 15$.

V. Fiat nunc $h = \frac{15(pp-21)}{\square}$. Sumatur $p = 4$, erit $h = \frac{-15.5}{\square} = -3$, ergo aequatio $7xx - 3yy = -3zz$ quod actu evenit si $x = 0$ et $y = z$, atque hinc sequitur ipsam aequationem propositam esse possibilem.

Nota. Revera autem est possibilis: si enim capiatur $z = 2$ et $y = 1$, fit

$$7xx + 113 = 456 \text{ sive } 7xx = 343 \text{ et } xx = 49 = \square.$$

Ita semper aequatio si fuerit possibilis, ad talem formam reduci poterit $axx + byy = azz$, cui manifesto satisfit sumendo $y = 0$ et $z = x$.

(Krafft.)

Judicium hoc reddi potest adhuc facilius hoc modo:

Cum sit $h = \frac{2.3.19(p^2+791)}{q^2}$, capiatur p ita, ut $p^2 + 791$ divisibile fiat per 2.3.19. Primo autem fit divisibile per 2, si $p = 2n + 1$; at vero per 3 fit divisibile, si $p = 3n \mp 1$. Utrumque igitur obtinetur, si $p = 6n \mp 1$. Restat, ut $p^2 + 791$ sit per 19 divisibile; quod fit, si p^2 per 19 divisum relinquat 7; sive debet esse $p^2 = 19n + 7$, ergo $19n + 7$ debet esse quadratum, quod fit, si $n = 3$, eritque $p = 8$. In genere ergo hoc fiet si $p = 19n \mp 8$; hoc est casibus $p = 8$, $p = 11$, $p = 27$, $p = 30$, $p = 46$, $p = 49$, $p = 65$, etc. inter quos numeros, reperitur statim 11, qui est formae $6n \mp 1$. Sumatur ergo $p = 11$, eritque $p^2 + 791 = 912 = 114.8$ ergo $h = \frac{114^2.8}{\square}$ et sublatis quadratis $h = 2$. Res ergo eo redit, an sit $7x^2 + 113y^2 = 2z^2$. Quia hic est $h = 2$, sumatur iterum $h = \frac{2(p^2+791)}{q^2}$. Ponatur $p = 7$, erit $h = \frac{2.840}{q^2} = 105$. Si sumsissemus $p = 3$, prodiisset

$\frac{2.800}{qq}=1$, et jam quaeritur, utrum possit esse $7x^2+113y^2=z^2$. Sumatur ergo $h=\frac{1(p^2+791)}{qq}$ et sumatur $p=0$, erit $h=\frac{7.113}{qq}$, et cum quaestio sit de forma $7x^2+113y^2=7.113z^2$, debet esse $x=113v$, ideoque $7.113v^2+y^2=7z^2$. Felicissime succedit, si in aequatione $h=p^2+791$ capiatur $p=7.4$. Tum erit

$$h=\frac{7^2.16+7.113}{qq}=\frac{7.225}{qq}=7,$$

ergo ventum est ad $7x^2+113y^2=7z^2$, quod fit si $y=0$ et $x=z$, ergo proposita aequatio est possibilis.

Haec solutio isti innititur principio: si fuerit $fx^2+gy^2=hz^2$, multiplicetur utrinque per p^2+fgq^2 fietque

$$h(p^2+fgq^2)z^2=fp^2x^2+gp^2y^2+f^2gq^2x^2+fg^2q^2y^2=f(px+gqy)^2+g(py-fqx)^2.$$

Ergo si ponatur $x'=px+gqy$ et $y'=py-fqx$, erit $fx'^2+gy'^2=h(p^2+fgq^2)z^2$; adeoque si aequatio proposita fuerit possibilis, etiam haec erit possibilis et vicissim.

Jam sumto $q=1$, habebitur praecedens forma $h(p^2+fg)$. Si nunc p ita sumi potest, ut p^2+fg divisorem habeat h, quod semper eveniet valore $p<\frac{1}{2}h$, et ponatur $p^2+fg=hh'$, ita ut loco h habeatur h^2h', sive omisso quadrato simpliciter h'. Sicque loco h prodiit novus valor h' illo multo minor; cum enim sit $p<\frac{1}{2}h$, erit $hh'<\frac{1}{4}h^2+fg$, ideoque $h'<\frac{1}{4}h+\frac{fg}{h}$. Sin autem pro p talis valor non detur, indicio id erit, aequationem propositam esse impossibilem; non autem hoc judicium inverti potest; dantur enim casus, quibus aequatio nihilominus est impossibilis; veluti evenit in hoc exemplo $2x^2+3y^2=7z^2$, ubi $f=2$, $g=3$ et $h=7$. Hinc novus valor orietur $h=7(p^2+6)$ et sumto $p=1$ fit $h=1$, unde novus valor erit $h=1(p^2+6)$, qui dat valores 7, 10, 15, etc., qui autem omnes nullo modo satisfaciunt; nam facile ostendi potest, aequationem $2x^2+3y^2=z^2$ esse impossibilem, sive $2x^2+3y^2$ quadratum esse non posse, vel enim x est divisibile per 3, vel non. Priori casu y non erit divisibile per 3, quia alioquin tota aequatio per 9 dividi posset, et posito $x=3v$, formula erit $18v^2+3y^2$, quae divisibilis est per 3, non vero per 9, ideoque quadratum esse nequit. At si $x=3v\pm1$, erit x^2 numerus formae $3n+1$, ideoque $2x^2=3n+2$ et ipsa formula $2x^2+3y^2$ erit numerus formae $3n+2$, quae forma quadratum esse nequit.

Simili modo judicari poterit, utrum hujusmodi aequatio generalior $fx^2+gxy+hy^2=kz^2$ possibilis sit nec ne. Multiplicetur enim utrinque per $p^2+gpq+fhq^2$, ut habeatur

$$(p^2+gpq+fhq^2)(fx^2+gxy+hy^2)=k(p^2+gpq+fhq^2)z^2,$$

ubi notandum, prius productum semper reduci posse ad formam $fX^2+gXY+hY^2$, quod cum non tam facile adpareat, per factores irrationales sequenti modo ostendetur.

Quaerantur factores formulae $fx^2+gxy+hy^2$, quod fit ponendo hanc formulam $=0$ et radicem extrahendo, unde fit $x=\frac{-gy\mp y\sqrt{(g^2-4fh)}}{2f}$, unde factores erunt

$$\frac{1}{4f}(2fx+gy+y\sqrt{(g^2-4fh)})(2fx+gy-y\sqrt{(g^2-4fh)})$$

sive $\frac{1}{f}(fx+\frac{1}{2}gy+y\sqrt{(\frac{1}{4}g^2-fh)})(fx+\frac{1}{2}gy-y\sqrt{(\frac{1}{4}g^2-fh)})$,

et posito brevitatis gratia $\frac{1}{4}g^2-fh=l$, ut fiat

$$fx^2+gxy+hy^2=\frac{1}{f}(fx+\frac{1}{2}gy+y\sqrt{l})(fx+\frac{1}{2}gy-y\sqrt{l}).$$

Simili modo erit $$p^2+gpq+fhq^2=(p+\frac{1}{2}gq+q\sqrt{l})(p+\frac{1}{2}gq-q\sqrt{l}) \text{ et}$$

$$fX^2+gXY+hY^2=\frac{1}{f}(fX+\frac{1}{2}gY+Y\sqrt{l})(fX+\frac{1}{2}gY-Y\sqrt{l});$$

haec ergo forma aequalis esse debet producto ex binis praecedentibus, quod fiet aequando alterutrum factorem producto ex binis praecedentibus, scilicet

$$fX+\frac{1}{2}gY+Y\sqrt{l}=(fx+\frac{1}{2}gy+y\sqrt{l})(p+\frac{1}{2}gq+q\sqrt{l});$$

sic enim sumto $\sqrt{l}$ negative, sponte fiet

$$fX+\frac{1}{2}gY-Y\sqrt{l}=(fx+\frac{1}{2}gy-y\sqrt{l})(p+\frac{1}{2}gq-q\sqrt{l});$$

sufficiet ergo alterutram ita evolvisse, ut membra rationalia et irrationalia seorsim inter se aequentur. Tum igitur fiet

$$fX+\frac{1}{2}gY=(fx+\frac{1}{2}gy)(p+\frac{1}{2}gq)+lgy$$

$$=pfx+\frac{1}{2}gpy+\frac{1}{2}fgqx+\frac{1}{2}g^2qy-fhqy$$

$$Y=fqx+gqy+py,$$

qui posterior valor in priore substitutus praebet

$$fX=pfx-fhqy, \quad \text{hinc} \quad X=px-hqy \quad \text{et} \quad Y=fqx+gqy+py.$$

Hoc igitur demonstrato ex dato valore k alius investigetur k', ut sit $k'=k(p^2+gpq+fhq^2)$, omissis factoribus quadratis, capiatur autem $q=1$, ut fiat $k'=k(p^2+gp+fh)$, et si aequatio est possibilis, loco p semper ejusmodi valorem reperire licet, ut formula $p^2+gp+fh$ factorem habeat k, quae posita $=kk'$ dabit novum valorem k'; quod si succedit, talis valor ipsius p semper dabitur minor, quam $\frac{1}{2}h$, dum scilicet p tam negative quam positive accipiatur, et sic valor k' multo minor erit quam k, unde continuo ad minores valores pervenietur, donec judicium facile reddatur.

Res exemplo illustretur: $5x^2+16xy+7y^2$, ubi $f=5$, $g=16$, $h=7$. Quaeramus casum possibilem, quo $k=7$, quippe qui oritur, si $x=1$ et $y=1$, ita ut sit $5x^2+16xy+7y^2=7z^2$, qui autem maxime est obvius sumendo $x=0$ et $y=z$. Ergo alium eligamus sitque $5x^2+16xy+7y^2=59z^2$ ut sit $k=59$. Jam quaeratur $k'=k(p^2\mp16p+35)$ et capiatur p ita, ut factor 59 tollatur, quod fit si $p=10$, $k'=59.295=5.59^2$, unde $k'=5$ qui casus est obvius sumendo $y=0$ et $z=x$.

Problema. Invenire numeros f et g, ut fiat $fx^2+gy^2=p^2+fg$.

Solutio. Erit ergo $fg-fx^2-gy^2=-p^2$; addatur x^2y^2 eritque $(f-y^2)(g-x^2)=x^2y^2-p^2$. Fiat $f-y^2=xy-p$, erit $g-x^2=xy+p$, ideoque $f=y^2+xy-p$ et $g=x^2+xy+p$, unde si f detur, ob $p=y^2+xy-f$, erit $g=x^2+y^2+2xy-f$, sive $f+g=(x+y)^2=\square$. Quoties ergo $f+g$ fuerit quadratum problemati satisfit; satisfiet ergo quoque, dummodo fuerit $fm^2+gn^2=\square$.

Theorema. Si fuerit $fx^2+gy^2=sz^2$ casu, quo $s=h$; tum etiam aequatio subsistere potest, quoties fuerit $s=h\mp4nfg$, dummodo hic numerus fuerit primus.

Hujus theorematis demonstratio etiamnum desideratur.

Exemplum. Sit $2x^2+3y^2=sz^2$ quod fieri potest si $s=5$. Idem ergo praestari potest si fuerit $s=5+24n$, unde hi numeri primi oriuntur: 5, 29, 53, 101, 149, 173, 197, 269, etc. Cum ergo sit $2x^2+3y^2=101z^2$, ita ut in superiori calculo sit $h=101$; erit $s=101(p^2+6)$. Fiat p^2+6 per 101 divisibile, sive $p^2=101n-6$, unde nascetur haec progressio arithmetica

0	1	2	3	4	5	6	
-6	95	196	297	398	499	600	etc.

ex qua vero illi numeri n valores excluduntur, qui habent sequentes formas

$$3\alpha+1, \quad 4\alpha, \quad 4\alpha+1, \quad 5\alpha+1, \quad 5\alpha+2.$$

Casui nostro satisfit si $p=14$, unde fit $s=2$; qui vero casus $2x^2+3y^2=2z^2$ est obvius; fit enim $y=0$.

eodem casu fit $s=149$; unde alius $149(p^2+6)$, ideoque $149n-6$ debet esse quadratum; unde excluduntur:

$$3\alpha+1,\quad 4\alpha,\quad 4\alpha+1,\quad 5\alpha+1,\quad 5\alpha+2,$$

remanent pro n ergo 3α, $3\alpha+2$, etc. et in numeris 3, 14, 15, 23, ubi $p=21$ satisfacit, seu $n=3$; $s=149.3.149=3$, unde iterum nascitur casus obvius. Omnes autem numeri primi pro s, quibus formula $2x^2+3y^2=sz^2$ subsistere potest, continentur in his duabus formulis $24n+5$ et $24n+11$, quibus adjungi debent 2 et 3 et praeterea nulli alii satisfaciunt, ita, ut satisfacientes ordine sint:

2, 3, 5, 11, 29, 53, 59, 83, 101, 107, 131, 149, 173, 179, 197.

Aliud judicium, utrum talis aequatio $fx^2+gy^2=hz^2$ *sit possibilis.*

Dividantur omnia quadrata per numerum h et notentur residua, quae sint 1, a, b, c, d, etc. et quadratum x^2 det residuum a, y^2 vero det b, sicque formula fx^2+gy^2 dabit residuum $af+bg$, quod cum per h debeat esse divisibile, fieri poterit $af+bg=0$, ideoque $b=\frac{-af}{g}$; ergo quodvis residuum si per $\frac{-f}{g}$ multiplicetur, iterum erit residuum. Quia autem $\frac{-f}{g}$ est fractus, ejus loco scribatur $\frac{nh-f}{g}$, ubi n ita sumatur, ut $nh-f$ fiat divisibile per g et quotus sit k, qui si inter residua reperiatur, aequatio erit possibilis; sin secus, impossibilis. Sic proposita aequatione $2x^2+3y^2=29z^2$, ubi $f=2$, $g=3$ et $h=29$, quaerantur residua quadratorum per 29 divisorum, quae sunt numero 14, nempe:

1, 4, 9, 16, 25, 7, 20, 6, 23, 13, 5, 28, 24, 22.

Quaeratur ergo $\frac{29n-2}{3}=9$ posito $n=1$. Quia ergo 9 inter residua occurrit, haec forma est possibilis. Sin autem proponatur $2x^2+3y^2=17z^2$, quadrata per 17 divisa dant residua

1, 4, 9, 16, 8, 2, 15, 13.

Nunc debet esse $\frac{17n-2}{3}=$ numero integro 5, qui cum non sit inter residua, indicat aequationem esse impossibilem.

Hoc vero judicium non certum videtur, nam si aequatio hac forma exhibeatur $17z^2-2x^2=3y^2$, ubi $f=17$, $g=-2$ et $h=3$, residuum quadratorum est unicum 1; at vero $\frac{3n-17}{-2}=7$, si $n=1$, et denuo per 3 dividendo prodit 1, quod est residuum, et tamen aequatio est impossibilis.

Notari meretur aequatio $7x^2+2y^2=23z^2$, quia ipse numerus 23 non in forma $7a^2+2b^2$ continetur, siquidem a et b sint integri; at si $a=\frac{1}{3}$ et $b=\frac{10}{3}$, fit utique $\frac{207}{9}=23$. Per regulam primam autem ex 23 prodit alius $23(p^2+14)$. Sumatur $p=3$ proditque unitas.

(*J. A. Euler.*)

Ut dubium circa criterium postremum tollatur, observandum est primum, criterium eo redire, num inter residua quadratorum per h divisorum occurrat numerus $-fg$, sive $nh-fg$, qui si non occurrat, aequatio $fxx+gyy=hzz$ certe est impossibilis; sin autem occurrat, plus inde non sequitur, quam vel hanc ipsam aequationem $fxx+gyy=hzz$, vel istam $xx+fgyy=hzz$ esse possibilem; unde fieri potest, ut prior non sit possibilis, tum autem certo posterior fit possibilis.

A. m. T. I. p. 201—207.

43.

(*W. L. Krafft.*)

Problema. Formulam mx^3+n quadratum reddere ex casu cognito $ma^3+n=bb$.

Solutio. Ponatur $x=a+y$ et formula proposita fiet

$$bb+3maay+3mayy+my^3=\square,$$

cujus radix ponatur $b+\frac{3maa}{2b}y$; hujus quadratum

$$bb+3maay+\frac{9mma^4}{4bb}yy \text{ praebet}$$

$$3a+y=\frac{9ma^4}{4bb}=\frac{9a(bb-n)}{4bb}$$

unde $y=-\frac{3a}{4}-\frac{9an}{4bb}$, ergo $x=\frac{a}{4}-\frac{9an}{4bb}$. Sin autem radix ponatur $b+\frac{3maa}{2b}y+pyy$, erit

$$3mayy+my^3=2bpyy+\frac{9mma^4}{4bb}yy+\frac{3mpaay^3}{b}+ppy^4.$$

Jam fiat $3ma=2bp+\frac{9mma^4}{4bb}=2bp+\frac{9am(bb-n)}{4bb}$, ergo

$$2bp=3ma-\frac{9am(bb-n)}{4bb}=\frac{3ma}{4}+\frac{9amn}{4bb}, \text{ ergo}$$

$$p=\frac{3ma}{8b}+\frac{9amn}{8b^3}=\frac{3ma}{8b}\left(1+\frac{3n}{bb}\right).$$

Superest haec aequatio $1=\frac{3paa}{b}+\frac{ppy}{m}$, ergo

$$\frac{ppy}{m}=1-\frac{3paa}{b}=1-\frac{9ma^3}{8bb}-\frac{27a^3mn}{8b^4}=-\frac{1}{8}+\frac{9n}{8bb}-\frac{27n}{8bb}+\frac{27nn}{8b^4}$$

$$=-\frac{1}{8}-\frac{9n}{4bb}+\frac{27nn}{8b^4}, \text{ ergo } y=\frac{m}{8pp}\left(-1-\frac{18n}{bb}+\frac{27nn}{b^4}\right).$$

(Lexell.)

Annotatio ad superiorem formulam $mx^3+n=\square$, *ubi ex dato casu* $ma^3+n=bb$ *ope transformationis elicuimus novum casum.*

Omni attentione dignum videtur, quod si n fuerit numerus quadratus $=kk$, ex casu cognito $ma^3+kk=bb$ immediate duo alii elici queant hoc modo. Ponatur $x=az$, ut habeatur $ma^3z^3+kk=\square$, hoc est

$$(bb-kk)z^3+kk=\square=yy, \quad \text{ita ut sit} \quad (bb-kk)z^3=yy-kk=(y+k)(y-k).$$

Resolvetur ergo formula $(bb-kk)z^3$ etiam in duos factores, quod duplici modo fieri potest:

I. Sit unus factor $(b+k)zz=y+k$, eritque alter $(b-k)z=y-k$, haec aequatio ab illa subtracta relinquit $(b+k)zz-(b-k)z=2k$, cujus una radix manifesto est $z=1$, unde pro altera fit

$$z=\frac{-2k}{b+k}, \quad \text{ergo} \quad x=\frac{-2ak}{b+k}.$$

II. Sit jam prior factor $(b-k)zz=y-k$, et alter dabit $(b+k)z=y+k$, unde fit differentia

$$(b-k)zz-(b+k)z=-2k,$$

cujus una radix est $z=1$, et altera $\quad z=\frac{2k}{b-k} \quad$ et $\quad x=\frac{2ak}{b-k}$.

Unde conficimus istud egregium THEOREMA:

Si formula mx^3+kk fuerit quadratum casu $x=a$, ita ut sit $ma^3+kk=bb$, tum etiam quadratum erit his duobus casibus:

$$\text{primo: } x=\frac{-2ak}{b+k}, \quad \text{et altero: } x=\frac{+2ak}{b-k}.$$

Exempli gratia, cum formula $90x^3+1$ fiat quadratum casu $x=\frac{1}{2}$; fiet enim $90\cdot\frac{1}{8}+1=\frac{49}{4}$, ubi est $a=\frac{1}{2}$, $k=1$ et $b=\frac{7}{2}$, bini casus derivati erunt $x=-\frac{2}{9}$ et $x=+\frac{2}{5}$; ex priore enim fit $\frac{-10.8}{81}+1=\frac{1}{81}$, et ex posteriore $\frac{90.8}{5^3}+1=\frac{18.8}{25}+1=\frac{169}{25}$.

A. m. T. I. p. 120. 121.

44.

Omni attentione digna est haec formula $181^2+7=32^3$; nam etsi $32=5^2+7$, tamen nullo modo est $181+\sqrt{-7}=(5+\sqrt{-7})^3$, verum tamen est

$$181+\sqrt{-7}=\left(\frac{1+3\sqrt{-7}}{8}\right)(5+\sqrt{-7})^3.$$

Notandum autem est $\frac{1+3\sqrt{-7}}{8}\cdot\frac{1-3\sqrt{-7}}{8}=1$; unde patet evolutionem illam per factores imaginarios profundiorem investigationem requirere.

Problema. Invenire in integris quadratum et cubum, quorum differentia sit valde parva, veluti

$$32^3-181^2=7 \quad \text{et} \quad 253^2-40^3=9.$$

Cum proxime esse debeat $x^2=y^3$, ponatur $x=p^3+a$, hincque fit

$$y=(p^3+a)^{\frac{2}{3}}=pp+\frac{2}{3}\cdot\frac{a}{p}-\frac{aa}{9p^4}\ \text{etc.}$$

unde si p et a ita sumantur, ut haec formula proxime aequetur numero integro y, problema erit solutum; veluti sumto $a=3$ et $p=2$, formula illa dat $y=5$ et $x=11$, fit autem 11^2 proxime $=5^3$.

A. m. T. I. p. 127.

45.

(*J. A. Euler.*)

Si debeat esse $13xx+12=\square$, valores pro x erunt

$$1,\quad 2,\quad 13,\quad 23,$$

quorum ordo ita se habet

$$-23,\quad -2,\quad +1,\quad +13,\quad \ldots\quad p,\quad q,\quad r$$

existente $r=11q-p$, ubi numerus 11 inde oritur, quod sit $\frac{11}{2}=\sqrt{(13\cdot\frac{9}{4}+1)}$.

Si debeat esse $5xx+44=\square$, valores pro x erunt

$$1,\quad 2,\quad 5,\quad 7,\quad 14,\quad 19,\quad 37,\quad 50,$$

quorum ordo ita se habet

$$-50,\quad -19,\quad -7,\quad -2,\quad +1,\quad +5,\quad +14,\quad +37,\quad \ldots\quad p,\quad q,\quad r$$

et $r=3q-p$, propter $\frac{3}{2}=\sqrt{(5\cdot\frac{1}{4}+1)}$.

Ut $3xx-143=\square$ debet esse $x=7$, 8, 9, 12, 16, 23, 28, quae multitudo est notatu digna et inde venit quod $143=11\cdot13$.

A. m. T. I. p. 135. 136.

46.

Problema. Datis numeris a et b, invenire omnes numeros x, ut haec formula $ax+b$ fiat quadratum.

Ponatur $x=ayy+2py+q$ et quadratum esse debet

$$aayy+2apy+aq+b=\square, \text{ quod fit, si } p=\sqrt{(aq+b)}.$$

Cognito ergo unico casu, qui sit $aq+b=pp$, erit

$$x=ayy\pm2py+q,$$

quae formula omnes soluliones continet.

Exemplum. Sit $a=7$ et $b=2$, et formula nostra $7x+2$. Quia $7.1+2=3^2$, erit $q=1$ et $p=3$; ergo omnes casus sunt $x=7yy\pm 6y+1$, quae formula praebet hos numeros pro x:

Existente y	prodit x
0	1
1	$7\pm 6+1$; 2 vel 14
2	29 ± 12; 17 vel 41
3	64 ± 18; 46 vel 82
4	113 ± 24; 89 vel 137
etc.	etc.

Lex progressionis valorum pro x:

	1,	2,	14,	17,	41,	46,	82,	89,	137,	etc.	
Diff.	1	12	3	24	5	36	7	48	9	60	etc.

A. m. T. I. p. 214.

47.

Zwei Trigonalzahlen zu finden, deren Produkt wieder eine Trigonalzahl sei.

Also
$$\frac{xx+x}{2}\cdot\frac{yy+y}{2}=\frac{zz+z}{2},\quad \text{oder}\quad x(x+1)\,y(y+1)=2z(z+1),\quad \text{oder}$$
$$pq\,.\,x(x+1)\,y(y+1)=2z(z+1)\,.\,pq.$$

Nun mache man $px(y+1)=2qz$ und $qy(x+1)=p(z+1)$, so wird aus dem ersten Satze
$$z=\frac{px(y+1)}{2q},\quad \text{und aus dem andern:}\quad z=\frac{qy(x+1)}{p}-1.$$

Daher
$$2qqy(x+1)-2pq=ppx(y+1),$$
welches sich auch so darstellen lässt
$$xy(2qq-pp)+2qqy-2pq-ppx=0.$$

Es sei nun $2qq-pp=a$, so ist
$$axy+2qqy-ppx-2pq=0,\quad \text{und hieraus}\quad y=\frac{ppx+2pq}{ax+2qq},$$
also
$$ay=\frac{appx+2apq}{ax+2qq}\quad \text{und}\quad ay-pp=\frac{2apq-2ppqq}{ax+2qq}.$$

Es sei nunmehr $2ppqq-2apq=2pq(pq-a)=fg$, so haben wir $pp-ay=\frac{fg}{ax+2qq}$. Nun setze man $ax+2qq=f$, so wird $pp-ay=g$, folglich $y=\frac{pp-g}{a}$ und $x=\frac{f-2qq}{a}$, wo leicht zu machen, dass $a=1$ sei.

Exempel. Man nehme $p=7$, $q=5$, $a=1$, so ist $fg=34.70=4.5.7.17$. Daher wird
$$x=f-50\quad \text{und}\quad y=49-g.$$

Es sei $g=20$, $f=119$, so ist $\qquad x=69 \quad$ und $\quad y=29$;

also die zwei Trigonalzahlen 35.69 und 15.29. Da nun $z=\frac{px(y+1)}{2}=7.69.3=1449$, so ist
$$\frac{zz+z}{2}=1449.725,\quad \text{welches in der That}=69.35.29.15\ \text{ist}.$$

Es sei ferner $f = 85 = 5.17$, so wird $g = 4.7 = 28$, $x = 35$, $y = 21$; mithin die beiden Trigonalzahlen 35.18 und 11.21. Es ist aber $z = 7.7.11 = 539$, also

$$\frac{zz+z}{2} = 539.270 = 35.18.11.21.$$

A. m. T. I. p. 254.

48.

(*N. Fuss I.*)

Problema. Invenire numeros integros x et y, ut fiat $axx - byy = A$.

Solutio. Primo notandum est hoc fieri non posse, nisi fuerit $A = aff - bgg$; deinde quaerantur per problema Pellianum numeri m et n, ut fiat $mm = abnn + 1$, sive $m = \sqrt{(abnn+1)}$. Cum ergo sit $mm - abnn = 1$, erit quoque $(mm - abnn)^{\lambda} = 1$; ponere igitur licebit

$$axx - byy = (aff - bgg)(mm - abnn)^{\lambda},$$

quae forma in factores irrationales resoluta dabit

$$x\sqrt{a} + y\sqrt{b} = (f\sqrt{a} + g\sqrt{b})(m + n\sqrt{ab})^{\lambda},$$

quod posterius productum evolvatur et termini signo $\sqrt{a}$ affecti aequentur ipsi $x\sqrt{a}$; reliqui termini signo $\sqrt{b}$ affecti aequentur ipsi $y\sqrt{b}$, hocque modo tam x quam y per numeros integros determinabitur, quod exemplo illustremus:

Exemplum. Quaerantur numeri x et y, ut fiat $3xx - yy = 2$, ubi $a = 3$ et $b = 1$; erit autem $2 = 3ff - gg$ sumendo $f = 1$ et $g = \pm 1$; quia igitur $ab = 3$, fiet $m = \sqrt{(3nn+1)}$. Sumendo $n = 1$ et $m = 2$, unde nostra formula erit $x\sqrt{3} + y = (\sqrt{3} + 1)(2 + \sqrt{3})^{\lambda}$

$$\begin{array}{lll}
\text{si } \lambda = 0 & \text{erit} & x\sqrt{3} + y = \sqrt{3} + 1 \\
\lambda = 1 & \text{»} & x\sqrt{3} + y = 3\sqrt{3} + 5 \\
\lambda = 2 & \text{»} & x\sqrt{3} + y = 11\sqrt{3} + 19 \\
\lambda = 3 & \text{»} & x\sqrt{3} + y = 41\sqrt{3} + 71 \\
\lambda = 4 & \text{»} & x\sqrt{3} + y = 153\sqrt{3} + 265.
\end{array}$$

Ceterum valores tam ipsius x quam ipsius y constituunt series recurrentes, quarum ultimus quisque terminus per $2m$ multiplicatus demto penultimo, praebet sequentem; sic in nostro exemplo, ubi $2m = 4$, litterae x et y ita procedant

$$x = 1, \quad 3, \quad 11, \quad 41, \quad 153$$
$$y = 1, \quad 5, \quad 19, \quad 71, \quad 265.$$

Occasione problematis Pelliani, seu formulae $m = \sqrt{(abnn+1)}$, praeter casus, ubi est vel $ab = \alpha\alpha \pm 1$, vel $ab = \alpha\alpha \pm 2$, etiam sequentes casus generaliores locum habent, scilicet si fuerit $ab = \alpha\alpha\beta\beta \pm \beta$, fiet $nn = 4\alpha\alpha$ et $n = 2\alpha$ et $m = 2\alpha\alpha\beta \pm 1$. Deinde si fuerit $ab = \alpha\alpha\beta\beta \pm 2\beta$, erit $n = \alpha$ et $m = \alpha\alpha\beta \pm 1$.

A. m. T. I. p. 277.

49.

Problema. Invenire duos numeros p et q, ut fiat $(pp+1)^2 + (qq+1)^2 = \square$.

Solutio. Ponatur $pp + 1 = xx - yy$ et $qq + 1 = 2xy$ eritque $pp = xx - yy - 1$ et $qq = 2xy - 1$. Sit jam $p = x - z$, erit $2xz - zz = yy + 1$, unde fit

$$x=\frac{zz+yy+1}{2z},\quad \text{hinc}\quad qq=\frac{yzz+y^3+y}{z}-1.$$

Statuatur nunc $y=nz$ fietque $qq=nzz+n^3zz+n-1$. Capiatur ergo $n=2$, ut sit $y=2z$, erit $qq=10z^2+1$ cui satisfit:

1) Si $z=\frac{2}{3}$; tum erit $qq=\frac{49}{9}$, ergo $q=\frac{7}{3}$. Porro $y=\frac{4}{3}$, unde $x=\frac{29}{12}$ et $p=\frac{7}{4}$. Tum igitur formula

$$\left(\frac{49}{16}+1\right)^2+\left(\frac{49}{9}+1\right)^2,\quad \text{erit quadratum radicis}\quad xx+yy=\frac{841}{144}+\frac{16}{9}=\frac{1097}{144}.$$

2) Sumatur $z=\frac{2}{9}$, erit $qq=\frac{121}{81}$, hinc $q=\frac{11}{9}$; tum vero $y=\frac{4}{9}$ et $x=\frac{5zz+1}{2z}=\frac{101}{36}$, ergo

$$p=\frac{101}{81}-\frac{2}{9}=\frac{83}{81}.$$

3) Sumatur $z=6$, erit $qq=361$ et $q=19$; porro $y=12$ et $x=\frac{181}{12}$ et $p=\frac{109}{12}$; ergo

$$\left(\frac{109^2}{144}+1\right)^2+(361+1)^2=\left(\frac{181^2}{144}+144\right)^2.$$

A. m. T. I. p. 279.

50.

PROBLEMA DIFFICILLIMUM,

quo quatuor biquadrata quaeruntur, quorum summa itidem sit biquadratum, sive ut sit $A^4+B^4+C^4+D^4=E^4$, siquidem fieri potest, sequenti modo tractandum videtur:

Statuatur scilicet $\quad A^2=\frac{pp+qq+rr-ss}{n};\quad B^2=\frac{2ps}{n};\quad C^2=\frac{2qs}{n}\quad$ et $\quad D^2=\frac{2rs}{n};\quad$ tum enim fiet

$$E^2=\frac{pp+qq+rr+ss}{n},$$

ita, ut haec quinque formulae quadrata reddi debeant. Incipiamus a prima et ultima, ita, ut reddi debeat

$$\frac{pp+qq+rr}{n}\pm\frac{ss}{n}=\square.$$

Cum igitur sit $aa+bb\pm 2ab=\square$, statuatur

$$\frac{pp+qq+rr}{n}=aa+bb\quad \text{et}\quad \frac{ss}{n}=2ab,$$

fit ergo $ss=2nab=\square$. Sit $2n=\alpha\beta$ et statuatur $a=\alpha ff$ et $b=\beta gg$ eritque

$$ss=\alpha\alpha\beta\beta ffgg,\quad \text{ergo}\quad s=\alpha\beta fg=2nfg;$$

tum vero esse debet

$$\frac{pp+qq+rr}{n}=\alpha\alpha f^4+\beta\beta g^4.$$

Pro reliquis conditionibus faciamus $\frac{2ps}{n}=4pfg=4xx$, unde $p=\frac{xx}{fg}$, porro $\frac{2qs}{n}=4qfg=4yy$, hinc $q=\frac{yy}{fg}$, pro quarta $\frac{2rs}{n}=4rfg=4zz$, habemus $r=\frac{zz}{fg}$. Hinc superest ista aequatio

$$\frac{pp+qq+rr}{n}=\alpha\alpha f^4+\beta\beta g^4,\quad \text{sive}\quad pp+qq+rr=\frac{1}{2}\alpha\beta(\alpha\alpha f^4+\beta\beta g^4),$$

et loco p, q, r valores inventos substituendo

$$\frac{x^4}{f^2g^2}+\frac{y^4}{f^2g^2}+\frac{z^4}{f^2g^2}=\frac{1}{2}\alpha\beta(\alpha\alpha f^4+\beta\beta g^4),\quad \text{sive}\quad x^4+y^4+z^4=\frac{1}{2}\alpha\beta ffgg(\alpha\alpha f^4+\beta\beta g^4).$$

Quodsi jam hic restituamus $\alpha ff = a$ et $\beta gg = b$, colligitur $x^4 + y^4 + z^4 = \frac{1}{2} ab(aa + bb)$. Quaeritur ergo num huic aequationi satisfieri possit; tum autem fiet

$$n = \frac{1}{2}\alpha\beta, \quad s = 2nfg \text{ vel } \alpha\beta fg, \quad \text{et} \quad p = \frac{xx}{fg}, \quad q = \frac{yy}{fg} \quad \text{et} \quad r = \frac{zz}{fg},$$

atque hinc porro $\qquad A = a - b$ et $E = a + b$, $B = 2x$, $C = 2y$ et $D = 2z$,

verum ne his quidem ambagibus est opus, cum enim fieri debeat $B^4 + C^4 + D^4 = E^4 - A^4$. Statuatur

$$A = a - b \text{ et } E = a + b, \quad \text{fietque} \quad E^4 - A^4 = 8a^3b + 8ab^3 = 8ab(aa + bb),$$

ergo utrinque per 16 dividendo prodit

$$\frac{1}{2} ab(aa + bb) = \frac{B^4 + C^4 + D^4}{16} = x^4 + y^4 + z^4.$$

A. m. T. I. p. 281.

51.

Notatu digna est haec formula: $1 + z - z^3$, quae fit quadratum sumto $z = \frac{11}{9}$, qui tamen valor per regulas vulgares non elicitur. Hinc ista quaestio:

Numerum 2 dividere in duas partes x et $2 - x$, quarum productum $2x - xx$ sit numerus formae $z^3 - z$; tum enim erit $1 - 2x + xx = 1 + z - z^3$, ideoque $1 - x = \sqrt{(1 + z - z^3)}$. Sumto ergo $z = \frac{11}{9}$, erit $1 - x = \frac{17}{27}$, hinc $x = \frac{10}{27}$, et altera pars $2 - x = \frac{44}{27}$, quarum productum est

$$2x - xx = \frac{440}{27^2}, \text{ at vero } z^3 - z = \frac{1331}{729} - \frac{11}{9} = \frac{440}{27^2}.$$

A. m. T. I. p. 295.

52.

Theorema I. Si p denotet numerum primum quemcunque, talis aequatio $z^3 = py^3 \pm ppx^3$ semper est impossibilis.

Demonstratio. Quia enim esse deberet z^3 divisibile per p, ideoque $z = pA$, unde fieret

$$ppA^3 = y^3 \pm px^3, \quad \text{sive} \quad y^3 = ppA^3 \mp px^3;$$

foret igitur etiam y divisibilis per p. Sit ergo $y = pB$, unde fieret $ppB^3 = pA^3 \mp x^3$, hinc ergo etiam x divisibile esse debet per p; hincque ponatur $x = pC$; unde fiet $ppC^3 = A^3 - pB^3$; foret igitur eodem modo $A = pD$, foretque $ppD^3 = pC^3 + B^3$, tum vero etiam B per p divisibilis esse deberet, porro etiam C, D, etc. in infinitum. Hoc ergo modo singulae litterae z, y, x non solum per p, sed etiam per pp, per p^3 atque adeo per p^∞ deberent esse divisibiles; quod cum sit absurdum, veritas theorematis est evicta.

Theorema II. Si numeri a, b, c fuerint primi ad p, ita ut nullus eorum per p sit divisibilis, tum etiam haec aequatio $az^3 \pm bpy^3 \pm cppx^3 = 0$ semper est impossibilis.

Demonstratio. Quia enim a per p non est divisibile, z deberet esse divisibile, tum vero etiam pari modo y et x, sicque ad eandem aequationem perveniretur, unde patet impossibilitas, uti casu praecedente.

Corollarium. Eadem demonstratio quoque habet locum, si p fuerit productum ex duobus vel pluribus numeris primis diversis, veluti si sit $p = 2.3$, vel 3.5, vel $3.5.7$, etc.

Theorema III. Si p sit vel numerus primus, vel productum ex aliquot numeris primis diversis, ... vero numeri a, b, c, d, etc. sint numeri ad p primi, tum etiam haec aequatio semper est impossibilis:

$$az^4 \pm bpy^4 \pm cppx^4 \pm dp^3v^4 = 0.$$

Quia ob rationes superiores singuli numeri z, y, x, v, etc. non solum per p, sed per omnes potestates ipsius p deberent esse divisibiles. Taliaque theoremata ad potestates altiores extendi possunt.

NB. Hae autem demonstrationes vim perderent suam, si esset $p = 1$, quia omnes plane numeri divisibiles sunt per omnes potestates ipsius 1.

A. m. T. II. p. 10. 11.

53.

Problema. Reddere hanc formulam quadratum: $(A + Bz)(a + bz + czz + dz^3)$.

Solutio. Statuatur hoc quadratum $= (A + Bz)^2 (p + qz)^2$ fietque

$$\left.\begin{matrix} App \\ -a \end{matrix}\quad \begin{matrix} +2Apq \\ +Bpp \\ -b \end{matrix}\right\} z \quad \left.\begin{matrix} +Aqq \\ +2Bpq \\ -c \end{matrix}\right\} zz \quad \left.\begin{matrix} +Bqq \\ -d \end{matrix}\right\} z^3 = 0,$$

ubi duae solutiones sunt considerandae, primo si $pp = \frac{a}{A}$, sumatur $q = \frac{b - Bpp}{2Ap}$, tum erit $z = \frac{c - Aqq - 2Bpq}{Bqq - d}$.
Altera solutio locum habet si $qq = \frac{d}{B}$; tum sumatur

$$p = \frac{c - Aqq}{2Bq} \quad \text{eritque} \quad z = \frac{a - App}{2Apq + Bpp - b}.$$

Problema. Si proposita fuerit haec formula $(A + Bz + Czz)(a + bz + czz)$, eam reddere quadratum.

Solutio. Ponatur hoc quadratum $= pp(a + bz + czz)^2$ fietque

$$A + Bz + Czz = ppa + ppbz + ppczz.$$

Hic ergo si fuerit $pp = \frac{A}{a}$, statim fit $z = \frac{ppb - B}{C - cpp}$. Secundo si fuerit $pp = \frac{C}{c}$, erit $z = \frac{app - A}{B - bpp}$.

In genere autem si satisfaciat valor $z = f$, quo casu fit $pp = \frac{A + Bf + Cff}{a + bf + cff}$, tum semper alius valor potest inveniri; quoniam enim habetur haec aequatio quadratica

$$zz - \frac{B - bpp}{C - cpp} z + \frac{A - app}{C - cpp} = 0,$$

eaque per hypothesin radicem habeat $z = f$; erit quoque $z = g$ existente

$$\text{tam} \quad f + g = \frac{bpp - B}{C - cpp} \quad \text{quam} \quad fg = \frac{A - app}{C - cpp},$$

unde duplici modo alter valor g reperitur. Hoc adhuc clarius ita ostendi potest. Cum esse del

$$A + Bz + Czz = pp(a + bz + czz),$$
$$\text{tum vero} \quad A + Bf + Cff = kk(a + bf + cff),$$

semper alius valor pro z assignari potest, existente pariter $p = k$. Dividatur prior aequatio per posteriorem fietque

$$\frac{A + Bz + Czz}{A + Bf + Cff} = \frac{a + bz + czz}{a + bf + cff};$$

subtrahendo utrinque unitatem et dividendo per $z - f$ prodit

$$\frac{B + C(z + f)}{A + Bf + Cff} = \frac{b + c(z + f)}{a + bf + cff},$$

unde f facile definitur.

Hac methodo insuper duo alii valores reperiri possunt. Cum enim sit

$$\frac{A+Bz+Czz}{A+Bf+Cff}=\frac{a+bz+czz}{a+bf+cff},$$

multiplicetur utrinque per $\frac{f}{z}$, ut habeatur

$$\frac{Af+Bfz+Cfzz}{Az+Bfz+Cffz}=\frac{af+bfz+cfzz}{az+bfz+cffz}$$

et sublata unitate erit
$$\frac{A(f-z)+Cfz(z-f)}{Az+Bfz+Cffz}=\frac{a(f-z)+cfz(z-f)}{az+bfz+cffz},$$

sive
$$\frac{A-Cfz}{A+Bf+Cff}=\frac{a-cfz}{a+bf+cff},$$

unde tertius desumitur valor. Porro multiplicetur utrinque per $\frac{ff}{zz}$, ut sit

$$\frac{Aff+Bffz+Cffzz}{Azz+Bfzz+Cffzz}=\frac{aff+bffz+cffzz}{azz+bfzz+cffzz}$$ et unitate utrinque sublata $$\frac{A(f+z)+Bfz}{A+Bf+Cff}=\frac{a(f+z)+bfz}{a+bf+cff}.$$

Horum valorum quilibet pro f assumtus praebebit duos novos valores, ita ut hoc modo infiniti valores reperiri queant. Sit $A=2$, $B=3$, $C=-1$, $a=3$, $b=-1$, $c=2$, ut debeat esse $\frac{2+3z-zz}{3-z+2zz}=\square$. Cui primo satisfacit $z=1$. Sit ergo $f=1$, erit secundo $\frac{3-z-1}{4}=\frac{2(z+1)-1}{4}$, unde $z=1$; tertio $2+z=3-2z$, unde $z=\frac{1}{3}$; quarto $2+2z+3z=3+3z-z$, unde $z=\frac{1}{3}$.

Interim tamen haec methodus nihil plane juvat, sed tantum duos valores ostendit. Cum enim f sit numerus definitus, aequatio $\frac{A+Bz+Czz}{A+Bf+Cff}=\frac{a+bz+czz}{a+bf+cff}$ manifesto est aequatio quadratica determinata, quae tantum duos admittit valores. Interim tamen haec methodus cum successu adhibitur in resolutione hujus formulae simplicioris $a+bz+czz=pp$, casusque constet, quo $a+bf+cff=kk$, erit igitur $\frac{a+bz+czz}{a+bf+cff}=\frac{pp}{kk}$. Jam sumatur primo $pp=kk$, fietque $b+c(z+f)=0$, unde $z=\frac{-b-cf}{c}$. Deinde sumatur $\frac{pp}{kk}=\frac{zz}{ff}$ eritque

$$ppff=kkzz,\quad \text{ideoque}\quad aff+bffz+cffzz=azz+bfzz+cffzz,$$

unde fit
$$a(f+z)+bfz=0\quad \text{atque}\quad z=\frac{-af}{a+bf},$$

qui posterior valor loco f assumtus denuo novum valorem praebet, et ita porro.

Verum hoc casu solutio generalis ita reperiri potest. Sumatur $p=k+\nu(z-f)$, ut sit

$$\frac{a+bz+czz}{a+bf+cff}=\frac{kk+2k\nu(z-f)+\nu\nu(z-f)^2}{kk}.$$

Subtrahatur utrinque unitas eritque

$$\frac{b+c(z+f)}{kk}=\frac{2k\nu+\nu\nu(z-f)}{kk},\quad \text{unde reperitur}\quad z=\frac{2k\nu-f\nu\nu-b-cf}{c-\nu\nu},$$

unde prior oritur posito $\nu=0$, posterior vero posito $\nu=\frac{k}{f}$.

A. m. T. II. p. 155. 156.

54.

Theorema. Si fuerit p numerus primus formae $4n+1$, semper dabitur numerus x, minor quam n, ut fiat $px-1=\square$.

Demonstratio. Cum sit $p=4n+1$, erit $p=aa+bb$. Jam quaeratur fractio $\frac{c}{d}$ proxime aequalis fractioni $\frac{a}{b}$, ita, ut sit $ad-bc=\pm1$, eritque $x=cc+dd$. Semper enim numeri c et d infra semisses numerorum a et b assignari possunt; tum autem erit $px-1=(ac+bd)^2$. Erit enim

$$px = (aa + bb)(cc + dd) = (ac + bd)^2 + (ad - bc)^2$$

at $ad - bc = \pm 1$ per hypothesin, unde $px - 1 = (ac + bd)^2$.

Exemplum. Sit $p = 193$, erit $a = 12$ et $b = 7$, porro $c = 5$ et $d = 3$, unde fit

$$x = 34 \quad \text{eritque} \quad px - 1 = 81^2.$$

A. m. T. II. p. 167.

55.

Theorema numericum profundissimae indaginis.

Si m, n et z denotent numeros integros positivos, tum ista formula

$$4mnz - m - n$$

nunquam evadere potest quadratum.

Hoc theorema inde est derivatum, quod inter divisores formae $mxx + yy$ occurrat formula $4mz + 1$, unde sequitur, formulam $4mz - 1$ nunquam esse posse divisorem illius formae $mxx + yy$, vel saltem hujus $m + yy$, unde haec aequatio

$$(4mz - 1)n = m + yy$$

semper erit impossibilis. Sit $\sqsupset\sqsubset$ signum impossibilitatis eritque $(4mz - 1)n - m \sqsupset\sqsubset yy$ sive $(4mz - 1)n - m \sqsupset\sqsubset \square$. Verum hoc fundamentum nondum est rigide demonstratum, ideoque demonstratio hujus theorematis plurimum desideratur. Interim tamen evidens est ejus veritas casibus, quibus est $m + n = 4i + 2$, quia tum fit $4mnz - 4i - 2$ numerus impariter par, a quadrato abhorrens. Dein etiam casu $m + n = 4i + 1$, quia tum prodit forma $4mnz - 4i - 1$, sive forma $4A - 1$, quae nunquam esse potest quadratum. Demonstrandi igitur tantum restant duo casus, alter, quo $m + n = 4i$, alter vero, quo $m + n = 4i + 3$, vel $4i - 1$. Pro casu priore $m + n = 4i$ sumi poterit $m = 2i - k$ et $n = 2i + k$, unde erit $(2i - k)(2i + k)z - i \sqsupset\sqsubset \square$, sive $(4ii - kk)z - i \sqsupset\sqsubset \square$. Pro altero casu, quo $m + n = 4i - 1$ sumi poterit

$$m = 2i - k \quad \text{et} \quad n = 2i + k - 1, \quad \text{eritque} \quad 4(2i - k)(2i + k - 1)z - 4i + 1 \sqsupset\sqsubset \square,$$

sive hoc modo

$$((4i - 1)^2 - (2k - 1)^2)z - 4i + 1 \sqsupset\sqsubset \square.$$

Hinc innumerae formae speciales derivari possunt, veluti ex priore forma $(4ii - kk)z - i \sqsupset\sqsubset \square$, unde casu $i = 1$

prodit $\quad 4z - 1 \sqsupset\sqsubset \square, \quad 3z - 1 \sqsupset\sqsubset \square$, qui per se sunt manifesti

casu $i = 2$: $16z - 2 \sqsupset\sqsubset \square$, $15z - 2 \sqsupset\sqsubset \square$, $12z - 2 \sqsupset\sqsubset \square$, $7z - 2 \sqsupset\sqsubset \square$

casu $i = 3$: $36z - 3 \sqsupset\sqsubset \square$, $35z - 3 \sqsupset\sqsubset \square$, $32z - 3 \sqsupset\sqsubset \square$, $27z - 3 \sqsupset\sqsubset \square$, $20z - 3 \sqsupset\sqsubset \square$, $11z - 3 \sqsupset\sqsubset \square$

casu $i = 4$: $64z - 4 \sqsupset\sqsubset \square$, $63z - 4 \sqsupset\sqsubset \square$, $60z - 4 \sqsupset\sqsubset \square$, $55z - 4 \sqsupset\sqsubset \square$, $48z - 4 \sqsupset\sqsubset \square$, $39z - 4 \sqsupset\sqsubset \square$, $28z - 4 \sqsupset\sqsubset \square$, $15z - 4 \sqsupset\sqsubset \square$.

Hic autem veritas singularum ostendi potest, at vero ex principiis diversissimis. Eodem modo formulae speciales ex altero casu oriundae

$$((4i - 1)^2 - (2k - 1)^2)z - 4i + 1 \sqsupset\sqsubset \square$$

sunt

	casu $i = 1$	casu $i = 2$	casu $i = 3$
	$8z - 3 \sqsupset\sqsubset \square$	$48z - 7 \sqsupset\sqsubset \square$	$120z - 11 \sqsupset\sqsubset \square$
		$40z - 7 \sqsupset\sqsubset \square$	$112z - 11 \sqsupset\sqsubset \square$
		$24z - 7 \sqsupset\sqsubset \square$	$96z - 11 \sqsupset\sqsubset \square$
			$72z - 11 \sqsupset\sqsubset \square$
			$40z - 11 \sqsupset\sqsubset \square$

A. m. T. II. p. 211. 212.

56.

Proposita hac formula ad quadratum reducenda: $(qq-pp)^2+(ppqq-1)^2=\square$, statuatur $q=p+z$, et pervenietur ad aequationem, unde per regulas cognitas reperitur

$$z=\frac{-4p(p^4-1)}{3p^4-1}, \text{ unde fit } q=\frac{-p(p^4-3)}{3p^4-1},$$

ubi p pro lubitu accipi potest, si modo excludantur casus $p=0$ et $p=\pm 1$. Ita sumto $p=2$ fit $q=\frac{26}{47}$. Si $p=3$ fit $q=\frac{117}{121}$. Si $p=\frac{3}{2}$ erit $q=\frac{99}{454}$. Ita sumto $p=2$ et $q=\frac{26}{47}$ erit $pp-qq=\frac{120.68}{47^2}$, et ob $pq=\frac{52}{47}$ erit $ppqq-1=\frac{5.99}{47^2}$. Quadratum ergo fieri debet $120^2.68^2+5^2.99^2=15^2(8^2.68^2+33^2)$, quod continetur in forma $4aabb+(aa-bb)^2$. Fit enim $8.68=2ab$ ergo $ab=4.68=16.17$ et $33=aa-bb=(a+b)(a-b)$.

Proposita tali formula $qq(pp-1)^2+pp(qq-1)^2=\square$, duplex solutio institui potest:

prior: ponatur $q=np+n-1$, tum enim erit $q+1=(p+1)n$,

altera: ponatur $q=\frac{np+1}{p+n}$, tum enim erit $q+1=\frac{(n+1)(p+1)}{p+n}$ et $q-1=\frac{(n-1)(p-1)}{p+n}$.

Praeterea notetur, hanc formam ad praecedentem $(qq-pp)^2+(ppqq-1)^2$ reduci ponendo $p=fg$ et $q=\frac{f}{g}$, unde etiam solutio praecedentis formulae hic adhiberi potest. Fluunt autem istae formulae ex solutione hujus problematis $aa+bb=\square$, $aa+cc=\square$, $bb+cc=\square$. Primo enim sumatur $b=\frac{pp-1}{2p}.a$, erit $aa+bb=\left(\frac{pp+1}{2p}\right)^2 aa$ et $c=\frac{qq-1}{2q}.a$. Tertia formula evadet $qq(pp-1)^2+pp(qq-1)^2$. Altera solutio ita se habet: Sumatur $a=2fg$ et $b=ff-gg$, satisfiet primae conditioni. Pro secunda statuatur

$$c=ffgg-1; \text{ erit enim } aa+cc=2ffgg+f^4g^4+1.$$

Tertia ergo postulat, ut sit

$$(ff-gg)^2+(ffgg-1)^2=\square.$$

A. m. T. III. p. 9.

57.

Problema. Formulam $2x^4-y^4=zz$ ad hanc $8p^4+q^4=rr$ reducere.

Solutio. Ponatur $2x^4+y^4=\nu$, erit $\nu\nu-z^4=8x^4y^4$, unde fit $8x^4y^4+z^4=\nu\nu$, sicque erit $p=xy$, $q=z$ et $r=\nu=2x^4+y^4$.

Generalius ergo hoc fieri potest, nempe si $\alpha x^4-\beta y^4=zz$ posito $\alpha x^4+\beta y^4=\nu$, erit

$$\nu\nu-z^4=8\alpha\beta x^4y^4, \quad \text{ideoque } \nu\nu=z^4+8\alpha\beta x^4y^4.$$

Problema. Formulam $8p^4+q^4=rr$ ad formam $2x^4-y^4=zz$ reducere.

Solutio. Cum ergo sit $8p^4=rr-q^4=(r+qq)(r-qq)$, manifestum est esse q et r numeros impares. Hinc sequitur, numerorum $r+qq$ et $r-qq$ alterum fore impariter parem, alterum pariter parem, unde nascuntur duo casus:

I. Sit $r+qq$ impariter par $=2\alpha$, alter vero $r-qq$ pariter par $=4\beta$; erit ergo $8p^4=8\alpha\beta$, ideoque $\alpha\beta=p^4$, unde quia α et β sunt primi inter se, uterque debet esse biquadratum. Sit ergo $\alpha=s^4$ et $\beta=t^4$, et $p=st$ et $r+qq=2s^4$ et $r-qq=4t^4$, unde oritur $2qq=2s^4-4t^4$, sive $q^2=s^4-2t^4$.

II. Sit $r-qq$ impariter par $=2\alpha$ et $r+qq$ pariter par $=4\beta$, eritque $8p^4=8\alpha\beta$, ideoque $p^4=\alpha\beta$. Sit nunc $\alpha=s^4$ et $\beta=t^4$, eritque $p=st$; ac nunc $r+qq=4t^4$ et $r-qq=2s^4$, ideoque $qq=2t^4-s^4$. Posteriore ergo tantum casu reductio praescripta fieri potest. Interim tamen formula $f^4+8g^4=hh$ semper ad formam $2x^4-y^4$ reduci potest. Quod si enim sumatur $x=f^3+2fgg-gh$ et $y=f^3-4fgg+gh$, semper erit $2x^4-y^4=zz$, existente $z=f^6+f^4gg+24ffg^4-8g^6-6f^3gh$.

ANALYSIS, qua haec reductio est inventa. Posito $2x^4 - y^4 = zz$, debet esse

$$xx = pp + qq, \quad yy = pp + 2pq - qq, \text{ tum enim fiet } z = qq + 2pq - pp.$$

Hic ergo p et q ita definiri debent, ut xx et yy fiant quadrata, quod sequenti modo praestari potest. Cum sit $yy - xx = 2q(p - q) = (y + x)(y - x)$, jam statuatur $y + x = \frac{2a}{b} \cdot q$ et $y - x = \frac{b}{a}(p - q)$. Sic enim fiet $yy - xx = 2q(p - q)$. Addantur jam quadrata, fiet

$$2yy + 2xx = \frac{4aa}{bb}qq + \frac{bb}{aa}pp - \frac{2bb}{aa}pq + \frac{bb}{aa}qq.$$

At vero ex primis formulis fiet $2yy + 2xx = 4pp + 4pq$, qui valor illi aequatus et multiplicatione facta per $aabb$ dat

$$(b^4 - 4aabb)pp - 2pq(b^4 + 2aabb) + (b^4 + 4a^4)qq = 0.$$

Hinc radicem extrahendo fit $\frac{p}{q} = \cdot\cdot\cdot\cdot$;

THEOREMA. Si fuerit $ma^4 - nb^4 = cc$, inde assignari potest talis forma $x^4 - mny^4 = zz$.

DEMONSTRATIO. Posito enim $ma^4 + nb^4 = \Delta$, erit $\Delta\Delta = c^4 + 4mna^4b^4$. At in altera formula si ponatur $xx = pp + mnqq$ et $yy = 2pq$, fiet $z = pp - mnqq$. Jam statuatur $p = rr$ et $q = 2ss$, ut fiat $y = 2rs$, hinc fiet $xx = r^4 + 4mns^4$. Facta ergo comparatione erit $x = \Delta$, $r = c$, $s = ab$, unde fit $y = 2abc$, $z = c^4 - 4mna^4b^4$. Hinc ergo necesse est ut fiat

$$x^4 - mny^4 = zz.$$

A. m. T. III. p. 129. 131.

58.

OBSERVATIO. Ut formula $\frac{pq(pp - qq)}{rs(rr - ss)}$ fiat quadratum, sumatur

$$\begin{array}{lll} & p = aa + bb & r = aa + bb \\ & q = 2aa - bb & s = 2bb - aa \\ \text{hinc} & p + q = 3aa & r + s = 3bb \\ & p - q = 2bb - aa & r - s = 2aa - bb \end{array}$$

substituendo fit formula $$\frac{pq(pp - qq)}{rs(rr - ss)} = \frac{aa}{bb}.$$

Aliter, sumi etiam potest

$$\begin{array}{lll} & p = bb - 2aa & r = 2bb - 4aa \\ & q = 6aa & s = bb + 4aa \\ \text{hinc} & p + q = bb + 4aa & r + s = 3bb \\ & p - q = bb - 8aa & r - s = bb - 8aa \end{array}$$

ac substituendo: $$\frac{pq(pp - qq)}{rs(rr - ss)} = \frac{aa}{bb}.$$

Ita sumi potest $p = 7$, $q = 6$, $r = 14$ et $s = 13$, eritque $pq(pp - qq) = 546$, $rs(rr - ss) = 4914$

hinc formula $$= \frac{546}{4914} = \frac{1}{9} = \left(\frac{1}{3}\right)^2.$$

A. m. T. I. p. 295.

59.

EVOLUTIO GENERALIOR formulae: $$\frac{pq(pp - qq)}{rs(rr - ss)} = \square = n.$$

Hic ponatur $q = \alpha t$ et $s = \beta t$, tum vero $p = \nu r$, unde reperitur

$$\frac{tt}{rr} = \frac{\alpha\nu^3 - n\beta}{\alpha^3\nu - n\beta^3} = \left(\frac{\nu}{\alpha} - z\right)^2,$$

unde oritur haec aequatio

$$0 = n\beta - n\beta^3 zz + \frac{2n\beta^3\nu z}{\alpha} + \alpha^3\nu zz - 2\alpha\alpha\nu\nu z - \frac{n\beta^3\nu\nu}{\alpha\alpha},$$

unde patet si $\nu = 0$, fore $z = \pm\frac{1}{\beta}$; at si $z = 0$, tum erit $\nu = \pm\frac{\alpha}{\beta}$, unde sequentes valores inveniuntur ope harum formularum

$$z + z' = \frac{2\nu}{\alpha}, \quad \nu + \nu' = \frac{\alpha z(2n\beta^3 + \alpha^3 z)}{n\beta^3 + 2\alpha^3 z}.$$

At vero si sumamus $\nu = \infty$, erit

$$z = \frac{-n\beta^3}{2\alpha^4}, \quad \nu = \frac{4\alpha^8 - nn\beta^8}{3n\alpha^3\beta^5}.$$

Quia hic litteras α et β pro lubitu assumere licet, fortasse hinc novi valores eliciuntur, quos praecedens methodus non dat.

Si ex. gr. $\alpha = 1$ et $\beta = 2$, ut sit $q = t$ et $s = 2t$, tum vero

$$\frac{p}{r} = \nu \quad \text{et} \quad \frac{t}{r} = \nu - z, \quad \text{erit} \quad z + z' = 2\nu \quad \text{et} \quad \nu + \nu' = \frac{z(z+16)}{2z+8}.$$

Hinc sumto $\nu = 0$, erit $z = \pm\frac{1}{2}$; at si $z = 0$ erit $\nu = \pm\frac{1}{2}$. Praeterea si $\nu = \infty$, erit $z = -4$ et sequens $\nu = -\frac{21}{8}$, ex quibus casibus sequentes valores oriuntur

$$\nu = \infty, \quad z = -4, \quad \nu = -\frac{21}{8}, \quad z = -\frac{5}{4}, \quad \nu = -\frac{8}{11}, \quad z = -\frac{9}{44}, \quad \nu = \frac{403}{8.167},$$

tum vero

$$z = 0, \quad \nu = \frac{1}{2}, \quad z = 1, \quad \nu = \frac{6}{5}, \quad z = \frac{7}{5}, \quad \nu = \frac{19}{18},$$

porro

$$z = 0, \quad \nu = -\frac{1}{2}, \quad z = -1, \quad \nu = 2, \quad z = -3, \quad \nu = -\frac{35}{2}, \quad z = -32, \quad \nu = \frac{117}{14}, \quad \text{etc.}$$

$$\nu = 0, \quad z = \frac{1}{2}, \quad \nu = \frac{11}{12}, \quad z = \frac{4}{3}, \quad \nu = \frac{5}{4}, \quad z = \frac{7}{6}, \quad \nu = \frac{64}{93}, \quad \text{etc.}$$

$$\nu = 0, \quad z = -\frac{1}{2}, \quad \nu = -\frac{31}{23}, \quad z = -\frac{12}{7}, \quad \nu = -\frac{17}{4}.$$

Casu $n = 1$, valores ν et z ita se habebunt:

$$\nu = 0, \quad 1, \quad 0, \quad 1, \quad \frac{8}{7}, \quad 1, \quad \frac{104}{105}, \quad 1$$

$$z = 1, \quad 1, \quad -1, \quad 3, \quad -\frac{5}{7}, \quad \frac{19}{7}, \quad -\frac{11}{15},$$

tum vero

$$z = 0, \quad 2, \quad -\frac{4}{5}, \quad \frac{14}{5}, \quad -\frac{8}{11}, \quad 0, \quad -2, \quad 4, \quad -\frac{2}{3}, \quad \frac{8}{3}$$

$$\nu = 1, \quad \frac{3}{5}, \quad 1, \quad \frac{57}{55}, \quad 1, \quad -1, \quad 1, \quad \frac{5}{3}, \quad 1, \quad \frac{55}{57}.$$

Tum

$$\nu = \infty, \quad 1, \quad \frac{7}{8}, \quad 1, \quad \frac{105}{104}, \quad 1, \quad \frac{1455}{1456}, \quad 1, \quad \frac{11.19.97}{16.7.181} = \frac{20271}{20272}$$

$$z = -\frac{1}{2}, \quad \frac{5}{2}, \quad -\frac{3}{4}, \quad \frac{11}{4}, \quad -\frac{19}{26}, \quad \frac{71}{28}, \quad -\frac{41}{56}, \quad \frac{153}{56}.$$

Ita ex casu $\nu = \frac{1455}{1456} = \frac{3.5.97}{16.7.13}$ valores pro p, q, r, s erunt: $p = 3.5.97$, $q = 2521$, $r = 16.7.13$, $s = 2521$, $p + q = 8.7.71$, $p - q = 2.13.41$, $r + s = 41.97$, $r - s = 3.5.71$, ubi factores utrinque se mutuo destruunt.

A. m. T. III. p. 143.

60.

Problema. Invenire duo triangula rectangula in numeris, quorum areae

$$A = pq(pp-qq) \quad \text{et} \quad B = rs(rr-ss)$$

datam inter se teneant rationem scil. $a:b$, ita ut sit $\frac{A}{B} = \frac{a}{b}$.

Hoc problema methodo directa frustra tractatur, unde ad solutiones particulares confugere necesse est, cujusmodi sunt sequentes:

I. Sumatur $r = p$ et $s = p - q$, erit $r + s = 2p - q$ et $r - s = q$, unde fit

$$B = p(p-q)(2p-q)q$$

hincque $\frac{A}{B} = \frac{p+q}{2p-q} = \frac{a}{b}$, ergo $bp + bq = 2ap - aq$, ideoque $\frac{p}{q} = \frac{a+b}{2a-b}$, unde haec solutio nascitur

$$\begin{aligned} p &= a + b & \qquad r &= a + b \\ q &= 2a - b & \qquad s &= 2b - a. \end{aligned}$$

II. Sit $r = 2p$ et $s = p + q$, erit $r + s = 3p + q$ et $r - s = p - q$, hinc $B = 2p(p+q)(3p+q)(p-q)$, ergo

$$\frac{A}{B} = \frac{q}{6p+2q} = \frac{a}{b}, \text{ sicque erit } bq = 6ap + 2aq, \text{ indeque } \frac{p}{q} = \frac{b-2a}{6a},$$

quocirca capiatur

$$\begin{aligned} p &= b - 2a & \qquad r &= 2b - 4a \\ q &= 6a & \qquad s &= b + 4a. \end{aligned}$$

III. Sit $r = 2p$ et $s = p - q$, erit $r + s = 3p - q$, et $r - s = p + q$, hinc

$$\frac{A}{B} = \frac{q}{6p-2q} = \frac{a}{b}, \text{ sicque } bq = 6ap - 2aq, \text{ ergo } \frac{p}{q} = \frac{b+2a}{6a},$$

quocirca capiatur pro ista solutione

$$\begin{aligned} p &= b + 2a & \qquad r &= 2b + 4a \\ q &= 6a & \qquad s &= b - 4a. \end{aligned}$$

IV. Sit $r = p + q$ et $s = p$, erit $r + s = 2p + q$ et $r - s = q$, hincque

$$\frac{A}{B} = \frac{p-q}{2p+q} = \frac{a}{b}, \text{ unde colligitur ob } bp - bq = 2ap + aq, \ \frac{p}{q} = \frac{a+b}{b-2a}$$

ergo capiantur

$$\begin{aligned} p &= a + b & \qquad r &= 2b - a \\ q &= b - 2a & \qquad s &= a + b. \end{aligned}$$

V. Sit $r = p + q$ et $s = q$, erit $r + s = p + 2q$ et $r - s = p$, unde fit

$$\frac{A}{B} = \frac{p-q}{p+2q} = \frac{a}{b}, \text{ inde } bp - bq = ap + 2aq, \text{ hinc } \frac{p}{q} = \frac{2a+b}{b-a},$$

quocirca capere debemus

$$\begin{aligned} p &= 2a + b & \qquad r &= a + 2b \\ q &= b - a & \qquad s &= b - a. \end{aligned}$$

VI. Sit $r = p + q$ et $s = 2q$, erit $r + s = p + 3q$ et $r - s = p - q$, unde fit

$$\frac{A}{B} = \frac{p}{2p+6q} = \frac{a}{b}, \text{ unde ob } bp = 2ap + 6aq, \text{ erit } \frac{p}{q} = \frac{6a}{b-2a},$$

ideoque hic capere oportet

$$\begin{aligned} p &= 6a & \qquad r &= 4a + b \\ q &= b - 2a & \qquad s &= 2b - 4a. \end{aligned}$$

VII. Sit $r = p - q$ et $s = q$, erit $r + s = p$ et $r - s = p - 2q$ et erit

$$\frac{A}{B} = \frac{p+q}{p-2q} = \frac{a}{b}, \text{ ideoque } bp + bq = ap - 2aq, \text{ hinc } \frac{p}{q} = \frac{b+2a}{a-b},$$

itaque ut capiatur necesse est

$$\begin{aligned} p &= b + 2a & \qquad r &= 2b + a \\ q &= a - b & \qquad s &= a - b. \end{aligned}$$

VIII. Sit $r = p - q$ et $s = 2q$, erit $r + s = p + q$ et $r - s = p - 3q$, hincque

$$\frac{A}{B} = \frac{p}{2p - 6q} = \frac{a}{b}, \text{ unde ob } bp = 2ap - 6aq \text{ invenitur } \frac{p}{q} = \frac{6a}{2a - b},$$

ideoque capere oportet

$$p = 6a \qquad r = b + 4a$$
$$q = 2a - b \qquad s = 4a - 2b.$$

IX. Sit $r = q$ et $s = p - q$, erit $r + s = p$ et $r - s = 2q - p$, unde fit

$$\frac{A}{B} = \frac{p+q}{2q - p} = \frac{a}{b}, \text{ ideoque } bp + bq = 2aq - ap, \text{ ergo } \frac{p}{q} = \frac{2a - b}{b + a},$$

quocirca sumatur

$$p = 2a - b \qquad r = b + a$$
$$q = b + a \qquad s = a - 2b.$$

X. Sit denique $r = 2q$ et $s = p - q$, erit $r + s = p + q$ et $r - s = 3q - p$, unde reperitur

$$\frac{A}{B} = \frac{p}{6q - 2p} = \frac{a}{b}; \text{ hinc ob } bp = 6aq - 2ap, \text{ erit } \frac{p}{q} = \frac{6a}{b + 2a},$$

quo notato manifestum est, ut esse debeat

$$p = 6a \qquad r = 2b + 4a$$
$$q = b + 2a \qquad s = 4a - b.$$

Has jam omnes solutiones in sequenti tabella uni conspectui exponamus.

	p	q	r	s
I	$a+b$	$2a-b$	$a+b$	$2b-a$
II	$b-2a$	$6a$	$2b-4a$	$b+4a$
III	$b+2a$	$6a$	$2b+4a$	$b-4a$
IV	$a+b$	$b-2a$	$2b-a$	$a+b$
V	$2a+b$	$b-a$	$a+2b$	$b-a$
VI	$6a$	$b-2a$	$4a+b$	$2b-4a$
VII	$b+2a$	$a-b$	$2b+a$	$a-b$
VIII	$6a$	$2a-b$	$b+4a$	$4a-2b$
IX	$2a-b$	$b+a$	$b+a$	$a-2b$
X	$6a$	$b+2a$	$2b+4a$	$4a-b$

Hic numeri p et q dicuntur genitores trianguli A, et r et s genitores trianguli B, de quibus notandum, si qui eorum prodeant negativi, eos in positivos converti posse, dummodo majores litteris p et r, minores vero litteris q et s tribuantur. Quo observato aliquot exempla evolvamus:

Exemplum 1. Sit $a = 1$ et $b = 1$, exclusis triangulis inter se similibus, oritur haec una solutio:

$$p = 6 \qquad r = 5$$
$$q = 1 \qquad s = 2.$$

Exemplum 2. Sit $a = 2$ et $b = 1$ et solutiones orientur in hac tabella contentae

p	q	r	s
12	3	9	6
12	5	10	7
5	1	4	1.

Deletis autem iis casibus, qui bis occurrunt, sequens tabella exhibet solutiones diversas:

	p	q	r	s	
I	$a+b$	$2a-b$	$a+b$	$2b-a$	α
V	$2a+b$	$b-a$	$a+2b$	$b-a$	β
II	$b-2a$	$6a$	$2b-4a$	$b+4a$	γ
III	$b+2a$	$6a$	$2b+4a$	$b-4a$	δ
α	$3a$	$2b-a$	$3b$	$2a-b$	1
β	$a+2b$	$3a$	$3b$	$b+2a$	5
γ	$b+4a$	$b-8a$	$3b$	$8a-b$	2
δ	$b+8a$	$b-4a$	$3b$	$8a+b$	3

Inter has octo solutiones quaelibet habet suam sociam, quae scilicet ex numeris genitoribus $p+q$ et $p-q$ nascitur, quas igitur paribus litteris graecis insignivimus.

Exemplum 1. Sit $a=2$ et $b=1$, octo solutiones ita se habebunt:

	p	q	r	s
α	3	3	3	0
α	6	0	3	3
β	5	1	4	1
β	6	4	5	3
γ	12	3	9	6
γ	15	9	15	3
δ	12	5	10	7
δ	17	7	17	3

Exemplum 2. Sit $a=3$ et $b=2$, et oriuntur solutiones in hac tabula expressae:

	p	q	r	s
α	5	4	5	1
α	9	1	6	4
β	8	1	7	1
β	9	7	8	6
γ	{18	4	14	8}
	{9	2	7	4}
γ	11	7	11	3
δ	{18	8	16	10}
	{9	4	8	5}
δ	13	5	13	3

Exemplum 3. Si $a=1$ et $b=1$, sequentes oriuntur solutiones:

	p	q	r	s
α	2	1	2	1
α	3	1	3	1
β	3	0	3	0
β	3	3	3	3
γ	6	1	5	2
γ	7	5	7	3
δ	6	3	6	3
δ	9	3	9	3

Si ambarum arearum productum AB debeat esse quadratum, tantum sumi oportet pro numeris a et b quadrata; sit igitur $a=4$ et $b=1$ eruntque solutiones

	p	q	r	s
α	7	5	5	2
α	12	2	7	3
β	{ 9	3	6	3 }
	3	1	2	1
β	4	2	3	1
γ	24	7	17	14
γ	31	17	31	3
δ	{ 24	9	18	15 }
	8	3	6	5
δ	11	5	11	1

A. m. T. I. p. 296—298.

Nota Editorum. Huic praecedenti fragmento in *Adversariorum* Tomo I Patris manu inscriptum est: «Omnia haec jam redacta» (*Dieses ist schon ausgeführt*); cum tamen in nullo cognitorum Euleri operum has investigationes detegere nobis contigerit, quae hanc ob rem et in recentissima editione *Commentationum arithmeticar.* desunt, esse utique potest eas in quapiam rarissima seu oblita collectione typis expressas reperiri. Hic saltem sufficiet remittere lectorem ad commentationem, cujus fragmentum supra in pag. 101 hujusce tomi Opp. posthum. reperitur, et in qua idem fere, aut simile argumentum tractatum fuisse videtur.

61.

Theorema. Haec formula $aax^4+bxxyy+ccy^4$, quadrato aequanda, semper reduci potest ad productum quatuor factoribus simplicibus constans, pariter quadrato aequandum.

Demonstratio. Formula proposita aequetur huic quadrato $\left(axx+cyy.\frac{p}{q}\right)^2$, fietque

$$bqqxx+ccqqyy=2acpqxx+ccppyy, \quad \text{unde fit} \quad \frac{xx}{yy}=\frac{cc(q+p)(q-p)}{q(2acp-bq)}=\square.$$

Quadratum ergo esse debet $(q+p)(q-p)q(2acp-bq)$. Simili modo, si radicem illius formulae posuissemus

$$cyy+axx.\frac{r}{s}, \quad \text{prodiisset} \quad \frac{xx}{yy}=\frac{s(2acr-bs)}{aa(s+r)(s-r)};$$

quadratum ergo debet esse $(s+r)(s-r)s(2acr-bs)$. Deinde, quia per utramque positionem est

$$axx+cyy.\frac{p}{q}=cyy+axx.\frac{r}{s}, \quad \text{erit} \quad \frac{xx}{yy}=\frac{cs(q-p)}{aq(s-r)},$$

ideoque debet esse $cs(q-p).aq(s-r)=\square$.

Corollarium. Pro $\frac{xx}{yy}$ nacti sumus sequentes tres valores:

$$\frac{cc(q+p)(q-p)}{q(2acp-bq)}, \quad \frac{s(2acr-bs)}{aa(s+r)(s-r)}, \quad \frac{cs(q-p)}{aq(s-r)},$$

ex quorum comparatione relatio inter rationes $r:s$ et $p:q$ deduci potest. Erit enim

$$r:s=\left(1+\frac{b}{ac}\right)q-p:q+p; \quad \text{vel erit etiam} \quad p:q=\left(1+\frac{b}{ac}\right)s-r:s+r.$$

A. m. T. III. p. 136.

62.

Problema. Invenire quatuor quadrata aa, bb, cc, dd, ut haec fractio fiat quadratum $\frac{aacc - bbdd}{aadd - bbcc}$.

Solutio duplex dari potest: Pro priore ponatur $c = ab$, $d = aa - 2bb$, eritque

$$ac + bd = 2b(aa - bb), \quad ac - bd = 2b^3, \quad ad + bc = a(aa - bb), \quad ad - bc = a(aa - 3bb),$$

ergo fieri debet $\frac{4b^4}{aa(aa - 3bb)} = \square$, sive tantum $aa - 3bb = \square$. Pro altera solutione fiat $a = cc + 2dd$, b … tum enim fiet $ac + bd = c(cc + 3dd)$, $ac - bd = c(cc + dd)$, $ad + bc = 2d(cc + dd)$, $ad - bc = 2d^3$, ergo fieri deb… $\frac{cc(cc + 3dd)}{4d^4} = \square$, sive tantum $cc + 3dd = \square$.

A. m. T. III. p. 150.

63.

Problema. Ad quadratum reducere hanc formulam $\frac{aabb - ccdd}{aacc - bbdd}$.

Solutio. Ponatur $b = ad$, erit formula $\frac{dd(a^4 - cc)}{aa(cc - d^4)} = \frac{a^4 - cc}{cc - d^4}$. Ponatur porro $c = aa - 2dd$, erit formula

$$\frac{4aadd - 4d^4}{a^4 - 4aadd + 3d^4}, \text{ quae demto quadrato in numeratore fit } \frac{aa - dd}{a^4 - 4aadd + 3d^4} = \frac{1}{aa - 3dd}.$$

Sumatur $a = pp + 3qq$ et $d = 2pq$, erit forma $\frac{1}{(pp - 3qq)^2}$, hincque porro prodit

$$c = p^4 - 2ppqq + 9q^4 \quad \text{et} \quad b = 2pq(pp + 3qq).$$

Hic quaelibet positio solutionem suppeditat praecedentis problematis (*).

Sit $p = 1$ et $q = 1$, erit $a = 4$, $b = 8$, $c = 8$, $d = 2$.

Sit $p = 2$ et $q = 1$, erit $a = 7$, $b = 28$, $c = 17$, $d = 4$,

unde oritur solutio supra data problematis praecedentis.

Sit $p = 1$ et $q = 2$, erit $a = 13$, $b = 52$, $c = 137$, $d = 4$.

Sit $p = 3$ et $q = 1$, erit $a = 12$, $b = 72$, $c = 72$, $d = 6$.

Sit $p = 2$ et $q = 3$, erit $a = 31$, $b = 372$, $c = 673$, $d = 12$.

Sequenti autem modo praecedens problema ad praesens reducitur: Cum esse debeat $A^4 - B^4 = C^4 - D^4$, ponatur $A + B = \alpha x$ et $A - B = \beta y$, tum vero $C + D = \gamma x$, $C - D = \delta y$,

$$\text{fiet} \quad \alpha\beta(\alpha\alpha xx + \beta\beta yy) = \gamma\delta(\gamma\gamma xx + \delta\delta yy).$$

Hinc oritur $\frac{xx}{yy} = \frac{\gamma\delta^3 - \alpha\beta^3}{\alpha^3\beta - \gamma^3\delta}$, unde haud difficulter superior derivatur.

A. m. T. III. p. 161. 162.

(*) Resolutio hujus aequationis $A^4 - B^4 = C^4 - D^4$. *Comment. arithm* T. I p. 473.

64.

Theorema. Si fuerit $X = (a - bx)^2(p - qx)^2 + (c - dx)^2(r - sx)^2 - nn(a - bx)^2(c - dx)^2$, statim sex valores habentur, quibus X fit quadratum.

Primo enim fiet $X = (a - bx)^2(p - qx)^2$, si fuerit $c - dx = 0$, ideoque $x = \frac{c}{d}$, et si fuerit

$$(r - sx)^2 - nn(a - bx)^2 = 0, \quad \text{hoc est} \quad r - sx = \pm n(a - bx).$$

Simili modo fiet $X = (c - dx)^2(r - sx)^2$ faciendo $a - bx = 0$, seu $x = \frac{a}{b}$; tum vero si $p - qx = \pm n(c - dx)$.

A. m. T. III. p. 166.

65.

Theorema. Ut in quadrilatero, circulo inscripto, quatuor latera a, b, c, d cum ambabus diagonalibus x et y numeris rationalibus exprimantur, necesse est, ut hoc productum $(ab+cd)(ac+bd)(ad+bc)$ reddatur quadratum. Quod fiet sumtis quinque numeris pro lubitu f, g, h, p, q, si capiatur

$$a=fgh(qq-pp),\quad b=g(fp+gq)^2-hhqq,\quad c=2fghpq+h(ff+gg-hh)qq,\quad d=f(gp+fq)^2-hhqq,$$

tum enim erit

$$x=f(fg(pp+qq)+(ff+gg-hh)pq)$$
$$y=g(fg(pp+qq)+(ff+gg-hh)pq).$$

Sin autem insuper requiratur, ut etiam area quadrilateri fiat rationalis, tum hanc formulam quadratum esse oportet

$$(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a),$$

quod autem per illas formulas nullo modo effici potest. At vero sequens problema generaliter resolvi potest:

> Dato circulo polygonum quotcunque laterum inscribere, cujus omnia latera una cum omnibus diagonalibus, atque adeo area numeris rationalibus exprimantur.

Solutio. (Fig. 1.) Posito radio circuli $=1$ sint arcus $AB=2A$, $BC=2B$, $CD=2C$, etc., eritque latus

$$AB=2\sin A,\quad BC=2\sin B,\quad CD=2\sin C,\quad \text{etc.}$$

Tantum ergo opus est, ut horum angulorum sinus sint rationales, simulque etiam cosinus, ut etiam diagonales fiant rationales, si quidem est

$$AC=2\sin(A+B)=2\sin A\cos B+2\cos A\sin B.$$

At vero si fuerit $\sin A=\frac{2ab}{aa+bb}$, erit $\cos A=\frac{aa-bb}{aa+bb}$; tales igitur formulae pro sinibus et cosinibus accipiantur, hocque modo non solum omnia latera, sed etiam diagonales fient rationales atque adeo area, cum posito centro O sit area $\Delta AOB=\sin A\cos A$, quod de omnibus reliquis valet. Possunt enim singuli hi anguli $2A$, $2B$, etc. usque ad ultimum pro lubitu assumi, ultimi vero sinus erit sinus summae reliquorum, et cosinus $=-$ cosinui summae reliquorum.

A. m. T. III. p. 159. 160.

66.

(*Lexell.*)

Theorema. Si a fuerit numerus quicunque non quadratus, et b et c numeri quicunque ad illum primi, tum ista formula

$$a(bbx^4+aaccy^4)$$

nunquam esse potest quadratum.

Demonstratio. Hic assumi potest numerum a etiam per nullum quadratum esse divisibilem, si enim esset $a=\alpha ff$, quadratum esse deberet $\alpha(bbx^4+\alpha\alpha ccf^4y^4)$, ubi si loco fy scribatur y, habetur formula prior, quin etiam numeri x et y sunt primi inter se. Quoniam igitur a est factor nostrae formae, necesse est, ut alter factor $bbx^4+aaccy^4$ etiam habeat factorem a, sed pars posterior jam habet factorem a, ergo pars prior erit divisibilis per a, ex quo x factorem habebit a, ideoque y non erit divisibile per a. Ponatur ergo $x=az$, atque nunc haec forma quadratum esse debebit

$$a(bba^4z^4+aaccy^4),\quad \text{seu}\quad a(bbaaz^4+ccy^4).$$

Quod ob eandem rationem fieri nequit, nisi y esset divisibile per a, qui casus cum jam sit exclusus, formula nostra nullo modo quadratum esse poterit.

A. m. T. I. p. 51.

67.

Varia conamina aequationis $a^\lambda+b^\lambda=c^\lambda$ impossibilitatem casu $\lambda>2$ demonstrandi.

1. (*Lexell.*)

Theorema. Non dantur tres numeri x, y, z, ut fiat $xxy+xzz+yyz=0$.

Sumi potest numeros x, y, z communem divisorem non habere; si enim haberent, per divisionem ex hac aequatione tolleretur; interim tamen bini communem divisorem habere debent. Hinc ponatur a maximus communis divisor numerorum x et y, b ipsorum x et z, et c ipsorum y et z, atque tum bini horum a, b, c erunt inter se primi. Ponatur igitur $x = ap$, $y = aq$, eruntque p et q primi inter se. Deinde sit $x = br$ et $z = bs$, denique $y = ct$ et $z = cu$, ita ut sit $x = ap = br$, $y = aq = ct$, $z = bs = cu$, quibus valoribus substitutis formula nostra est:

$$abcprt + abcpsu + abcqts = 0, \quad \text{sive} \quad prt + psu + qst = 0.$$

Cum autem sit $ap = br$, sive $\frac{p}{r} = \frac{b}{a}$, erit $p = lb$, $r = la$; deinde $\frac{q}{t} = \frac{c}{a}$, unde $q = mc$ et $t = ma$, et $\frac{s}{u} = \frac{c}{b}$, $s = nc$, $u = nb$, ita ut sit $x = lab$, $y = mac$, $z = nbc$; ubi notandum numeros mc, lb esse inter se primos, nec non lb et nc, et ma et nc. Aequatio autem nostra hanc habebit formam:

$$llmaab + nnlbbc + mmncca = 0.$$

Hinc ergo lb divisor esse deberet membri $mmncca$, quod ob conditiones memoratas esse nequit.

2. (*J. A. Euler.*)

NB. Haec demonstratio non succedit. Caeterum hoc theorema huc redit, ut demonstretur esse non posse $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 0$. Hoc autem sequenti modo demonstrari posse videtur:

Posito $z = \frac{xy}{v}$, et nostra aequatio fiet $\frac{x}{y} + \frac{v}{x} + \frac{y}{v} = 0$, quae forma similis est propositae. Cum numeri x, y, z sint inaequales, sit z maximus, y medius et x minimus, sive negative, sive positive. Jam cum sit $z = \frac{xy}{v}$, manifestum est fore $v < x$. Unde patet, si terni numeri z, y et x satisfecerint, tum etiam hos y, x et v satisfacturos, quorum y jam erit maximus, x medius et v minimus. Ponatur jam $y = \frac{xv}{u}$, eritque $u < v$, quare etiam hi tres numeri x, v et u satisfacerent. Si porro ponatur $x = \frac{uv}{t}$, erit $t < u$, atque etiam hi v, u et t satisfacerent. Hocque modo continuo ad numeros minores perveniretur; quare cum in minimis hujusmodi numeri non dentur, etiam in maximis tales non dantur. Manifestum vero est hos numeros semper fore integros.

Corollarium. Hoc modo demonstrari posset fieri non posse $\frac{x}{y} + \frac{y}{x} = 0$. Si enim $y > x$ et ponatur $y = \frac{xx}{v}$, erit $v < x$, et tum prodit $\frac{v}{x} + \frac{x}{v} = 0$, quae posito denuo $x = \frac{vv}{u}$ daret $u < v$ et $\frac{v}{u} + \frac{u}{v} = 0$, quo pacto iterum ad numeros continuo minores perveniretur.

Eodem modo etiam demonstrari potest esse non posse $\frac{v}{x} + \frac{x}{y} + \frac{y}{z} + \frac{z}{v} = 0$. Posito enim $z = \frac{vy}{u}$, ubi si z fuerit numerus maximus et v minimus, ($u <$ erit v) similis aequatio prodit scilicet $\frac{u}{v} + \frac{v}{x} + \frac{x}{y} + \frac{y}{u} = 0$ ex numeris minoribus formata; hocque modo continuo minores invenire liceret.

Corollarium 2. Cum igitur aequatio $xxy + xzz + yyz = 0$ sit impossibilis, inde vero prodeat

$$z = \frac{-yy \pm \sqrt{(y^4 - 4x^3 y)}}{2x},$$

sequitur formulam $y^4 - 4x^3 y$ quadratum nunquam esse posse.

Corollarium 3. Ex aequatione supra allata pro quatuor numeris sequitur

$$vvyz + xxzv + yyxv + zzxy = 0$$

hinc

$$vv = \frac{-(xxz + yyx)v - zzxy}{yz}$$

et

$$v = \frac{-x(xz + yy) \pm \sqrt{(xx(xz + yy)^2 - 4z^3 yyx)}}{2yz},$$

unde haec formula $xx(xz + yy)^2 - 4z^3 yyx$ nunquam quadratum fieri potest.

Sit verbi gratia $z=x$ et haec formula fiet

$$xx(xx+yy)^2-4x^4yy, \quad \text{vel} \quad (xx+yy)^2-4xxyy$$

NB. Verum nostrum theorema in casu quatuor numerorum non amplius locum habet, quia utique in minimis numeris casus dantur possibiles: veluti si fuerit $z=x$ et $y=-v$. Quod ergo de quatuor numeris hic dictum est, neutiquam valet.

Theorema. Neque summa neque differentia duorum cuborum potest esse cubus.

Demonstratio I. Si p, q et r denotent numeros integros, sive positivos sive negativos, demonstrandum est hanc aequationem nullo modo subsistere posse:

$$p^3+q^3+r^3=0.$$

Tum enim dividendo per pqr foret

$$\frac{pp}{qr}+\frac{qq}{pr}+\frac{rr}{pq}=0, \quad \text{ideoque etiam} \quad \frac{ppq}{qqr}+\frac{qqr}{prr}+\frac{prr}{ppq}=0$$

atque hinc etiam si ponamus $ppq=x$, $qqr=y$ et $rrp=z$, foret

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=0.$$

Hoc autem nunquam fieri posse ante est demonstratum.

Demonstratio II. Demonstrabo hic hanc formulam $ab(a\pm b)$ cubum esse non posse. Primo enim numeri a et b non solum integri sed etiam primi inter se assumi possunt. Quare cum hi tres factores a, b et $a\pm b$ sint inter se primi, unusquisque foret cubus, unde posito $a=x^3$, $b=y^3$, foret $x^3\pm y^3=$ cubo. Quod autem formula $ab(a\pm b)$ cubus esse nequit, ita ostendo: Si esset cubus, ejus radix statui posset $\frac{m(a\pm b)}{n}$. Tum ergo foret $\quad ab(a\pm b)=\frac{m^3(a\pm b)^3}{n^3}$, vel $n^3ab=m^3(a\pm b)^2=m^3(aa\pm 2ab+bb)$.

Hoc enim si esset, numeri a et b forent inaequales. Sit igitur a major et b minor, et ponatur $a=\frac{bb}{c}$, eritque

$c<b$, tum autem foret $\frac{n^3b^3}{c}=m^3\left(\frac{b^4}{cc}\pm\frac{2b^3}{c}+bb\right)$, sive $n^3bc=m^3(bb\pm 2bc+cc)$, ubi $b>c$,

ergo si porro ponatur $b=\frac{cc}{d}$, erit $c>d$, hincque iterum foret $n^3cd=m^3(cc\pm 2cd+dd)$; hocque modo continuo ad numeros minores perveniretur. Unde quia res in minimis numeris non succedit, etiam in maximis succedere non posset.

NB. Hic vero vitium ingens inest, quoniam ob numeros a et b inter se primos, c non est integer, neque etiam sequentes d, e, etc. Quocirca ex parvitate horum numerorum nihil concludi potest. Interim tamen etiam ne prior demonstratio valet, etsi enim omnes tres numeri non habent communem divisorem, tamen bini quivis necessario communem habent factorem. Quamobrem ex aequalitate $\frac{y}{z}=\frac{-xx-zy}{xy}$ concludi nequit, esse z partem ipsius xy, quia fortasse fractio $\frac{y}{z}$ ad minores terminos reduci potest, cujus demum denominator divisor esse debet ipsius xy.

A. m. T. I. p. 51—54.

3. (*Lexell.*)

Theorema Fermatii, quo neque summa cuborum potest esse cubus, neque summa duarum potestatum quintarum potestas quinta esse potest, nec in genere summa duarum potestatum altiorum similis potestas altior, facile ita transformari potest, ut certae formulae quadrata esse nequeant. Si enim $a^5+b^5=c^5$, ponatur

$$x+y=a^5 \quad \text{et} \quad x-y=b^5, \quad \text{foretque} \quad 2x=a^5+b^5=c^5 \quad \text{et} \quad xx-yy=a^5b^5 \quad \text{et} \quad 4xx=c^{10};$$

hinc igitur foret $\frac{xx-yy}{4xx}=\frac{a^5b^5}{c^{10}}$, ideoque potestas quinta, pro qua scribatur

$$\frac{z^5}{x^5}, \text{ seu } \frac{xz^5}{x^6}, \text{ ita ut foret } \frac{xx-yy}{4xx}=\frac{xz^5}{x^6};$$

multiplicetur per $4x^6$, fietque

$$x^6-x^4yy=4xz^5, \quad \text{sive} \quad x^6-4xz^5=x^4yy=\square.$$

Quare si demonstrari posset formulam x^6-4xy^5 quadratum esse non posse, simul demonstratum est formulam a^5+b^5 potestatem quintam esse non posse. Si enim esset x^6-4xy^5 quadratum, ob factores $x(x^5-4y^5)$ inter se primos, uterque quadratum esse deberet. Sit igitur $x=pp$, et alter factor $p^{10}-4y^5$ deberet esse quadratum, puta qq; foret ergo

$$p^{10}-qq=4y^5=4r^5s^5=(p^5+q)(p^5-q), \quad \text{ideoque} \quad p^5+q=2r^5 \quad \text{et} \quad p^5-q=2s^5,$$

unde addendo foret $2p^5=2r^5+2s^5$, sive $p^5=r^5+s^5$.

Simili modo formula $a^3+b^3=c^3$ transformabitur in hanc aequivalentem $x^4-4xy^3=\square$. Hoc postremum theorema etiam hoc modo repraesentari potest, ut nunquam fieri queat

$$x^3+(x+a)^3=(x+b)^3, \quad \text{ubi manifesto} \quad b>a.$$

Foret ergo $x^3=(x+b)^3-(x+a)^3=3(b-a)xx+3(bb-aa)x+b^3-a^3$,

demonstrandum ergo est hanc aequationem nunquam habere radicem rationalem. Ad hoc observetur, cum posterius membrum factorem habeat $b-a$, etiam x^3 talem factorem habere debet, et perspicuum est $b-a$ vel esse cubum, vel noncuplum cubi.

Sit primo $b-a=f^3$, et erit $x^3=3f^3xx+3f^3(b+a)x+f^3(bb+ab+aa)$.

Ponatur ergo $x=fy$, eritque $y^3=3ffyy+3(b+a)fy+bb+ab+aa$, ideoque y debet esse factor formulae $bb+ab+aa$.

Sit secundo $b-a=9f^3$, et ultimum membrum fieret (ob $b=9f^3+a$)

$$9^3f^9+3.9^2af^6+3.9aaf^3=27f^3(27f^6+9af^3+aa),$$

unde fit $x^3=27f^3xx+27f^3(b+a)x+27f^3(27f^6+9af^3+aa)$.

Ponatur $x=3fy$,* erit $y^3=9ffyy+3fy(b+a)+27f^6+9af^3+aa$.

Pro utroque casu limites assignari possunt; pro priore enim manifesto est $y>3ff$, et pro altero $y>9ff$; qui limites sunt nimis parvi; nimis magni autem hoc modo reperientur: Consideretur aequatio in genere

$$y^3=\alpha yy+\beta y+\gamma,$$

ubi α, β, γ sint positivi, ac primo erit $y>\alpha$; deinde cum sit $y=\alpha+\frac{\beta}{y}+\frac{\gamma}{yy}$, si in membro posteriori loco y scribatur α, hoc membrum fit nimis magnum, erit ergo $y<\alpha+\frac{\beta}{\alpha}+\frac{\gamma}{\alpha\alpha}$; ponatur hic limes $=\lambda$, sit $y<\lambda$, eritque vicissim $y=\alpha+\frac{\beta}{\lambda}+\frac{\gamma}{\lambda\lambda}$.

Exemplum. Sit pro casu priori $f=1$ et $a=1$, erit $b=2$ et $x=y$, hinc $y^3=3yy+9y+7$, statim $y>3$, hinc $y<7$, hinc $y>4\frac{3}{7}$, $y<5\frac{12}{31}$, radix ergo rationalis deberet esse 5, quae cum non sit divisor ultimi termini . . . . . .

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4. (W. L. Krafft.)

Problema. Invenire numeros x et y inter se primos, ut formula x^3+ny^3 fiat numerus quadratus.

Solutio. Si hi numeri non essent primi inter se, quaestio foret levissima; posito enim $x=pr$ et $y=qr$ formula nostra prodit $r^3(p^3+nq^3)$, quae aequetur quadrato r^4ss, ita ut hinc statim fiat

$$r=\frac{p^3+nq^3}{ss}, \quad \text{unde fit} \quad x=\frac{p^4+npq^3}{ss} \quad \text{et} \quad y=\frac{p^3q+nq^4}{ss}.$$

quod uti est facillimum, ita casus, quo x et y sint inter se primi, maxime est difficilis. Formulae istius factor simplex est $x+y\sqrt[3]{n}$, et si α denotet unam radicem cubicam unitatis, ita ut sit $\alpha^3=1$, quam constat $\frac{-1+\sqrt{-3}}{2}$, tum tertia radix cubica erit $\alpha\alpha$. Hinc formulae nostrae x^3+ny^3 alius factor simplex erit $x+\alpha y\sqrt[3]{n}$, ac tertius $x+\alpha\alpha y\sqrt[3]{n}$, ita ut formula nostra futura sit productum horum trium factorum

$$(x+y\sqrt[3]{n})(x+\alpha y\sqrt[3]{n})(x+\alpha\alpha y\sqrt[3]{n}),$$

qui singuli factores reddantur quadrata, hoc modo, quo statim patet, si unus fuerit quadratus, etiam reliquos fore quadratos.

Posito enim $x+y\sqrt[3]{n}=(p+q\sqrt[3]{n}+r\sqrt[3]{nn})^2$, per naturam rei fiet

$$x+\alpha y\sqrt[3]{n}=(p+\alpha q\sqrt[3]{n}+\alpha\alpha r\sqrt[3]{nn})^2 \quad \text{et} \quad x+\alpha\alpha y\sqrt[3]{n}=(p+\alpha\alpha q\sqrt[3]{n}+\alpha r\sqrt[3]{nn})^2.$$

Productum ergo, quod est x^3+ny^3, etiam erit quadratum, et quidem rationale, quippe cujus radix erit

$$p^3+nq^3+nnr^3-3npqr.$$

Tantum igitur opus est, primam illam positionem supra datam evolvi, ex qua consequimur

$$x+y\sqrt[3]{n}=pp+2pq\sqrt[3]{n}+2pr\sqrt[3]{nn}$$
$$+2nqr+nrr\sqrt[3]{n}+qq\sqrt[3]{nn},$$

unde statim sequitur fore $x=pp+2nqr$, $y=2pq+nrr$.

Praeterea vero esse oportet $2pr+qq=0$, unde $r=-\frac{qq}{2p}$. Sumatur ergo $p=2aa$ et $r=-bb$, fietque $q=2ab$, consequenter valores satisfacientes sunt

$$x=4a(a^3-nb^3) \quad \text{et} \quad y=b(8a^3-nb^3).$$

Aliter. Si ponatur $p=aa$ et $r=-2bb$, erit $q=2ab$ et $x=a(a^3-8nb^3)$ et $y=4b(a^3+nb^3)$, ubi a et b pro lubitu assumere licet.

Exemplum. Quaerantur duo cubi inter se primi x^3 et y^3, quorum summa fiat quadratum, cujusmodi quidem statim sunt obvii 1 et 8. Hic ob $n=1$, erit

$$x=a(a^3-8b^3) \quad \text{et} \quad y=4b(a^3+b^3).$$

Sit $a=3$, $b=1$, erit $x=57$, $y=112$, quorum cuborum summa fit quadratum, cujus radix $=1261$.

Sit $a=2$, $b=-1$, erit $x=32$, $y=-28$, sive $x=8$, $y=-7$.

In hac tamen solutione, etsi generalis videtur, casus quo $x=1$ et $y=2$ non continetur, cujus ratio sine dubio in eo est quaerenda, quod hoc casu numerus n ipse sit cubus, ideoque irrationalitas evanescat. Quod clarius patebit ex solutione magis directa, nam ut x^3+y^3 fiat quadratum, ponatur $x+y=p$ et $x-y=q$, ut sit $x=\frac{p+q}{2}$ et $y=\frac{p-q}{2}$, unde fit

$$x^3+y^3=\frac{p^3+3pqq}{4}=\frac{(pp+3qq)p}{4},$$

quae formula ut reddatur quadrata, debet esse $p(pp+3qq)$ quadratum, unde si hi duo factores sint inter se primi, uterque factor quadratum esse debet. Posterius vero tantum locum habet, si p divisibile sit per 3. Hinc duos casus evolvi convenit.

I. Sint hi factores inter se primi, atque ut $pp+3qq$ fiat quadratum, vidimus sumi debere $p=ff-3gg$ et $q=2fg$; at vero ut et p fiat quadratum, capiatur $f=hh+3kk$ et $g=2hk$. Ergo solutio hinc nata erit

$$x=\frac{h^4+4h^3k-6hhkk+12hk^3+9k^4}{2}, \quad y=\frac{h^4-4h^3k-6hhkk-12hk^3+9k^4}{2}.$$

II. Sit $p = 3r$ et formula nostra erit $r(3rr + qq)$; fiat igitur $q = ff - 3gg$ et $r = 2fg$; fiet

$$3rr + qq = (ff + 3gg)^2.$$

Jam ut et r fiat quadratum, sumatur $f = 2hh$ et $g = kk$, ut fiat $r = 4hhkk$, ideoque $p = 12hhkk$ et $q = 4h^4 - 3k^4$. At vero etiam alia solutio pro hoc casu locum habet, ponendo $q = \frac{3gg - ff}{2}$ et $r = fg$; tum vero etiam $f = hh$ et $g = kk$. Si $h = 1 = k$, erit $f = 1$ et $g = 1$, hinc $q = 1$ et $r = 1$, ergo $p = 3$, $x = 2$ et $y = 1$, qui est casus cognitus.

In genere autem $q = \frac{3k^4 - h^4}{2}$ et $r = hhkk$, $p = 3hhkk$,

$$x = \frac{3k^4 + 6hhkk - h^4}{4}, \quad y = \frac{6hhkk - 3k^4 + h^4}{4}.$$

Supra observavimus, ut foret $a^3 + b^3 = c^3$, fore quoque $x^4 - 4xz^3 = \square$ et vicissim. Cum ergo quadratum esse debeat $x(x^3 - 4z^3)$, unde uterque factor debet esse quadratum. Reddatur primo posterior $x^3 - 4z^3$ quadratum, pro quo casu est $n = -4$, unde colligitur $x = a(a^3 + 32b^3)$ et $y = 4b(a^3 - 4b^3)$. Ut ergo et x fiat quadratum, debet esse $a(a^3 + 32b^3) = \square$, ergo uterque factor deberet esse $\square$, ideoque $a^3 + 32b^3 = \square$. Loco $2b$ scribamus $-c$ et formula erit $a(a^3 - 4c^3)$; quocirca si in maximis numeris formula $x(x^3 - 4z^3)$ esset $\square$, hoc modo ad aliam similem formulam deveniretur $a(a^3 - 4c^3)$ etiam quadratum, ubi numeri a et c manifesto multo forent minores, quam illi x et y. Deinde ex his a et c simili modo deduceremur ad alios multo minores, puta d et e, ita ut similis forma $d(d^3 - 4e^3)$ esset $\square$ et ita porro; unde certe proditura esset in minimis numeris talis forma quadratum; quare cum in minimis numeris talis forma non detur, ne in maximis quidem talis existit. Casus autem obvius, quo $e = 0$, hic nullam facit exceptionem; ad eum enim perveniri non potest, nisi jam in prima forma fuerit $z = 0$, qui casus ne in quaestionem quidem cadit.

Problema. Reddere formulam $x^3 + ny^3$ cubum.

Solutio. Statim manifestum est, ad hoc statui oportere

$$x + y\sqrt[3]{n} = (p + q\sqrt[3]{n} + r\sqrt[3]{nn})^3;$$

tum enim ipsius formulae $x^3 + ny^3$ radix cubica erit $p^3 + nq^3 + nnr^3 - 3npqr$. Facta autem evolutione reperietur

$$x + y\sqrt[3]{n} = \left.\begin{matrix} p^3 \\ + 6npqr \\ + nq^3 \\ + nnr^3 \end{matrix}\right. \left.\begin{matrix} + 3ppq \\ + 3nprr \\ + 3nqqr \end{matrix}\right\} \sqrt[3]{n} \left.\begin{matrix} + 3ppr \\ + 3pqq \\ + 3nqrr \end{matrix}\right\} \sqrt[3]{nn}$$

hinc

$$x = p^3 + 6npqr + nq^3 + nnr^3$$

$$y = 3ppq + 3nprr + 3nqqr$$

$$0 = 3ppr + 3pqq + 3nqrr,$$

ex qua aequatione fit

$$p = \frac{-qq \pm \sqrt{(q^4 - 4nqr^3)}}{2r},$$

unde quadratum esse deberet formula $q(q^3 - 4nr^3)$, ideoque uterque factor seorsim. Sit ergo $q = ss$ debetque esse $s^6 - 4nr^3 = \square = tt$, sive $s^6 - tt = 4nr^3 = 4nf^3g^3$. Fiat ergo $s^3 + t = 2f^3$ et $s^3 - t = 2ng^3$, unde $s^3 = f^3 + ng^3$, quae formula similis est ipsi propositae, ubi litterae f et g sine dubio multo sunt minores quam x et y. Quare si in minimis numeris talis casus non datur, ne in maximis quidem dabitur.

p. 99—103.

5. *(Lexell.)*

Ad Theorema Fermatii supra memoratum, quo aequalitas $a^\lambda + b^\lambda = c^\lambda$ locum habere nequit praeter casus $\lambda = 1$ et $\lambda = 2$, reductio ibi tradita hoc modo facillime obtinetur: Si esset $c^\lambda = a^\lambda + b^\lambda$, foret

$$c^{2\lambda}-4a^{\lambda}b^{\lambda}=(a^{\lambda}-b^{\lambda})^2=\square,$$

ideoque pro ab posito d, talis formula $c^{2\lambda}-4d^{\lambda}$ deberet esse quadratum, cujus igitur impossibilitatem ostendi oportet, praeter casus $\lambda=1$ et $\lambda=2$.

Corollarium. Simili modo ex formula $b^{\lambda}=c^{\lambda}-a^{\lambda}$ deducitur

$$b^{2\lambda}+4c^{\lambda}a^{\lambda}=(c^{\lambda}+a^{\lambda})^2=\square,$$

quin etiam $a^{2\lambda}+4c^{\lambda}b^{\lambda}=\square$, quae ergo formulae etiam sunt impossibiles. Demonstratio pro casu saltem $\lambda=3$ ita tentetur: Cum sit $a^3+b^3=c^3$, erit $(a+b)(aa-ab+bb)=c^3$, quos factores ut primos inter se spectemus, cum casus, quo divisorem communem habent 3, nullam novam difficultatem implicet. Sit igitur uterque cubus $a+b=p^3$ et $aa-ab+bb=P^3$, fietque $c=Pp$; tum vero erit $p^6-P^3=3ab$, deinde ob $b^3=c^3-a^3=(c-a)(cc+ac+aa)$, fiat iterum $c-a=q^3$ et $cc+ac+aa=Q^3$, fietque $b=Qq$ et $Q^3-q^6=3ac$,

denique ob $a^3=c^3-b^3=(c-b)(cc+bc+bb)$ sit $c-b=r^3$ et $cc+bc+bb=R^3$, unde $a=Rr$ et $R^3-r^6=3bc$. Introductis igitur litteris p, q, r et P, Q, R, ob $c=Pp$, $b=Qq$ et $a=Rr$, sequentes conditiones sunt adimplendae:

I. $p^3=Rr+Qq$, II. $q^3=Pp-Rr$, III. $r^3=Pp-Qq$,

IV. $P^3=RRrr-RrQq+QQqq$, V. $Q^3=PPpp+PpRr+RRrr$, VI. $R^3=PPpp+PpQq+QQqq$,

quibus praeterea adjungere licet

VII. $p^6-P^3=3QqRr$, VIII. $Q^3-q^6=3PpRr$, IX. $R^3-r^6=3PpQq$.

Denique etiam notasse juvabit $Q^3-P^3=(c-b)(a+c-b)=(Pp+Qq)(Rr+Pp-Qq)$. Totum ergo negotium huc redit, ut in his conditionibus contradictio detegatur.

p. 113.

6.

Theorema demonstrandum. Non dantur plures quantitates rationales veluti A, B, C, D, etc. quarum summa $A+B+C+D$ etc. per productum $ABCD$ etc. multiplicata producat unitatem. Sive si hoc signum $\mathrel{\pm\!\!\!=}$ denotet impossibilitatem aequalitatis, theorema hoc complectitur sequentes formas:

I. $AB(A+B)\mathrel{\pm\!\!\!=}1$, II. $ABC(A+B+C)\mathrel{\pm\!\!\!=}1$, III. $ABCD(A+B+C+D)\mathrel{\pm\!\!\!=}1$, etc.

Hae formae etiam ita exhiberi possunt

I. $A+B\mathrel{\pm\!\!\!=}\frac{1}{AB}$, II. $A+B+C\mathrel{\pm\!\!\!=}\frac{1}{ABC}$, III. $A+B+C+D\mathrel{\pm\!\!\!=}\frac{1}{ABCD}$, etc.

Hinc si postremae formulae fractae referantur littera O, sequentes formae sunt notatu dignae:

I. $A+B\mathrel{\pm\!\!\!=}O$ existente $ABO=1$, II. $A+B+C\mathrel{\pm\!\!\!=}O$ existente $ABCO=1$,

III. $A+B+C+D\mathrel{\pm\!\!\!=}O$ existente $ABCDO=1$, etc.

Porro quia litterae A, B, C, O sunt fractiones, si ponamus

$$A=\frac{a}{b},\quad B=\frac{b}{c},\quad C=\frac{c}{d},\quad \text{etc.}$$

sequentes habebuntur relationes impossibiles:

I. $\frac{a}{b}+\frac{b}{c}\mathrel{\pm\!\!\!=}\frac{c}{a}$; II. $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}\mathrel{\pm\!\!\!=}\frac{d}{a}$, III. $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}\mathrel{\pm\!\!\!=}\frac{e}{a}$.

At si hujus theorematis demonstratio haberetur, inde facile sequentia theoremata demonstrari possent:

Theorema I. Summa duorum cuborum esse nequit cubus, sive $p^3+q^3\mathrel{\pm\!\!\!=}r^3$.

Demonstratio. Facta divisione per pqr, ut habeatur

$$\frac{pp}{qr}+\frac{qq}{pr}=\frac{rr}{pq},$$

ubi si faciamus $\frac{pp}{qr}=A$, $\frac{qq}{pr}=B$, erit $AB=\frac{pq}{rr}=\frac{1}{O}$, sive $A+B=\frac{1}{AB}$, quod cum impossibile sit, hujus theorematis veritas est evicta.

Theorema II. Summa trium biquadratorum biquadratum esse nequit, sive $p^4+q^4+r^4=s^4$.

Demonstratio. Facta divisione per $pqrs$ habebitur

$$\frac{p^3}{qrs}+\frac{q^3}{prs}+\frac{r^3}{pqs}=\frac{s^3}{pqr}$$

$$A+B+C=O$$

ubi manifesto est $ABCO=1$. Quod cum sit impossibile, etiam hoc theorema est demonstratum.

Theorema III. Non dantur quatuor potestates quintae, quarum summa sit potestas quinta, sive

$$p^5+q^5+r^5+s^5=t^5.$$

Demonstratio. Facta divisione per productum $pqrst$ et comparatione cum superioribus litteris A, B, C, D, O instituta, hoc modo

$$\frac{p^4}{qrst}+\frac{q^4}{prst}+\frac{r^4}{pqst}+\frac{s^4}{pqrt}=\frac{t^4}{pqrs}$$

$$A+B+C+D=O$$

hic statim apparet esse $ABCDO=1$. Sicque etiam hoc theorema est demonstratum.

Theorema generale. Existente n exponente potestatis, non dantur $n-1$ tales potestates, quarum summa esset similis potestas.

Corollarium 1. Hinc multo minus $n-2$, vel $n-3$, vel $n-4$, etc. tales potestates dantur, quarum summa esset similis potestas. Hoc ergo modo theorema illud Fermatii in multo majori extensione adeo esset demonstratum.

Corollarium 2. Quia potestates impares aeque negativae ac positivae esse possunt, litterae illae p, q, r, s, sive A, B, C, D utcunque ratione signorum variare poterunt, id quod hoc modo referri potest:

$$\text{I.}\quad \pm p^3\pm q^3\pm r^3=0, \qquad \text{II.}\quad \pm p^5\pm q^5\pm r^5\pm s^5\pm t^5=0 \quad \text{etc.}$$

Corollarium 3. Hoc autem nullo modo valet pro potestatibus paribus, quoniam $-p^4$ non est potestas quarta, unde hoc theorema non ad hanc formam debet extendi: $p^4+q^4-r^4=s^4$, quandoquidem statim in oculos incurrit casu $q=r$ hanc aequationem subsistere non posse, quemadmodum modo supra vidimus talem formam revera resolvi posse.

Huic fragmento manu J. A. Euleri *inscriptum:* Hujus autem falsitas infra fusius ostendetur.

pag. 115. 116.

7.

Ecce quatuor numeri, quorum tam summa quam productum unitati aequatur:

$$+\frac{4}{3},\quad +\frac{3}{2},\quad -\frac{1}{3},\quad -\frac{3}{2},$$

unde superior illa conjectura omni fundamento destituitur.

Problema. Invenire quotcunque numeros, quorum summa multiplicata per productum producat unitatem.

1. Si desiderentur duo tales numeri, ut sit $ab(a+b)=1$, ponatur $a=\alpha b$ eritque $\alpha b^3(\alpha+1)=1$, sive $b^3=\frac{1}{\alpha(\alpha+1)}$, sicque $\alpha(\alpha+1)$ debet esse cubus, quod fieri nequit.

2. Si tres desiderentur numeri $abc(a+b+c)=1$, ponatur $a=\alpha(b+c)$, ideoque

$$a+b+c=(1+\alpha)(b+c), \quad \text{ergo} \quad \alpha(\alpha+1)bc(b+c)^2=1.$$

Nunc ponatur $b=\beta c$ eritque

$$(b+c)^2=cc(1+\beta)^2, \quad \text{ideoque} \quad \alpha\beta(\alpha+1)(\beta+1)^2c^4=1, \quad \text{sive} \quad \frac{1}{c^4}=\alpha\beta(\alpha+1)(\beta+1)^2.$$

Sumatur $\alpha=\beta\beta+2\beta$, et debet esse

$$\frac{1}{c^4}=\beta\beta(\beta+2)(\beta+1)^4, \quad \text{sive} \quad \frac{1}{c^4(\beta+1)^4}=\beta\beta(\beta+2).$$

Sit $\beta=pp-2$, unde $\quad \frac{1}{c^4(\beta+1)^4}=pp(pp-2)^2, \quad$ ergo $\quad \frac{1}{cc(\beta+1)^2}=p(pp-2),$

quod fit quadratum I. si sumatur $p=2$; tum enim erit $\frac{1}{c(\beta+1)}=2$; deinde $\beta=2$, $\alpha=8$, $c=\frac{1}{6}$, $b=\frac{1}{3}$, $a=4$. Consequenter tres numeri quaesiti 4, $\frac{1}{3}$, $\frac{1}{6}$, quorum summa est $\frac{9}{2}$ et productum $\frac{2}{9}$. Deinde $p(pp-2)$ fit quadratum sumendo $p=\frac{9}{4}$. Erit enim $p(pp-2)=\frac{9}{4}\cdot\frac{49}{16}$, hinc $\frac{1}{c(\beta+1)}=\frac{21}{8}$, porro $\beta=\frac{49}{16}$, $\alpha=\frac{49.81}{16^2}$, $c=\frac{8.16}{21.65}$, $b=\frac{7.8}{3.65}$, denique $a=\frac{49.81.8}{16^2.21}=\frac{7.27}{16.2}$, ergo tres numeri quaesiti $a=\frac{7.27}{32}$, $b=\frac{7.8}{3.65}$, $c=\frac{8.16}{21.65}$, quorum summa $\frac{65^2}{21.32}$, productum vero $\frac{21.32}{65^2}$.

Alia Solutio. Sumatur $\beta=pp-1$, ut fiat $\frac{1}{c^4p^4}=\alpha(\alpha+1)(pp-1)$, jam sumatur $\alpha=p-2$, unde $\frac{1}{c^4p^4}=(p-2)(p+1)(p-1)^2$, statuatur $(p-2)(p+1)=(p+1)^2qq$, unde $p-2=(p+1)qq$ et $p=\frac{2+qq}{1-qq}$, $p+1=\frac{3}{1-qq}$, $p-1=\frac{1+2qq}{1-qq}$, ideoque $\frac{1}{ccpp}=\frac{3q(1+2qq)}{(1-qq)^2}$. Superest ergo reddi quadratum $3q(1+2qq)$, quod manifesto fit sumto $q=\frac{1}{2}$; hinc $\frac{1}{cp}=2$, $p=3$, $\alpha=1$, $\beta=8$, $c=\frac{1}{6}$, ergo tres numeri sunt $a=\frac{3}{2}$, $b=\frac{4}{3}$, $c=\frac{1}{6}$, quorum summa est $=3$ et productum $=\frac{1}{3}$. Deinde solutionem praebet positio $q=\frac{1}{24}$.

Aliter, sumto statim $\alpha=1$, fit $\frac{1}{c^4p^4}=2\beta(\beta+1)^2$; sumatur $\beta=2pp$, fiet

$$\frac{1}{ccpp}=2p(2pp+1), \quad \text{cui satisfacit} \quad p=2.$$

3. Si desiderentur quatuor numeri, ut sit $abcd(a+b+c+d)=1$, ponatur

$$a=\alpha(b+c+d), \quad b=\beta(c+d) \quad \text{et} \quad c=\gamma d,$$

unde $b=\beta(\gamma+1)d$, $a=\alpha(\beta+1)(\gamma+1)d$ et summa omnium $(\alpha+1)(\beta+1)(\gamma+1)d$, productum vero $\alpha\beta\gamma(\beta+1)(\gamma+1)^2d^4$. Debet ergo esse

$$\alpha\beta\gamma(\alpha+1)(\beta+1)^2(\gamma+1)^3d^5=1.$$

Sumatur $\quad \gamma=\beta \quad$ eritque $\quad \alpha\beta\beta(\alpha+1)(\beta+1)^5d^5=1, \quad$ ideoque $\quad (\beta+1)^5d^5=\frac{1}{\alpha\beta\beta(\alpha+1)}$;

Sumatur porro $\quad (\beta+1)d=\frac{kk}{\alpha(\alpha+1)} \quad$ fietque $\quad k^{10}=\frac{\alpha^4(\alpha+1)^4}{\beta\beta}, \quad \beta=\frac{\alpha\alpha(\alpha+1)^2}{k^5},$

ubi α et k pro arbitrio sumi possunt; tum autem habebitur $\beta=\frac{\alpha\alpha(\alpha+1)^2}{k^5}$; hinc

$$d=\frac{kk}{\alpha(\alpha+1)(\beta+1)},\quad \gamma=\frac{\alpha\alpha(\alpha+1)^2}{k^5};\quad \alpha=\text{arbitr.},\quad c=\gamma d,\quad b=\beta(c+d)=\beta(\gamma+1)d,$$
$$a=\alpha(\beta+1)(\gamma+1)d.$$

Exemplum. $\alpha=1$, $k=1$, erit $\beta=1=\gamma$, $d=\frac{1}{10}$, $c=\frac{2}{5}$, $b=2$, $a=\frac{5}{2}$. Consequenter quatuor numeri sunt: $\frac{5}{2}$, 2, $\frac{2}{5}$, $\frac{1}{10}$ quorum summa est $=5$, et productum $=\frac{1}{5}$.

4. Si desiderentur quinque numeri, ut sit $abcde(a+b+c+d+e)=1$, ponatur
$$d=\delta e,\quad c=\gamma(\delta+1)e,\quad b=\beta(\gamma+1)(\delta+1)e,\quad a=\alpha(\beta+1)(\gamma+1)(\delta+1)e.$$
Hinc summa omnium $=(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)e$ et productum $\alpha\beta\gamma\delta(\beta+1)(\gamma+1)^2(\delta+1)^3e^5$,
$$\text{ergo}\quad \alpha\beta\gamma\delta(\alpha+1)(\beta+1)^2(\gamma+1)^3(\delta+1)^4e^6=1.$$
Sumatur $\delta=\beta$ et $\gamma=pp-1$, erit $\alpha\beta\beta(pp-1)(\alpha+1)p^6(\beta+1)^6e^6=1$.

Sit $p(\beta+1)e=\frac{1}{k}$ eritque $\alpha\beta\beta(pp-1)(\alpha+1)=k^6$. Fiat $\alpha(\alpha+1)(pp-1)=\alpha\alpha qq$, inde $\alpha=\frac{pp-1}{qq-pp+1}$, $\alpha\beta q=$ ergo $\beta=\frac{k^3}{\alpha q}$. Sit $p=2$ et $q=2$, hinc $\alpha=3$, $\beta=\frac{k^3}{6}$. Ponatur $k=2$, erit $\beta=\frac{4}{3}=\delta$, $e=\frac{3}{28}$, $d=$ $c=\frac{3}{4}$, $b=\frac{4}{3}$, $a=7$; consequenter quinque numeri 7, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{1}{7}$, $\frac{3}{28}$, quorum summa est $\frac{28}{3}$ et productum omnium $\frac{3}{28}$.

Conjectura igitur supra proposita maxime fallit, ita ex casu ultimo, quo volebamus demonstrare non dari quinque potestates sextas, quarum summa sit potestas sexta, tum demum demonstratio haberetur, si ostendi posset, quinque illos numeros a, b, c, d, e nunquam ita definiri posse, ut eorum quilibet, per quemcunque reliquorum divisus, praebeat potestatem sextam. Si igitur demonstrari posset omnes has fractiones
$$\frac{a}{b},\quad \frac{a}{c},\quad \frac{a}{d},\quad \frac{a}{e}$$
non esse posse potestates sextas, tum simul demonstratum esset, non dari quinque potestates sextas, potestati sextae aequales.

8. *(J. A. Euler.)*

Ad casum superiorem secundum pro tribus numeris, quo formula
$$\alpha\beta(\alpha+1)(\beta+1)^2$$
debet esse biquadratum, sumatur $\beta=4\alpha(\alpha+1)$, eritque formula
$$4\alpha\alpha(\alpha+1)^2(2\alpha+1)^4=\frac{1}{c^4},\quad \text{ergo}\quad 2\alpha(\alpha+1)(2\alpha+1)^2=\frac{1}{cc},\quad \text{sive}\quad 2\alpha(\alpha+1)=\frac{1}{cc(2\alpha+1)^2}.$$
Debet ergo $2\alpha(\alpha+1)$ esse quadratum. Ponatur ergo $2\alpha(\alpha+1)=\alpha\alpha pp$, erit $\alpha=\frac{2}{pp-2}$, hincque fit
$$\alpha\alpha pp=\frac{4pp}{(pp-2)^2}=\frac{1}{cc(2\alpha+1)^2},\quad \text{ergo}\quad \frac{2p}{pp-2}=\frac{1}{c(2\alpha+1)}=\frac{pp-2}{c(pp+2)},\quad \text{hinc}\quad c=\frac{(pp-2)^2}{2p(pp+2)}.$$
Porro $\beta=\frac{8pp}{(pp-2)^2}$, unde tres numeri erunt
$$\text{I.}\quad a=\frac{pp+2}{p(pp-2)},\qquad \text{II.}\quad b=\frac{4p}{pp+2},\qquad \text{III.}\quad c=\frac{(pp-2)^2}{2p(pp+2)}.$$

Exemplum. Si $p=2$ fit $a=\frac{3}{2}$, $b=\frac{4}{3}$, $c=\frac{1}{6}$. Summa $=3$, productum $=\frac{1}{3}$.

Eodem redeunt sequentes solutiones

$$1. \quad \alpha = \frac{8kk}{(2kk-1)^2} \quad \text{et} \quad \beta = 2kk, \quad \text{sive} \quad \beta = \frac{1}{2kk}.$$

$$2. \quad \alpha = \frac{1}{2kk-1} \quad \text{et} \quad \beta = \frac{(2kk-1)^2}{8kk}, \quad \text{sive} \quad \beta = \frac{8kk}{(2kk-1)^2}.$$

Nam si fuerit $\alpha(\alpha+1)\beta(\beta+1)^2 =$ biquadrato, casu $\beta = b$, tum erit etiam casu $\beta = \frac{1}{b}$.

3. Si $kx(mxx+n) = \square$ casu $x = a$, tum etiam erit quadratum casu $x = \frac{n}{ma}$.

Problema. Invenire tres numeros p, q et r ita, ut formula $(pp-qq)(qq-rr)$ fiat biquadratum: veluti evenit I. si $p=51$, $q=3$, $r=1$; II. si $p=14$, $q=13$, $r=11$; III. si $p=29$, $q=25$, $r=23$.

Hic notasse juvabit ex casu quovis cognito facile erui alios, scilicet

$$p' = p+2q-r, \quad q' = p+r, \quad r' = p-2q-r,$$

sive etiam $\quad p' = p+2q+r, \quad q' = p-r, \quad r' = p-2q+r.$

Hoc problema facillime ex praecedente, quo numeri illi a, b, c sunt inventi, resolvitur. Sumatur enim

$$q = a+b \quad \text{et} \quad p = \sqrt{\left(qq+\frac{4}{ab}\right)} \quad \text{et} \quad r = a-b.$$

Exemplum. Sumto $a = \frac{3}{2}$, $b = \frac{4}{3}$, erit $a+b = q = \frac{17}{6}$, $r = \frac{1}{6}$ et $p = \sqrt{\left(\frac{289}{36}+2\right)} = \frac{19}{6}$; sive $p=19$, $q=17$ et $r=1$; hincque alii reperiuntur

$$p' = 52, \quad q' = 20, \quad r' = 16, \quad \text{sive}$$
$$p' = 13, \quad q' = 5, \quad r' = 4,$$

$$\text{vel etiam} \quad p' = 54, \quad q' = 18, \quad r' = 14, \quad \text{sive}$$
$$p' = 27, \quad q' = 9, \quad r' = 7.$$

Ex solutione generali sumatur $a = \frac{8kk}{2k(kk+2)}$ et $b = \frac{(kk-2)^2}{2k(kk+2)}$, fietque

$$q = \frac{(kk+2)^2}{2k(kk+2)}, \quad r = \frac{k^4-12kk+4}{2k(kk+2)} \quad \text{et} \quad p = \sqrt{\left(\frac{(kk+2)^2}{4kk}+\frac{2(kk+2)^2}{(kk-2)^2}\right)} = \frac{(kk+2)^2}{2k(kk-2)}, \quad \text{vel}$$

$$p = (kk+2)^3, \quad q = (kk-2)(kk+2)^2, \quad r = (kk-2)(k^4-12kk+4).$$

Analysis, qua haec solutio innititur, ita se habet:

Inventis ternis numeris a, b, c, ut supra, sumatur $q = a+b$ et $r = a-b$ et $p = a+b\pm 2c$; tum enim fiet

$$pp-qq = 4cc+4c(a+b) = 4c(a+b+c), \quad \text{at} \quad qq-rr = 4ab,$$

$$\text{ergo} \quad (pp-qq)(qq-rr) = 16abc(a+b+c) = 16.$$

Hinc porro colligimus $p' = 2a+4b+2c$, vel

$$p' = a+2b+c, \quad q' = a+c, \quad r' = a-c,$$

$$\text{vel etiam} \quad p' = 2a+b+c, \quad q' = b+c, \quad r' = b-c.$$

Alia analysis. Loco a, b, c scribantur $\frac{x}{s}$, $\frac{y}{s}$ et $\frac{z}{s}$, ut debeat esse

$$xyz(x+y+z) = s^4.$$

Jam fiat primo $s^4 = (x+y+z)^2pp$, eritque $xyz = (x+y+z)pp$, hinc

$$z = \frac{(x+y)pp}{xy-pp} \quad \text{et} \quad x+y+z = \frac{xy(x+y)}{xy-pp}, \quad \text{ergo} \quad ss = \frac{xyp(x+y)}{xy-pp}.$$

Ponatur porro $x = nqq$ et $y = nrr$, fietque

$$ss = \frac{n^3 qqrrp(qq+rr)}{nnqqrr - pp}, \quad \text{sive} \quad \frac{ss}{nnqqrr} = \frac{np(qq+rr)}{nnqqrr - pp}.$$

Ponatur nunc $nqr - p = k(qq+rr)$ eritque $p = nqr - k(qq+rr)$, ergo

$$\frac{ss}{nnqqrr} = \frac{nnqr - nk(qq+rr)}{k(2nqr - k(qq+rr))} = \frac{n(k(qq+rr) - nqr)}{k(k(qq+rr) - 2nqr)}.$$

Casus I. Sit $n = 2k$, erit $\dfrac{ss}{nnqqrr} = \dfrac{qq+rr-2qr}{qq+rr-4qr} = \dfrac{(q-r)^2}{qq+rr-4qr}$.

Sicque quadratum esse debet $qq + rr - 4qr$, cujus radix ponatur $q + \frac{f}{g}r$, ita ut fiat

$$rr - 4qr = \frac{2f}{g}qr + \frac{ff}{gg}rr, \quad \text{vel} \quad ggr - 4ggq = 2fgq + ffr,$$

$$\text{vel} \quad (gg - ff)r = (4gg + 2fg)q, \quad \text{sive} \quad \frac{q}{r} = \frac{gg-ff}{4gg+2fg}.$$

Sumatur ergo $q = gg - ff$ et $r = 4gg + 2fg$, eritque

$$\frac{ss}{nnqqrr} = \frac{(q-r)^2}{(q + \frac{f}{g}r)^2}, \quad \text{ideoque} \quad \frac{s}{nqr} = \frac{q-r}{q + \frac{f}{g}r} = \frac{3gg+ff+2fg}{gg+ff+4fg} \quad \text{et} \quad s = \frac{2k(gg-ff)(4gg+2fg)(3gg+ff+2fg)}{gg+ff+4fg}.$$

Quocirca erit $p = 2kqr - k(qq+rr) = -k(q-r)^2$, ideoque $x = 2kqq$, $y = 2krr$, $z = \dfrac{(x+y)pp}{xy - pp}$.

Casus II. Sumatur $n = k$, erit $\dfrac{ss}{kkqqrr} = \dfrac{qq+rr-qr}{(q-r)^2}$. Sit $\sqrt{(qq+rr-qr)} = q + \frac{f}{g}r$, erit

$$rr - qr = \frac{2f}{g}qr + \frac{ff}{gg}rr, \quad \text{vel} \quad ggr - ggq = 2fgq + ffr, \quad \text{hinc} \quad \frac{q}{r} = \frac{gg-ff}{gg+2fg}.$$

Sumatur ergo $q = gg - ff$ et $r = gg + 2fg$, ita ut sit

$$\frac{s}{kqr} = \frac{q + \frac{f}{g}r}{q - r} = \frac{gg+fg+ff}{ff+2fg}; \quad \text{hincque erit}$$

$$p = kqr - k(qq+rr), \quad \text{vel} \quad p = k(qr - qq - rr), \quad \text{indeque} \quad y = krr, \quad x = kqq, \quad z = \frac{(x+y)pp}{xy - pp}.$$

Aliter. Cum sit $ss = \dfrac{xyp(x+y)}{xy - pp}$, dividetur per xy, eritque $ss = \dfrac{p(x+y)}{1 - \frac{pp}{xy}}$. Sumatur $p = \dfrac{2xy}{x+y}$, fietque $ss = \dfrac{2xy(x+y)^2}{(x-y)^2}$. Superest ergo ut $2xy$ fiat quadratum. Sumatur $x = 2qq$ et $y = rr$ fietque

$$ss = \frac{4qqrr(2qq+rr)^2}{(2qq-rr)^2}, \quad \text{ideoque} \quad s = \frac{2qr(2qq+rr)}{2qq-rr}; \quad \text{hincque} \quad p = \frac{4qqrr}{2qq+rr},$$

$$\text{indeque} \quad x = 2qq, \quad q = rr, \quad z = \frac{pp(2qq+rr)}{2qqrr - pp}, \quad \text{vel} \quad z = \frac{8qqrr(2qq+rr)}{(2qq-rr)^2}.$$

Exemplum casus primi. Sit $g = 2$ et $f = 1$, erit $s = \dfrac{2.3.20.17}{13}k$, $q = 3$, $r = 20$, $p = 17^2 k$, $x = 2.9$ $y = 2.20^2 k$, $z = \dfrac{818.17^4 k^3}{409.13^2 kk}$, vel $z = \dfrac{2.17^4 k}{13^2}$, hinc

$$a = \frac{x}{s} = \frac{3.13}{17.20}, \quad b = \frac{y}{s} = \frac{13.20}{3.17}, \quad c = \frac{z}{s} = \frac{17^3}{3.13.20},$$

quorum productum est $\dfrac{13.17}{3.20}$ et summa . . . . falsa!

9. (*N. Fuss I.*)

Tentamen demonstrationis theorematis Fermatiani, quod esse nequeat $x^n + y^n = z^n$, statim ac n superat binarium.

Pro casu $n = 3$ res eo redit, ut demonstretur hanc formulam $ab(a+b)$ cubum esse non posse, ubi a et b sint primi inter se. Ponatur ergo $ab(a+b) = x^3$ eritque $4a^3b + 4aabb = 4ax^3$, sive $(aa+2ab)^2 = 4ax^3 + a^4$, unde $aa + 2ab = \sqrt{(4ax^3 + a^4)}$. Quoniam hic x et a non sunt numeri primi inter se, sit d maximus eorum communis divisor, ac ponatur $a = dp$ et $x = dz$, sicque p et z erunt primi inter se, et quia a et b etiam sunt primi inter se, erit quoque b primus ad d et p, tum igitur erit

$$2dbp + ddpp = dd\sqrt{(4pz^3 + p^4)}, \quad \text{ideoque} \quad \sqrt{(4pz^3 + p^4)} = \frac{2bp}{d} + pp.$$

Erit ergo $\frac{2bp}{d}$ numerus integer. Quia ergo b primus ad d, necesse est, ut $\frac{p}{d}$ sit integer; ponatur ergo $p = dq$, erit $\sqrt{(4dqz^3 + d^4q^4)} = 2bq + ddqq$. Unde quia radix factorem habet q, at z ad q primus, necesse ut d habeat factorem. Sit ergo $d = qr$ eritque $\sqrt{(4qqrz^3 + q^8r^4)} = 2bq + q^4rr$, seu $\sqrt{(4rz^3 + q^6r^4)} = 2b + q^3rr$. Sicque erit r factor quantitatis post signum, dum alter factor est $4z^3 + q^6r^3$, unde necesse est $r = \square$. Sit ergo $r = ss$, erit $\sqrt{(4z^3 + q^6s^6)} = \frac{2b}{s} + q^3s^3$. At vero $\frac{2b}{s}$ numerus integer esse non potest, unde patet aequationem nullo modo subsistere posse. Sicque impossibile erit, ut sit $ab(a+b) = x^3$, neque ergo unquam esse poterit $a^3 + b^3 = c^3$. Facile autem patet, hoc modo rem de altioribus potestatibus demonstrari posse. Verum haec conclusio maxime est incerta, cum fieri posset tam $s = 1$, quam $s = 2$. Ceterum theorema Fermatianum huc redit, ut demonstretur nunquam fieri posse, ut haec formula $1 + 4x^n$, vel etiam $1 - 4x^n$ unquam evadat quadratum, simul ac exponens n binarium superaverit; hic autem x omnes numeros rationales tam fractos, quam integros significare potest. Reducatur enim res ad numeros integros, ponendo $x = \frac{pq}{rr}$ et formula evadet

$$r^{2n} \pm 4p^nq^n = \square,$$

cujus radix statuatur $r^n + 2\nu$, ita ut ν primus ad r, erit

$$r^{2n} + 4p^nq^n = r^{2n} + 4\nu r^n + 4\nu\nu, \quad \text{unde erit} \quad r^n = \frac{4p^nq^n - 4\nu\nu}{4\nu}, \quad \text{sive} \quad p^nq^n = \nu(r^n + \nu),$$

qui duo factores sunt primi inter se, unde uterque debet esse potestas exponentis n. Capi ergo poterit $\nu = p^n$, tum autem erit $r^n + \nu = q^n$, ideoque $r^n + p^n = q^n$.

Quare si haec formula $1 \pm 4x^n$ fuerit impossibilis, etiam impossibile erit

$$\text{ut} \quad r^n + p^n = q^n.$$

A. m. T. II. p. 161.

10. (*J. A. Euler.*)

Ut fiat $x^3 + y^3 = \square$, sumatur $x + y = 3aabb$, $x - y = \frac{3a^4 - b^4}{2}$.

Summae vel differentiae duorum cuborum,

quae sint quadrata:

I. $2^3 + 1^3 = 3^2$, II. $8^3 - 7^3 = 13^2$, III. $65^3 + 56^3 = 671^2$, IV. $74^3 - 47^3 = 549^2$.

11. (*Lexell.*)

V. $37^3 + 11^3 = 228^2$, VI. $71^3 - 23^3 = 588^2$.

Proposito problemate, quo quaeruntur duo cubi inter se primi, quorum summa $x^3 + y^3$ sit quadratum, duo casus sunt perpendendi, alter, quo ambo numeri x et y sunt impares, alter vero, quo unus par, alter impar.

Pro *casu priori* erit $x=a+b$ et $y=a-b$, numerorum a et b altero existente pari, altero impari, hinc autem fit

$$x^3+y^3=2a^3+6abb=2a(aa+3bb),$$

ubi iterum duo casus occurrunt: *primo* vel $2a$ et $aa+3bb$ sunt primi inter se, quia $aa+3bb$ est impar, ergo uterque factor seorsim esse debet quadratum, unde patet a esse debere parem, b vero imparem; ponatur ergo $2a=4cc$, et quadratum insuper esse debet $aa+3bb=4c^4+3bb=\square$, quod facile fit; vel *secundo* $2a$ et $aa+3bb$ communem factorem habere possunt 3, quod fit si a sit divisibile per 3, existente a pari; sit ergo $a=6c$, fiet $12c(36cc+3bb)=\square$, hinc $4c(12cc+bb)=\square$. Sit ergo $c=dd$, fierique debet $12d^4+bb=\square$, quod facile fit. Pro *posteriori casu* poni debet $x=\frac{a+b}{2}$ et $y=\frac{a-b}{2}$, ubi uterque a et b impar; tum igitur quadratum esse debet $\frac{2a(aa+3bb)}{8}=\square$, sive $a(aa+3bb)=\square$. Hic iterum vel a non est divisibile per 3, vel divisibile per 3; illo casu sit $a=cc$, ideoque $c^4+3bb=\square$, hoc vero casu sit $a=3cc$, unde

$$3cc(9c^4+3bb)=\square, \quad \text{sive} \quad 3c^4+bb=\square.$$

A. m. T. I. p. 129.

12. (*N. Fuss I.*)

Si esse $a^3=b^3+c^3$, foret $a^6-4b^3c^3=(b^3-c^3)^2$. Hinc ergo si demonstrari posset nunquam esse $a^6-4d^3=\square$, theorema foret demonstratum. Quoniam igitur haec forma a^6-4d^3 continetur in hac A^2-dB^2, etiam ejus radix quadrata similem formam habeat necesse est, quae ergo sit $pp-dqq$. Ergo fit $a^3=pp+pq^3=p(p+q^3)$. Hinc prior factor p debet esse cubus $=r^3$, et alter factor r^3+q^3 pariter cubus. Unde si foret $b^3+c^3=$ cubo, alius casus hinc deduceretur $r^3+q^3=$ cubo.

A. m. T. II. p. 211.

13.

Observatio circa theorema Fermatii, quo affirmat, hanc aequalitatem $a^n+b^n=c^n$ semper esse impossibilem, simul ac exponens n excedat binarium, cujus autem demonstrationem nemo adhuc invenire potuit.

Reduci potest ista forma ad formulas, quae quadrata fieri debent. Multiplicetur enim formula proposita per $4a^n$ et utrinque addatur b^{2n}, prodibit

$$(2a^n+b^n)^2=4a^nc^n+b^{2n}=\square=BB.$$

Simili modo erit $4b^nc^n+a^{2n}=AA$, item $c^{2n}-4a^nb^n=CC$. Totum negotium ergo eo redit, num impossibilitas harum formularum ostendi possit. Ceterum apparet sufficere, casus examinare, quibus n est numerus primus; nam si $a^n+b^n=c^n$, erit etiam $a^{\lambda n}+b^{\lambda n}=c^{\lambda n}$, sicque n spectari poterit ut numerus impar, tum autem formula a^n+b^n factorem habet $a+b$. Debet ergo etiam esse $a+b=p^n$, similique modo $c-a=q^n$ et $c-b=r^n$. Quod si ergo hae conditiones cum praecedentibus conjungantur, impossibilitas fortasse facilius ostendi poterit. Non solum igitur ostendi oportet hanc formulam $4a^nc^n+b^{2n}$ non esse posse quadratum, ita, ut simul $a+b=p^n$.

Pro casu $a=1$ et $b=1$ fit illa formula $4c^n+1=\square$, quod in integris nunquam evenire posse ita ostendo, quod quidem manifestum est si n est par. Pro imparibus autem statim patet c non esse posse numerum imparem. Sit igitur par $=2d$, erit formula $2^{n+2}d^n+1$, cujus ergo radix esse debet $1+2^{n+1}s$,

$$2^{n+2}d^n+1=1+2^{n+2}s+2^{2n+2}ss, \quad \text{unde} \quad d^n=s+2^nss=s(2^ns+1),$$

qui factores cum sint primi inter se, debebit esse $s=t^n$, alter vero factor erit $2^nt^n+1=(2t)^n+1$, quod est impossibile.

A. m. T. III. p. 165–166.

14.

Theorema. Formula $1+2x^3$ nullo casu fit quadratum, neque in integris, neque in fractis, praeter casum $x=0$.

Demonstratio innititur huic fundamento, quod omnes cubi per 7 non divisibiles sint formae $7n \pm 1$. Hinc ergo omnes potestates sextae erunt formae $7n+1$. Deinde omnia quadrata sunt vel $7n$, vel $7n+1$, vel $7n+2$, vel $7n+4$, ita ut nulli numeri formae $7n+3$, $7n+5$, $7n+6$ sint quadrati. Jam forma $1+2x^3$ in integris quadratum esse non potest. Si enim x per 7 non sit divisibile, forma numeri $1+2x^3$ erit $7n+3$ et $7n+6$, quorum neuter quadratum esse potest. Sumto autem $x=7a$, erit $1+2.7^3.a^3=zz$. Foret ergo $2.7^3.a^3=zz-1=(z+1)(z-1)$, ergo factorum $z+1$ et $z-1$ alter debet esse cubus, alter duplex cubus. Sumamus $z-1=7^3.b^3$, ideoque $z=1+7^3.b^3$, unde $2a^3=2b^3+7^3.b^6$, unde patet esse debere $a=bc$, erit ergo $2c^3=2+7^3.b^3$.

At si x est numerus fractus, ejus denominator debet esse quadratum. Ponatur ergo $x=\frac{a}{bb}$, fieri debet $b^6+2a^3=\square$, ubi nisi $b=7n$, semper erit $b^6=7n+1$ et $a^3=7n\pm 1$ (si a non est $7n$), ergo

$$b^6+2a^3=7n+1\pm 2,$$

hoc est vel $7n+3$, vel $7n-1$, neutro casu quadratum. Sit $a=7c$ erit $b^6+2.7^3.c^3=zz$. Sit $z=b^3+2.7^3d^3$, hinc $c^3=2b^3d^3+2.7^3.d^6$ et sumto $c=dc$ erit $c^3=2b^3+2.7^3.d^3$, ergo c^3-2b^3 divisibile esset per 7, quod fieri nequit.

Verum rigida demonstratio postulat profundiores indagationes.

Theorema I. Si fuerit $2x^3+1=\square$, dari poterunt duo cubi, quorum summa vel differentia sit cubus quadruplus.

Demonstratio. Loco x scribamus $\frac{x}{yy}$ fietque $2x^3+y^6=zz$. Jam ponatur $x=ab$ fierique debet $zz-y^6=2a^3b^3$. Fiat ergo $z+y^3=2a^3$ et $z-y^3=b^3$, unde fit $2y^3=2a^3-b^3$, ergo $b^3=2(a^3-y^3)$. Fiat $b=2c$, erit $4c^3=a^3-y^3$.

Theorema II. Si dentur duo cubi, quorum summa vel differentia aequetur cubo quadruplo, dari poterit x, ut sit $2x^3+1=\square$.

Demonstratio. Sit $a^3+b^3=4c^3$, erit $4a^6+4a^3b^3+b^6=16a^3c^3+b^6=\square$. Jam sumatur $x=\frac{2ac}{bb}$ erit $\square=2x^3+1$.

Theorema III. Non dantur duo cubi, quorum summa vel differentia sit cubus quadruplus.

Demonstratio. Si enim fuerit $x^3+y^3=4z^3$, evidens est ambos numeros x et y esse debere impares, unde statui poterit $x=a+b$ et $y=a-b$, ita ut numerorum a et b alter sit par alter impar, unde fiet $2a^3+6abb=4z^3$, sive $a(aa+3bb)=2z^3$, ubi $aa+3bb$ erit numerus impar, unde patet a esse debere parem et b imparem. Hinc porro si ambo factores a et $aa+3bb$ fuerint primi inter se, debet esse $a=2p^3$ et $aa+3bb=q^3$. At vero si a sit $3c$, ambo factores communem habebunt divisorem 3, eritque

$$9c(3cc+bb)=2p^3q^3,$$

unde $9c$ debet esse duplus cubus veluti $2.27d^3$, ita ut $c=2.3d^3$, ideoque $a=2.9d^3$. Tum vero $bb+3cc$ debet esse cubus, unde casus duo sunt considerandi: prior, quo $a=2p^3$ et $aa+3bb=q^3$; alter, quo $a=2.9p^3$ et $aa+3bb=3q^3$, sive posito $a=3c$ debet esse $bb+3cc=r^3$. Quod autem uterque casus sit impossibilis, demonstrari potest ope sequentis lemmatis.

*

Lemma. Si fuerit $xx+3yy=$ cubo, certum est ejus radicem ejusdem fore formae, puta $pp+3qq$, ita ut $xx+3yy=(pp+3qq)^3$. Erit ergo $x+y\sqrt{-3}=(p+q\sqrt{-3})^3$, $x-y\sqrt{-3}=(p-q\sqrt{-3})^3$, hoc est

$$x+y\sqrt{-3}=p^3-9pqq+(3ppq-3q^3)\sqrt{-3},$$

$$\text{unde fit} \quad x=p^3-9pqq \quad \text{et} \quad y=3ppq-3q^3.$$

Demonstratio casus prioris: Cum igitur $aa+3bb=$ cubo, per lemma erit $a=p^3-9pqq$ et $b=3ppq-3q^3$. Quam ob rem debet esse $a=p^3-9pqq=2$ cubis, unde hoc productum $p(p-3q)(p+3q)$ cubo duplo aequari debet, et cum numerorum p et q alter sit par, alter impar, erit p par, ideoque $p=2$ cubis. At vero $p+3q$ et $p-3q=$ cubo. Ponatur ergo $p+3q=r^3$ et $p-3q=s^3$, erit $2p=r^3+s^3=4t^3$, quod fieri non potest, quia si darentur tales numeri $a^3+b^3=4c^3$, nunc darentur multo minores $r^3+s^3=4t^3$.

Demonstratio alterius casus. Cum fieri debeat $bb+3cc=$ cubo, erit $b=p^3-9pqq$ et $c=3ppq-3q^3$. Cum igitur $a=3c$, erit $a=9(ppq-q^3)=2.9s^3$, sive $q(pp-qq)=2s^3$, ubi q erit par, ideoque ponatur

$$p+q=t^3 \quad \text{et} \quad p-q=u^3, \quad \text{erit} \quad 2q=t^3-u^3=4v^3.$$

Unde si magni darentur numeri, etiam in minoribus dari deberent, ut fiat $2x^3+1=\square$, quod autem cum in minimis non fiat, etiam in maximis non succedet.

A. m. T. III. p. 167—169.

15.

Problema. Invenire duos cubos, quorum summa aequetur dato multiplo cujuspiam cubi, sive ut sit

$$x^3+y^3=nz^3.$$

Solutio. Ponatur $n=\alpha\beta\gamma$ et fiat $x=a+b$ et $y=a-b$; tum vero $z=2\nu$, erit $a(aa+3bb)=4\alpha\beta\gamma\nu^3$. Fiat $aa+3bb=(pp+3qq)^3$ et vidimus fore $a=p(pp-9qq)$ et $b=3q(pp-qq)$, esseque oportebit $a=\frac{4\alpha\beta\gamma\nu^3}{(pp+3qq)^3}$. Sumatur $\nu=fgh(pp+3qq)$, ut prodeat $a=4\alpha\beta\gamma f^3g^3h^3$. Cum igitur sit $a=p(p+3q)(p-3q)$, fiat $p=\alpha f^3$, $p+3q=2\beta g^3$ et $p-3q=2\gamma h^3$. Hinc erit $p=\beta g^3+\gamma h^3$ et $3q=\beta g^3-\gamma h^3$. Hinc ergo debet esse

$$\alpha f^3=\beta g^3+\gamma h^3,$$

quod si ergo hoc fieri potest, etiam aequatio proposita erit confecta. Ita sumtis f, g, $h=\pm1$, solutio locum habebit si fuerit $\alpha=\beta\pm\gamma$. Sumto $f=2$, $g=h=\pm1$ solutio locum habet quoties fuerit $8\alpha=\beta\pm\gamma$. Tali autem casu, quo $\alpha f^3=\beta g^3+\gamma h^3$, invento, erit $p=\alpha f^3$ et $q=\frac{\beta g^3-\gamma h^3}{3}$, unde porro deducitur $a=p(pp-9qq)$ et $b=3q(pp-qq)$, ex quibus denique $x=a+b$ et $y=a-b$. Tandem autem erit $z=2\nu=2fgh(pp+3qq)$.

Exemplum 1. Sit $\alpha=3$, $\beta=2$, $\gamma=1$, ideoque $n=6$, fiet $3f^3=2g^3+h^3$, quod fit si $f=1$, $g=1$, $h=1$, tum autem erit $p=3$ et $q=\frac{1}{3}$, unde deducitur $a=24$ et $b=\frac{80}{9}$. Erit ergo $a:b=27:10$. Sit ergo $a=27$ et $b=10$ eritque $x=37$, $y=17$, $z=\frac{56}{3}$. Cum jam sit $x^3+y^3=(x+y)(xx-xy+yy)$, erit $x+y=54$ et ob $x-y=20$, ergo $xx-2xy+yy=400$ et $xx-xy+yy=1029$, ergo $x^3+y^3=54.1029=6.3^3.7^3$.

Exemplum 2. Sit $\alpha=5$, $\beta=3$, $\gamma=1$, ideoque $n=15$, fiet $p=5f^3=3g^3+h^3$, quod fit si $h=2$ et $g=-1$ et $f=1$; tum erit $p=5$, $q=-\frac{11}{3}$, unde $a=5.96$ et $b=\frac{104.11}{9}$. Sumatur $a=540$, $b=143$, $x=683$, $y=397$ fietque $x^3+y^3=15z^3$.

Observatio maximi momenti. Arbitratus sum, si fuerit $xx-nyy=(pp-nqq)^3$, etiam fore

$$x+y\sqrt{n}=(p+q\sqrt{n})^3 \quad \text{et} \quad x-y\sqrt{n}=(p-q\sqrt{n})^3,$$

unde facta evolutione fiat $x=p^3+3npqq$ et $y=3ppq+nq^3$. At nunc se mihi casus obtulit maxime discrepans

$16^2-3.23^2=(1-3.2^2)^3$, unde deberet esse $16+23\sqrt{3}=(1+2\sqrt{3})^3$, quod autem neutiquam contingit. Similique modo deberet esse $16-23\sqrt{3}=(1-2\sqrt{3})^3$. Interim tamen productum priorum

$$16^2-3.23^2=(1-3.2^2)^3=37^2-3.30^2.$$

Revera igitur hoc remedium afferri debet: Si fuerit $xx-nyy=(pp-nqq)^3$, tum sumtis factoribus dabuntur numeri f et g, ut sit $x+y\sqrt{n}=(f+g\sqrt{n})(p+q\sqrt{n})^3$ atque $x-y\sqrt{n}=(f-g\sqrt{n})(p-g\sqrt{n})^3$, ubi necesse est, ut sit $ff-ngg=1$. Haec ergo applicemus ad casum observatum, ubi est $x=16$, $n=3$, $y=23$, deinde $p=1$, $q=2$, et facto calculo litterae f et g ita determinantur, ut sit $f=-\frac{1478}{1331}$ et $g=-\frac{371}{1331}$, unde revera fit $ff-3gg=1$. Unde patet hujusmodi coëfficientes nullo modo divinari posse.

Sequens autem consideratio me ad hunc casum deduxit: Quaesivi numeros x et y, ut $(x+y)(xx+yy)$ fiat cubus, et vidi esse debere $x+y=4A^3$ et $xx+yy=2B^3$. Posui ergo $xx+yy=2(aa+bb)^3$ et inveni $x=a(aa-3bb)$ et $y=b(3aa-bb)$. Hinc porro inveni hanc solutionem $y=-9$ et $x=13$, deinde ex hoc casu elicui $x=7.37.61$ et $y=9.13.229$. — At vero valores litterarum f et g multo simplicius exhiberi possunt, uti ex sequente problemate patebit:

Problema. Invenire numeros inter se primos x et y, ut sit $xx-3yy$ cubus.

Solutio. Primo haec conditio est adjicienda, ut numeri x et y sint primi inter se: si enim compositi admittantur, solutio esset facillima sumendo

$$x=a(aa-3bb) \quad \text{et} \quad y=b(aa-3bb); \text{ tum enim foret } xx-3yy=(aa-3bb)^3.$$

Ponatur igitur $xx-3yy=(pp-3qq)^3$ et sumtis factoribus fiat

$$x+y\sqrt{3}=(f+g\sqrt{3})(p+q\sqrt{3})^3$$

similique modo $$x-y\sqrt{3}=(f-g\sqrt{3})(p-q\sqrt{3})^3,$$

sic enim fiet $xx-3yy=(ff-3gg)(pp-3qq)^3$. Necesse igitur est, ut sit $ff-3gg=1$, quod infinitis modis fieri potest. Primo $f=1$ et $g=0$; secundo $f=2$ et $g=1$; tertio $f=7$ et $g=4$; quarto $f=26$ et $g=15$. et in genere

$$f+g\sqrt{3}=\frac{1}{2}(2+\sqrt{3})^n+\frac{1}{2}(2-\sqrt{3})^n$$

$$f-g\sqrt{3}=\frac{1}{2}(2+\sqrt{3})^n-\frac{1}{2}(2-\sqrt{3})^n.$$

His notatis cum sit $(p+q\sqrt{3})^3=p(pp+9qq)+3q(pp+qq)\sqrt{3}$. Ponatur brevitatis gr.

$p(pp+9qq)=P$ et $q(pp+qq)=Q$, ut fiat $(p+q\sqrt{3})^3=P+3Q\sqrt{3}$ et $(p-q\sqrt{3})^3=P-3Q\sqrt{3}$.

Hinc ergo erit

$$x+y\sqrt{3}=fP+(gP+3fQ)\sqrt{3}+9gQ, \text{ unde fit } x=fP+9gQ \text{ et } y=gP+3fQ;$$

ubi notetur litteras f et g tam negative quam positive accipi posse.

Sit nunc $p=1$ et $q=2$, erit $P=37$ et $Q=10$, ergo $x=37f+90g$ et $y=37g+30f$; quare sumto $f=1$ et $g=0$, erit $x=37$ et $y=30$. At sumto $f=2$ et $g=1$ erit $x=164$, $y=97$. Sumto vero $f=2$ et $g=-1$ erit $x=16$, $y=23$, qui est ipse casus supra tam difficilis visus. Hoc ergo modo omnes casus possibiles pro x et y erui poterunt, ut $xx-3yy$ fiat cubus, dummodo litteris f et g omnes valores tam positivi quam negativi successive tribuantur. Eodem modo problema generalius solvi potest, ut fiat $xx-nyy$ cubus, qui sit $(pp-nqq)^3$ et sumto

$$ff-ngg=1 \text{ erit } x+y\sqrt{n}=(f+g\sqrt{n})(p+q\sqrt{n})^3,$$

ubi erit $$(p+q\sqrt{n})^3=p(pp+3nqq)[=P]+q(3pp+nqq)\sqrt{n}[=Q\sqrt{n}].$$

Hinc ergo erit

$$x+y\sqrt{n}=fP+gP\sqrt{n}+fQ\sqrt{n}+ngQ,\quad \text{ideoque}\quad x=fP+ngQ \quad \text{et}\quad y=gP+fQ,$$

quod ergo infinitis modis fieri potest, si modo fuerit $ff-ngg=1$, id quod semper praestari potest quoties fuerit numerus positivus. At si n fuerit numerus negativus, evidens est formulam $ff+ngg$, saltem pro f et integris, aliter unitati aequalem esse non posse, nisi sit $f=1$ et $g=0$.

p. 169. 171.

68.

(Lexell.)

Problema. Datis numeris m et n, item a et b, invenire x et y, ut fiat

$$maa-nbb=nxx-myy,\quad \text{sive}\quad m(aa+yy)=n(bb+xx).$$

Solutio. Ponatur $x=mpa+qb$ et $y=qa+npb$, unde fiet

$$maa-nbb=m(mnppaa-qqaa)+n(qqbb-mnppbb)=maa(mnpp-qq)-nbb(mnpp-qq).$$

Oportet ergo sit $mnpp-qq=1$. Manifestum ergo est hoc problema solutionem non admittere, nisi numeri m et n sint summae duorum quadratorum. Quoties autem fuerint tales, ope problematis Pelliani semper invenire licet numeros p et q, ut fiat $mnpp=qq+1$, sive $mnpp-1=\square$.

(J. A. Euler.)

Excipiuntur tamen casus, quibus vel mn est ipse numerus quadratus, vel in duo quadrata inter se prima resolvi nequit; cujusmodi sunt: 8, 18, 20, 32, 40, 45, etc. Sic si $mn=13$, erit $p=5$ et $q=18$; nam $13.5^2=18^2+1$, et si $mn=125$, erit $p=61$ et $q=682$, nam $125.61^2=682^2+1$ etsi $125=100+25$ quae non sint prima inter se, sed notandum est esse $125=11^2+2^2$, quae utique sunt prima.

A. m. T. I. p. 129.

69.

(N. Fuss I.)

Problema. Resolvere aequalitatem $ab(a+b)^2=cd(c+d)^2$.

Solutio. Ponatur $m(a+b)=n(c+d)$ fierique debet $nnab=mmcd$. Porro sit

$$a=mmps,\quad c=nnqs,\quad b=qrs,\quad d=prs,$$

unde prior aequatio erit $m^3p+mqr=n^3q+npr$, unde fit $r=\frac{n^3q-m^3p}{mq-np}$. Ut ergo fractiones evitentur, sumatur $s=mq-np$ eritque in numeris integris

$$a=mmp(mq-np),\quad b=q(n^3q-m^3p),\quad c=nnq(mq-np),\quad d=p(n^3q-m^3p).$$

Hinc enim fit

$$a+b=n(nnqq-mmpp),\quad c+d=m(nnqq-mmpp),$$

quae solutio est generalis. Notetur autem, si litterae a, b, c, d sint quadrata, veluti $a=A^2$, $b=B^2$, $c=C^2$, $d=D^2$, tum aequalitatem propositam accipere hanc formam

$$AB(AA+BB)=CD(CC+DD);$$

ad quam igitur solvendam illae quatuor formulae quadrata fieri debent. Primo ergo quadratum erit

$$\frac{a}{c}=\frac{mmp}{nnq},\quad \text{ideoque}\quad \frac{p}{q}=\square,\quad \text{sive}\quad pq=\square.$$

Praeterea vero quadratum esse debet

$$\frac{a}{b}=\frac{mmp\,(mq-np)}{q\,(n^3q-m^3p)},\quad \text{sive}\quad \frac{mq-np}{n^3q-m^3p}=\square,$$

hocque modo omnes erunt quadrata, unde eadem solutio prodit, quae supra est data.

Resolutio succincta aequalitatis $(aa+bb)\,ab=(cc+dd)\,cd$.

Sumtis pro m et n numeris quibuscunque capiatur $\frac{f}{g}=\frac{mm+nn}{mm-nn}$; tum vero sumatur

$$p=4f^3+fgg-3g^3,\quad q=4f^3+fgg+3g^3\quad \text{et}\quad s=4f^3-5fgg,$$

tum habebitur $a=mp$, $b=ns$, $c=ms$, $d=nq$. Veluti si sumatur $m=3$ et $n=1$, erit $\frac{f}{g}=\frac{5}{4}$, ideoque $f=5$, $g=4$, hinc $4ff+gg=116$, ergo $p=388$, $q=772$, $s=100$, seu $p=97$, $q=193$, $s=25$, unde fit $a=291$, $b=25$, $c=75$, $d=193$, hic scilicet numeros p, q, s per 4 deprimere licuit, quod semper evenit quando g numerus par.

Duo numeri a et b assignari possunt, ut fiat $10ab\,(aa+bb)=53$, quod utique in numeris integris fieri nequit. Hoc autem evenit sumendo $a=\frac{27}{10}$ et $b=\frac{4}{15}$, tum fit $ab=\frac{54}{75}$ et $10ab=\frac{540}{75}=\frac{36}{5}$; tum ob $a=\frac{81}{30}$ et $b=\frac{8}{30}$ erit $aa+bb=\frac{265}{30}$, ideoque $10ab\,(aa+bb)=53$.

Resolutio hujus formulae $ab\,(maa+nbb)=cd\,(mcc+ndd)$.

Posito $b=pc$ et $d=qa$, reperitur $\frac{aa}{cc}=\frac{mq-np^3}{mp-nq^3}$. Posito $p=q\,(1+z)$ et $\frac{a}{c}=1-5z$, porro $q=\frac{k}{h}$, ambo numeri h et k arbitrio relinquuntur. Tum sumatur $\frac{f}{g}=\frac{mhh+nkk}{mhh-nkk}$, eritque

$$a=h\,(4f^3-5fgg),\quad b=k\,(4f^3+fgg+3g^3),\quad c=h\,(4f^3+fgg-3g^3),\quad d=k\,(4f^3-5fgg).$$

Cum enim sit $\frac{mhh}{nkk}=\frac{f+g}{f-g}$, erit

$$maa+nbb=mn\,(2ff-gg)\,(4ff-3gg)\,(4f^3+fgg-3g^3)$$
$$mcc+ndd=mn\,(2ff-gg)\,(4ff-3gg)\,(4f^3+fgg+3g^3)$$

erit igitur

$$\frac{ab}{cd}=\frac{4f^3+fgg+3g^3}{4f^3+fgg-3g^3}\quad \text{et}\quad \frac{maa+nbb}{mcc+ndd}=\frac{4f^3+fgg-3g^3}{4f^3+fgg+3g^3}.$$

Ceterum hic patet, permutatis numeris h et k sumtoque g negativo, litteras a et b abire in d et c.

Alia Resolutio formulae $\frac{aa}{cc}=\frac{p^3-q}{q^3-p}$,

pro qua supra posuimus $q=p(1+z)$.

Nunc vero ponamus $q=nn(p-1)+p$, tum enim prodit

$$\frac{aa}{cc}=\frac{p^3-p-nn(p-1)}{n^6(p-1)^3+3n^4p(p-1)^2+3nnpp(p-1)+p^3-p},$$

ubi commode per $p-1$ dividitur prodibitque

$$\frac{aa}{cc}=\frac{pp+p-nn}{n^6(p-1)^2+3n^4p(p-1)+3nnpp+pp+p}.$$

Hoc modo habentur duae formulae ad quadratum reducendae scilicet

$$n^6pp+n^6p+n^8\quad \text{et}\quad n^6pp+3n^4pp+3nn+pp-(2n^6+3n^4-1)p+n^6.$$

Necesse ergo est, ut sit $(nn+1)^3=\square$, ideoque etiam $nn+1$. Sit igitur $nn+1=mm$ eritque

$$\frac{aa}{cc}=\frac{pp+p-mm+1}{m^6pp-2mm(mm-1)^2p+(mm-1)^3};$$

hujusmodi autem binae formulae supra sunt resolutae. Ita si sumatur $m=2$, ut sit $q=4p-3$, hinc reperitur $p=\frac{8299}{64.3.5.7}$. Hae autem solutiones diversae erunt ab iis, quas prior solutio suppeditaverat. Ceterum hic statim ab initio scribi debuisset $mm-1$ loco nn.

Solutio generalior. Loco p et q scribatur $\frac{p}{r}$ et $\frac{q}{r}$ et formula resolvenda erit $\frac{p^3-qrr}{q^3-prr}$. Jam ponatur $p=1+\alpha z$, $q=1+\beta z$ et $r=1+\gamma z$ et habebimus

$$\frac{3\alpha-\beta-2\gamma+(3\alpha\alpha-2\beta\gamma-\gamma\gamma)z+(\alpha^3-\beta\gamma\gamma)zz}{3\beta-\alpha-2\gamma+(3\beta\beta-2\alpha\gamma-\gamma\gamma)z+(\beta^3-\alpha\gamma\gamma)zz}=\square.$$

Hic igitur tantum opus est, ut fiat

$$\frac{3\alpha-\beta-2\gamma}{3\beta-\alpha-2\gamma}=\frac{ff}{gg},\quad\text{unde}\quad 2\gamma=\frac{(3\alpha-\beta)gg-(3\beta-\alpha)ff}{gg-ff}.$$

Hoc modo prodit $\frac{p}{r}=1433$ et $\frac{q}{r}=473$. Veluti si $\alpha=2$ et $\beta=1$, fit $\gamma=\frac{5gg-ff}{gg-ff}=\frac{4gg}{gg-ff}+1$. Ergo si $g=1$ et $f=2$ erit $2\gamma=-\frac{1}{3}$, ideoque $\gamma=-\frac{1}{6}$, unde fit

$$\frac{3.64+433z+287zz}{3.16+132z+34zz}=\square.$$

Ceterum hic nil impedit, quominus sumatur vel $\alpha=0$, vel $\beta=0$, vel $\gamma=0$; tantum sumi non debet $\beta=\alpha$. Quovis autem casu simplicissima solutio ita reperitur: Cum fiat $\frac{A+Bz+Czz}{a+bz+czz}=\square$, in qua aequatione $\frac{A}{a}$ per hypothesin $=\square$, ponatur hoc $\square=\frac{A}{a}$, indeque prodit z. Sequens solutio imprimis est memorabilis, sumendo $p=(1+nn)z$, $q=1+z$, ac per artificium modo memoratum reperitur

$$z=\frac{(nn+1)(n^4-3nn+1)}{3n^4},\quad\text{unde fit}\quad p=\frac{n^6-2n^4+nn+1}{3nn}\quad\text{et}\quad q=\frac{n^6+n^4-2nn+1}{3n^4},$$

unde pro solutione formulae $ab(aa+bb)=cd(cc+dd)$ statim habetur $a=3n^5$, $b=n^6-2n^4+nn+1$, $c=3n^2$, $d=n(n^6+n^4-2nn+1)$, quandoquidem posueramus $b=cp$, $d=aq$, hinc autem colligitur $\frac{aa}{cc}=n^6$, hincque $\frac{a}{c}=n^3$. Quodsi jam pro casu simplicissimo sumatur $n=2$, fit $a=96$, $b=37$, $c=12$, $d=146$ hincque erit

$$ab=2^5.3.37,\quad cd=2^3.3.73,\quad aa+bb=5.29.73,\quad cc+dd=4.5.29.37.$$

Theorema. Ex qualibet resolutione aequationis $ab(aa+bb)=cd(cc+dd)$ semper alia solutio deduci potest.

Demonstratio. Quia $ab(aa+bb)=cd(cc+dd)$, erit $(a+b)^4-(a-b)^4=(c+d)^4-(c-d)^4$, hinc $(a+b)^4-(c+d)^4=(a-b)^4-(c-d)^4$, seu

$$(a+b+c+d)(a+b-c-d)(\square+\square)=(a-b+c-d)(a-b-c+d)(\square+\square).$$

Quamobrem si ponamus $a'=a+b+c+d$ et $b'=a+b-c-d$; dein etiam

$$c'=a+b-c-d\quad\text{et}\quad d'=a-b-c+d$$

erit $a'b'(a'a'+b'b')=c'd'(c'c'+d'd')$. Quia igitur erat $a=291$, $b=25$, $c=75$, $d=193$, erit $a'=584$, $b'=48$, $c'=384$, $d'=148$, qui per 4 depressi dant

$$a'=146,\quad b'=12,\quad c'=96,\quad d'=37,$$

quae est solutio posterior minima.

Formulae sat concinnae

pro resolutione formulae $ab(aa+bb)=cd(cc+dd)$

Sumtis pro lubitu binis quadratis ff et gg, capiatur $\frac{\alpha}{\beta}=\frac{3ff+gg}{3gg+ff}$, erit

$$a=f(\alpha+\beta)(\alpha\alpha-3\alpha\beta+\beta\beta)$$
$$b=g(\beta^3-5\alpha\beta\beta+4\alpha\alpha\beta-2\alpha^3)$$
$$c=g(\alpha+\beta)(\alpha\alpha-3\alpha\beta+\beta\beta)$$
$$d=f(\alpha^3-5\alpha\alpha\beta+4\alpha\beta\beta-2\beta^3)$$

vel si ponatur $(\alpha+\beta)(\alpha\alpha-3\alpha\beta+\beta\beta)=\Delta$, erit

$$a=f\Delta,\quad b=g(\Delta-3\alpha(\alpha-\beta)^2),\quad c=g\Delta,\quad d=f(\Delta-3\beta(\alpha-\beta)^2).$$

Pro numeris α et β construatur haec tabula:

f	g	α	β
2	1	13	7
3	1	7	3
3	2	31	21
4	1	49	19
4	3	57	43
5	1	19	7
5	2	79	37
5	3	21	13
5	4	91	73

Hinc si $f=3$ et $g=1$, erit $\alpha=7$ et $\beta=3$, hincque $\Delta=-50$, unde $a=150$, $b=386$, $c=50$, $d=582$, sive per 2 deprimendo $a=75$, $b=193$, $c=25$, $d=291$. Sit $f=5$ et $g=1$, erit $\alpha=19$ et $\beta=7$, unde $\Delta=286$, $a=1430$, $b=7922$, $c=286$, $d=13690$, sive $a=715$, $b=3961$, $c=143$, $d=6845$. Hinc per theorema alii reperiuntur hoc modo:

$$a'=2966,\quad b'=578,\quad c'=2487,\quad d'=864.$$

Problema Diophanteum. Cognito uno casu, quo haec formula $\frac{A+Bz+Czz+Dz^3}{E+Fz}$ fit quadratum, ex eo alium casum derivare.

Solutio. Ponamus esse casu $z=e$, $\frac{A+Be+Cee+De^3}{E+Fe}=kk$, tum sumatur

$$s=\frac{B+2Ce+3Dee-Fkk}{2k(E+Fe)},\quad \text{quo facto alius casus erit}\quad z=\frac{C+2De-Ess-2Fks}{Fss-D};$$

tum enim erit $$\frac{A+Bz+Czz+Dz^3}{E+Fz}=(k-es+sz)^2.$$

Hoc ergo modo ex unico casu innumerabiles alii successive deduci possunt.

Analysis. Ponatur $\frac{A+Bz+Czz+Dz^3}{E+Fz}=(k+s(z-e))^2$ et facta evolutione erit

$$A+Bz+Czz+Dz^3=Ekk+2kEs(z-e)+Ess(z-e)^2+Fkkz+2Fksz(z-e)+Fssz(z-e)^2,$$

hinc subtrahatur aequatio $A+Be+Cee+De^3=Ekk+Fekk$ et dividendo per $z-e$ prodit

$$B+C(z+e)+D(zz+ez+ee)=2Eks+Fkk+Ess(z-e)+2Fksz+Fssz(z-e).$$

Jam ponatur $z=e+\nu$, unde terminis ad eandem partem translatis erit

$$\left.\begin{array}{l} B + 2Ce + 3Dee - 2Eks - Fkk - 2Feks \\ + C\nu + 3De\nu - Ess\nu - Fsse\nu - 2Fks\nu \\ + D\nu\nu - Fss\nu\nu \end{array}\right\} = 0.$$

Nunc littera s ita determinetur, ut prima linea evanescat, hoc est ponendo

$$s = \frac{B + 2Ce + 3Dee - Fkk}{2k(E + Fe)}$$

tum dividendo per ν reperietur

$$\nu = \frac{C + 3De - Ess - Fsse - 2Fks}{Fss - D}, \quad \text{hincque} \quad z = \frac{C + 2De - Ess - 2Fks}{Fss - D},$$

tum igitur erit $\dfrac{A + Bz + Czz + Dz^3}{E + Fz} = (k - es + sz)^2.$

Aliud problema. Ex cognito casu, quo $\dfrac{A + Bz + Czz}{D + Ez + Fzz}$ fit quadratum, invenire alium casum, quo idem obtineatur.

Solutio. Sit $z = e$ casus ille cognitus, quo fiat $\dfrac{A + Be + Cee}{D + Ee + Fee} = kk$ et ponatur $\dfrac{A + Bz + Czz}{D + Ez + Fzz} = kk$, ut sit

$$A + Bz + Czz = Dkk + Ekkz + Fkkzz \quad \text{hinc subtrahatur aequatio}$$

$$A + Be + Cee = Dkk + Eekk + Feekk$$

et facta divisione per $z - e$ prodibit

$$B + C(z + e) = Ekk + Fkk(z + e), \quad \text{unde statim elicitur} \quad z = \frac{Ekk + Fkke - B - Ce}{C - Fkk}.$$

Verum hoc modo unicus alius valor reperitur, namque ex invento iterum pristinus valor prodiret.

A. m. T. II. p. 157—161.

Theorema. Hae duae formulae $ab(aa + bb)$ et $2cd(cc - dd)$ ita inter se conveniunt respectu factorum non quadratorum, ut altera in alteram transformari possit.

Prior enim in posteriorem transmutatur ponendo $a = 2cd$ et $b = cc - dd$. Vicissim autem posterior in priorem transmutatur ponendo $c = aa + bb$ et $d = 2ab$; tum enim fit

$$2cd(cc - dd) = 4ab(aa + bb)^3.$$

Tres numeri formae $xy(xx - yy)$ infinitis modis dari possunt, qui inter se prorsus sint aequales, scilicet

I. $x = ff + 3gg$ et $y = 4fg$; II. $x = ff + 3gg$ et $y = 3gg - ff + 2fg$;

III. $x = ff + 3gg$ et $y = 3gg - ff - 2fg$.

His numeris resolvitur problema, *quo quaeruntur tria triangula rectangula, quorum areae sint inter se aequales.*

Nam in triangulo ABC sumtis cathetis $AC = 2xy$ et $BC = xx - yy$, erit hypotenusa $AB = xx + yy$, area $= xy(xx - yy)$. Ex prima igitur forma fit area

$$4fg(ff + 3gg)(f - g)(f - 3g)(f + g)(f + 3g).$$

Ex secunda et tertia idem. Ratio investigationis haec est:

Ponatur $pr(p + r)(p - r) = qs(q + s)(q - s)$ et sumatur $q = p$, erit $r(pp - rr) = s(pp - ss)$, unde fit

$$pp = rr + rs + ss, \quad \text{sive} \quad pp - \frac{3}{4}ss = (r + \frac{1}{2}s)^2.$$

Erit ergo $2r + s = \sqrt{(4pp - 3ss)} = 2p - \frac{fs}{g}$, unde $\frac{p}{s} = \frac{ff + 3gg}{4fg}$.

Sumatur ergo $p = q = ff + 3gg$ et $s = 4fg$ eritque

$$2r + s = 2r + 4fg = \pm 2(ff + 3gg) - 4ff, \text{ hincque vel } r = 3gg - ff - 2fg, \text{ vel } r = 3(gg - ff) - 2fg.$$

Varii numeri formae $xy(xx - yy)$, qui eosdem factores non quadratos involvunt, sub littera F contentos, in hac tabella exhibuntur:

F	x	y
2.3	2	1
	25	24
2.3.5.7	5	2
	6	1
	8	7
2.3.5.11	6	5
	8	3
	27	22
2.7.11	9	2
	18	7
3.5.7.11	11	4
	16	5

Conjectura. Posito $xy(xx - yy) = A^2F$, inter numeros F videntur omnes numeri primi, vel ipsi, vel eorum dupla, vel ambo interdum occurrere, excepto scilicet binario, uti ex hac tabula colligere licet:

F	x	y
2.3	2	1
5	5	4
7	16	9
2.7	9	7
2.11	50	49
13	325	36
2.17	9	8
2.19	1250	289
23	156^2	133^2
2.23	121	23
29	29.13^2	70^2
31	1600	81

Problema. Invenire numeros p, q, x, y, ut sit $pq(pp - qq) = nxy(xx - yy)\,\square$, pro quolibet numero dato n.

Solutio tantum particularis tradi potest, et calculis satis molestis expeditis inveni sequentes valores

$$p = s^4 - 20nsstt - 8nnt^4 \qquad p + q = 2(ss - ntt)(ss - 4ntt)$$
$$q = (ss + 2ntt)(ss + 8ntt) \qquad p - q = 6ntt(5ss + 4ntt)$$
$$x = s^4 - 20nsstt - 8nnt^4 \qquad x + y = (5ss + 4ntt)(ss - 4ntt)$$
$$y = 4(ss - ntt)(ss + 2ntt) \qquad x - y = 3ss(ss + 8ntt).$$

Hic enim rejectis factoribus quadratis formula $xy(xx - yy)$ omnes continet factores alterius $pq(pp - qq)$, ac praeterea factorem n haec posterior continebit. Notandum hic est numerum n tam positive quam negative accipi posse. Deinde etiam valores singularum harum litterarum semper positivi capi possunt, etiamsi prodeant negativi.

Ita sumto $s=1$ et $t=1$ pro casu $n=2$ erit $p=71$, $q=85$, $x=71$, $y=20$. Hic loco p et q eorum semisumma et semidifferentia sumi possunt fietque $p=2.3.13$ et $q=7$. Hinc enim fiet

$$pq(pp-qq)=2.3.5.7.13.17.71 \quad \text{et} \quad xy(xx-yy)=3.5.7.13.17.71.$$

Sin autem sumatur $n=-2$ manente $s=t=1$, prodit $p=9$, $q=5.9$, sive $p=1$ et $q=5$, sive etiam $p=3$ et $q=2$, tum vero erit $x=9$, $y=4.9$, sive $x=1$ et $y=4$, tum enim erit

$$pq(pp-qq)=2.3.5 \quad \text{et} \quad xy(xx-yy)=3.5.$$

A. m. T. III. p. 121—123.

b) *Quaestiones ad resolutionem plurium aequationum ducentes.*

70.

(*W. L. Krafft.*)

Problema. Efficere, ut fiat $x+y+z=\square$ et $xyz=1$, ideoque $z=\frac{1}{xy}$.

Sequentes solutiones particulares prodierunt

$$\text{I.}\quad x=\frac{1}{2},\qquad y=\frac{9}{2},\qquad z=\frac{4}{9}$$

$$\text{II.}\quad x=\frac{2}{9},\qquad y=\frac{50}{9},\qquad z=\frac{81}{100}$$

$$\text{III.}\quad x=\frac{13}{392},\qquad y=\frac{117}{392},\qquad z=\frac{392^2}{9.13^2}$$

$$\text{IV.}\quad x=\frac{2}{5},\qquad y=\frac{18}{5},\qquad z=\frac{25}{36}$$

$$\text{V.}\quad x=-2,\qquad y=-\frac{2}{9},\qquad z=\frac{9}{4}$$

$$\text{VI.}\quad x=\frac{1}{6},\qquad y=\frac{1}{12},\qquad z=72.$$

Si ratio x ad y sumatur $a:b$, et ponatur $x=ma$ et $y=mb$, erit $z=\frac{1}{mmab}$, unde fieri debet

$$m(a+b)+\frac{1}{mmab}=\square,\quad \text{sive}\quad m^3(a+b)+\frac{1}{ab}=\square,$$

$$\text{seu}\qquad m^3aabb(a+b)+ab=\square.$$

A. m. T. I. p. 121.

71.

(*Lexell.*)

Ut formulae $aa+Mbb$ et $aa+Nbb$ quadrata reddi queant, posito $M=m\mu$ capiatur $N=(\alpha pp+m)(\alpha qq+\mu)$ sicque pro dato numero M infiniti valores idonei pro N reperiuntur, veluti si $M=1$, ideoque $m=\mu=\pm 1$ erit $N=(\alpha pp\pm 1)(\alpha qq\pm 1)$. Si $p=1$, erit $N=(\alpha\pm 1)(\alpha qq\pm 1)$, cujusmodi formulae sunt

$$\begin{array}{lll}
\text{pro } \alpha=1, & N=2(qq+1) & \\
\alpha=2, & N=3(2qq+1); & 2qq-1 \\
\alpha=3, & N=4(3qq+1); & 2(3qq-1) \\
\alpha=4, & N=5(4qq+1); & 3(4qq-1).
\end{array}$$

Ut formulae $aa + mmbb$ et $aa + nnbb$ reddi possint quadrata, numeros m et n ex talibus formis sumi oportet, (ubi quidem ratio inter m et n est definienda)

$$pq(rr-1),\quad pr(qq-1),\quad qr(pp-1),\quad p(qq-rr),\quad q(pp-rr),\quad r(pp-qq)$$
$$p(qqrr-1),\quad q(pprr-1),\quad r(ppqq-1),\quad ppqq-rr,\quad pprr-qq,\quad qqrr-pp,$$
$$ppqqrr-1,$$

quae formulae omnes ita sunt comparatae, ut si earum quadratis addatur idem quadratum $4ppqqrr$, proveniant quadrata.

A. m. T. I. p. 122.

Problema. Dato numero A, invenire conditiones numeri N, ut ambae istae formulae $xx + Ayy$, $xx + Nyy$ simul fieri possint quadrata.

Solutio. Ponatur $A = \mu\nu$, pro casu scilicet, quo habet factores, et priori formae satisfiet sumendo $x = \mu pp - \nu qq$ et $y = 2pq$, tum enim erit $xx + Ayy = (\mu pp + \nu qq)^2$. Simul vero etiam altera formula evadet quadratum, si fuerit $N = mmppqq + m(\mu pp - \nu qq)$, tum enim erit

$$xx + Nyy = (\mu pp - \nu qq)^2 + 4mppqq(\mu pp - \nu qq) + 4mmp^4q^4 = (\mu pp - \nu qq + 2mppqq)^2,$$

erit ergo $N = mmppqq + m(\mu pp - \nu qq)$, ubi m pro lubitu assumere licet. Quin etiam pro m fractiones assumere licet, ita ut pro N nihilominus prodeant numeri integri, sumto enim $m = n + \frac{\nu}{pp}$, tum enim fiet

$$N = \left(n + \frac{\nu}{pp}\right)(nppqq + \mu pp) = (npp + \nu)(nqq + \mu).$$

A. m. T. I. p. 128.

72.

(N. Fuss I.)

Problema Diophanteum. Invenire numerum x ut his duabus conditionibus satisfiat

$$xx + 2ax + mmc = \square \quad \text{et} \quad xx + 2bx + nnc = \square,$$

cujus solutio particularis est

$$x = \frac{(nna - mmb)^2 - mmnn(m-n)^2 c}{2mn(m-n)(na - mb)}.$$

Ita si proponantur hae duae formulae: $xx + 2ax + c = \square$ et $xx + 2bx + c = \square$, sumatur $m = 1$ et $n = -1$

$$\text{eritque } x = \frac{(a-b)^2 - 4c}{4(a+b)}.$$

A. m. T. II. p. 154.

73.

Problema. Resolvere has duas aequalitates:

$$(2 + 2a)xx + (2 - 2a)yy = 4AA \quad \text{et} \quad (2 + 2b)xx + (2 - 2b)yy = 4BB.$$

Solutio. Hinc ergo primo erit $4A^2 + 4B^2 = (4 + 2(a+b))xx + (4 - 2(a+b))yy$; posito ergo

$$a + b = 2c, \quad \text{erit} \quad A^2 + B^2 = (1 + c)xx + (1 - c)yy.$$

Deinde vero erit $4A^2 - 4B^2 = 2(a-b)(xx - yy)$, unde posito $a - b = 2d$, erit $A^2 - B^2 = d(xx - yy)$. Statuatur ergo $A + B = x + y$ eritque $A - B = d(x - y)$. Addantur quadrata eritque

$$2A^2 + 2B^2 = (x+y)^2 + dd(x-y)^2 = 2(1+c)xx + 2(1-c)yy,$$

quae evoluta fit $(1 + dd)xx + (1 - dd)2xy + (1 + dd)yy = 2(1+c)xx + 2(1-c)yy$, sive

$$(dd - 2c - 1)xx + 2(1 - dd)xy + (dd + 2c - 1)yy = 0,$$

$$\text{sive} \quad (dd - 1)(xx - 2xy + yy) - 2c(xx - yy) = 0,$$

quae commode per $x-y$ divisa fit $(dd-1)(x-y)-2c(x+y)=0$, unde reperitur

$$\frac{x}{y}=\frac{dd+2c-1}{dd-2c-1}.$$

Sumatur ergo $x=dd+2c-1$ et $y=dd-2c-1$, fietque

$$A+B=2(dd-1) \quad \text{et} \quad A-B=4cd, \quad \text{ergo} \quad A=dd+2cd-1 \quad \text{et} \quad B=dd-2cd-1.$$

Haec autem solutio tantum est particularis.

Problema. Resolvere has duas aequalitates:

$$xx+2axy+nccyy=A^2 \quad \text{et} \quad xx+2bxy+nddyy=B^2.$$

Solutio. Eliminetur littera n, quaerendo $A^2dd-B^2cc=xx(dd-cc)+2xy(add-bcc)$. Ponatur nunc $Ad+Bc=x(d+c)$, eritque $Ad-Bc=x(d-c)+2y\left(\frac{add-bcc}{d+c}\right)$, unde erit $2Ad=2dx+2y\cdot\frac{add-bcc}{d+c}$, unde fit $A=x+y\cdot\frac{add-bcc}{d(d+c)}$, quo valore substituto erit

$$2ax+nccy=2x\cdot\frac{add-bcc}{d(d+c)}+y\left(\frac{add-bcc}{d(d+c)}\right)^2,$$

$$\text{unde fit} \quad \frac{x}{y}=\frac{(add-bcc)^2-nccdd(d+c)^2}{2cd(d+c)(ad+bc)}.$$

A. m. T. II. p. 157.

74.

Resolutio aequationum

$$xx+2axy+cyy=A^2 \quad \text{et} \quad xx+2bxy+dyy=B^2.$$

Cum sit $A^2-B^2=2(a-b)xy+(c-d)yy$, sumatur $A-B=(a-b)y$, erit

$$A+B=2x+\frac{c-d}{a-b}y, \quad \text{hinc additis quadratis erit}$$

$$2(AA+BB)=4xx+4\cdot\frac{c-d}{a-b}xy+\left(\frac{c-d}{a-b}\right)^2yy$$
$$+(a-b)^2yy.$$

Cum igitur $2(AA+BB)=4xx+4(a+b)xy+2(c+d)yy$, inde sequitur

$$4\cdot\frac{c-d}{a-b}x-4(a+b)x+\left[\left(\frac{c-d}{a-b}\right)^2+(a-b)^2-2(c+d)\right]y=0, \qquad \text{hincque fit}$$

$$\frac{x}{y}=\frac{(c-d)^2+(a-b)^4-2(c+d)(a-b)^2}{4(a-b)(aa-bb-c+d)}.$$

Sumi ergo poterit $x=(c-d)^2+(a-b)^4-2(c+d)(a-b)^2$ et $y=4(a-b)(aa-bb-c+d)$. Haec solutio differt ab ea, quae supra est tradita, ubi loco c et d habuimus ncc et ndd, quod mirum non est, cum utraque solutio tantum sit particularis.

Eodem etiam modo hae aequalitates resolvi possunt

$$(2+a)xx+(nn-4)xy+(2-a)yy=A^2 \quad \text{et} \quad (2+b)xx+(nn-4)xy+(2-b)yy=B^2.$$

Facta enim simili operatione, dividi poterit per $x-y$, ac reperietur inde

$$x=n^4-8nn+(a-b)^2+2nn(a+b) \quad \text{et} \quad y=n^4-8nn+(a-b)^2-2nn(a+b).$$

In his autem formulis continetur casus supra tractatus, quando $n=2$.

A. m. T. II. p. 158.

75.

Problema. Invenire quatuor numeros positivos x, y, z, ν, inter se primos, quorum tam summa quam summa quadratorum sit biquadratum.

Solutio. Positis $x = aa + bb + cc - dd$, $y = 2ad$, $z = 2bd$, $\nu = 2cd$, erit

$$xx + yy + zz + \nu\nu = (aa + bb + cc + dd)^2.$$

Ut vero fiat biquadratum, sumatur $a = pp + qq + rr - ss$, $b = 2ps$, $c = 2qs$, $d = 2rs$ eritque

$$aa + bb + cc + dd = (pp + qq + rr + ss)^2.$$

Ut vero etiam ipsa summa $x + y + z + \nu$ fiat quadratum, hoc fiet sumendo $p = s + \frac{3}{2}r - q$. Hoc enim modo erit $\sqrt{(x + y + z + \nu)} = 2qq - 3qr - 2qs + \frac{13}{4}rr + 5rs + 2ss$. Quae quantitas ut denuo fiat quadratum, posito $q = r + t$ reperitur $r = \frac{uu - 2tt + 2ts - 2ss}{t + 3s + 3u}$; hocque modo problemati satisfiet. Hinc sequens exemplum

$p = 10$	$a = 3$	$x = 409$	$x^2 = 167281$	$x + y + z + \nu = 625 = 5^4$
$q = 2$	$b = 20$	$y = 24$	$y^2 = 576$	$xx + yy + zz + \nu\nu = 21^4$.
$r = 2$	$c = 4$	$z = 160$	$z^2 = 25600$	
$s = 9$	$d = 4$	$\nu = 32$	$\nu^2 = 1024$	

Problema. Invenire quinque numeros positivos et inter se primos x, y, z, ν, u, quorum tam summa quam summa quadratorum sit biquadratum.

Solutio. Statuatur $x = aa + bb + cc + dd - ee$, $y = 2ae$, $z = 2be$, $\nu = 2ce$, $u = 2de$ eritque summa

$$x^2 + y^2 + z^2 + \nu^2 + u^2 = (a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

Praeterea capiatur $a = pp + qq + rr + ss - tt$, $b = 2pt$, $c = 2qt$, $d = 2rt$, $e = 2st$; hocque modo fiet

$$x^2 + y^2 + z^2 + \nu^2 + u^2 = (p^2 + q^2 + r^2 + s^2 + t^2)^2.$$

Jam ut etiam ipsa summa fiat quadratum, sumi debet $p = -q - r + \frac{3}{2}s + t$ atque radix summae erit

$$pp + qq + rr + ss + tt + 2st,$$

quae denuo evadet quadratum posito $r = q + s + g$, si capiatur

$$s = \frac{6qq - 4qt + tt + 6gq - 2gt - 2ft + 2gg - ff}{3f - g},$$

ubi quatuor litterae q, t, f, g nostro arbitrio relinquuntur, quas facile ita accipere licet, ut numeri quaesiti fiant positivi.

A. m. T. III. p. 125. 126.

76.

Problema. Invenire duos numeros positivos et inter se primos, x et y, quorum summa sit quadratum, summa autem quadratorum cubus.

Tales numeri simpliciores sunt $x = 29601 = 9.11.13.23$, $y = 25624 = 8.3203$, unde $x + y = 235^2$, $xx + yy = 1153^3$.

Analysis. Cum debeat esse $xx + yy = p^3$, necesse est, ut sit $p = aa + bb$; tum vero evadit $x = a(aa - 3bb)$ et $y = b(3aa - bb)$. Hinc autem erit $x + y = a^3 + 3aab - 3abb - b^3 = (a - b)(aa + 4ab + bb) = \square$. Fiat igitur $a - b = cc$ eritque $x + y = cc(6bb + 6bcc + c^4) = \square$. Sit igitur $c^4 + 6bcc + 6bb = \left(cc + 3b\frac{f}{g}\right)^2$, unde haec relatio

$$\frac{b}{cc} = \frac{2g(f - g)}{2gg - 3ff} = \frac{2g(g - f)}{3ff - 2gg}.$$

Quia numeri x et y debent esse positivi, necesse est, ut sit $aa > 3bb$, sive $a > b\sqrt{3}$, ergo $b + cc > b\sqrt{3}$, sive $cc > b(\sqrt{3}-1)$, ideoque $\frac{b}{cc} < \frac{1}{\sqrt{3}-1} < \frac{\sqrt{3}+1}{2}$. Ponatur igitur

$$b = 2g(g-f) \quad \text{et} \quad cc = 3ff - 2gg, \quad \text{ergo} \quad 3ff - 2gg = \square.$$

Huic satisfit sumendo $f = 11$ et $g = 1$, vel etiam sumto $f = -3$ et $g = 1$, unde fit $c = 5$, $b = 8$, $a = 33$, hinc $x = 29601$ et $y = 25624$.

Si quaerantur tres numeri x, y, z, positivi et primi inter se, quorum summa sit quadratum, quadratorum vero summa cubus, tales numeri sunt

$$35 + 9 + 5 = 7^2 \quad \text{et} \quad 35^2 + 9^2 + 5^2 = 11^3;$$

Similî modo etiam $\quad 67 + 9 + 5 = 9^2 \quad \text{et} \quad 67^2 + 9^2 + 5^2 = 19^3;$

at vero methodus tales numeros inveniendi adhuc latet.

A. m. T. III. p. 128.

77.

Problema. Has duas formulas $xx + abyy$ et $xx + cdyy$ ad quadratum reducere.

Solutio. Pro priore ponatur $x = \zeta(app - bqq)$, erit $y = 2\zeta pq$; pro altera ponatur $x = \eta(crr - dss)$, eritque $y = 2\eta rs$. Ponatur igitur $\zeta pq = \eta rs = \zeta\eta fghk$, fiatque $p = \eta fg$, erit $q = hk$ et $r = \zeta fh$ et $s = gk$. Qui valores pro x substituti praebent

$$\frac{gg}{hh} = \frac{\zeta}{\eta} \cdot \frac{\zeta\eta cff + bkk}{\zeta\eta aff + dkk}. \text{ Quadratum ergo esse debet } \zeta\eta(\zeta\eta cff + bkk)(\zeta\eta aff + dkk),$$

vel posito $\zeta\eta = \vartheta$, quadratum esse debet $\vartheta(\vartheta cff + bkk)(\vartheta aff + dkk)$, vel loco ϑ scribamus $\frac{\mu}{\nu}$, fieri debet $\mu\nu(\mu cff + \nu bkk)(\mu aff + \nu dkk)$, quod si reddi queat quadratum, tunc etiam ambae formulae propositae fient quadrata. Ubi scilicet litterae f, k, μ, ν pro lubitu accipi possunt.

A. m. T. III. p. 136.

Ut hae duae formulae $xx + nyy$ et $yy + nxx$ simul quadrata reddi queant, necesse est, ut numerus n in sequenti formula contineatur:

$$n = \frac{(s + xx - yy)(s - xx + yy)(s + xx + yy)(s - xx - yy)}{4ssxxyy}.$$

Quodsi fuerit $n = aa + 3$, tum ambae formulae quadrata reddi possunt; tum enim sumto $x = a + 1$ et $y = a - 1$ fiet $nxx + yy = (aa + a + 2)^2$ et $nyy + xx = (aa - a + 2)^2$. Hinc solus casus $a = 1$ excipitur; tum enim ob $n = 4$ foret $y = 0$. Praeterea vero innumeri alii dantur casus pro n, pro quibus inveniendis quaeratur numerus $c = \frac{xx + yy}{2d} \cdot \frac{dd - 1}{xy}$, tum enim semper erit $n = cc - dd + 1$.

Theorema. Hae duae formulae $xx + nyy$ et $yy + nxx$ simul quadrata reddi nequeunt, nisi $n + 1$ sit summa duorum quadratorum.

Demonstratio. Posito $xx + nyy = pp$ et $yy + nxx = qq$ erit $pp + qq = (n + 1)(xx + yy)$. Constat autem summam duorum quadratorum $pp + qq$ alios divisores non continere posse, nisi qui ipsi sint summae duorum quadratorum. Quoties ergo $n + 1$ non fuerit summa duorum quadratorum, quod fit his numeris pro n assumtis:

2, 5, 6, 8, 10, 11, 13, 14, 17, 18, 20, 21, 22, 23, 26, etc.

ambae formulae simul quadrata fieri nequeunt.

Cum his formulis $7xx + yy = pp$ et $7yy + xx = qq$ satisfiat sumendo $x = 3$ et $y = 1$, unde fit $p = 8$ et $q = 4$, innumerabiles alii dabuntur valores pro x et y, ex quibus magno labore hos eruimus: $x = 1121$ et $y = 477$, unde fit $p = 3004$ et $q = 1688$, uti ex hoc schemate apparet:

$$\begin{array}{rlcrl} xx = & 3256641 & \qquad & 7xx = & 8796487 \\ 7yy = & \underline{1592703} & & yy = & \underline{227529} \\ qq = & 2849344 & & pp = & 9024016 \\ q = & 1688 & & p = & 3004. \end{array}$$

A. m. T. III. p. 146.

77.

Problema. Invenire quatuor quadrata xx, yy, zz, $\nu\nu$, ut sit

$$xxyy - zz\nu\nu = A^2, \quad xxzz - yy\nu\nu = B^2, \quad yyzz - xx\nu\nu = C^2.$$

Solutio. Quaeratur formula F, quae addita producat quadrata. Talis est

$$F = x^4 + y^4 + z^4 - 2xxyy - 2xxzz - 2yyzz + 2xx\nu\nu + 2yy\nu\nu + 2zz\nu\nu + \nu^4,$$

cui si addatur $4xxyy - 4zz\nu\nu$ prodit quadratum $(xx + yy - zz + \nu\nu)^2$.

Si addatur $4xxzz - 4yy\nu\nu$ oritur $(xx + zz - yy + \nu\nu)^2$
et addito $4yyzz - 4xx\nu\nu$ prodit $(yy + zz - xx + \nu\nu)^2$.

Hinc ergo facto $F = 0$, erit

$$A = \frac{xx + yy - zz + \nu\nu}{2}, \quad B = \frac{xx + zz - yy + \nu\nu}{2}, \quad C = \frac{yy + zz - xx + \nu\nu}{2}.$$

Fiat igitur $F = 0$, et per extractionem quaeratur $\nu\nu$, eritque

$$\nu\nu = -xx - yy - zz + 2\sqrt{(xxyy + xxzz + yyzz)}.$$

Ponatur ergo $\sqrt{(xxyy + xxzz + yyzz)} = S$, ut fiat $\nu\nu = 2S - xx - yy - zz$. Fingatur autem $S = xy + tz$, ita ut $SS = xxyy + 2xytz + ttzz = xxyy + xxzz + yyzz$, unde fit

$$z = \frac{2txy}{xx + yy - tt}, \quad \text{hincque} \quad S = \frac{xy\,(xx + yy + tt)}{xx + yy - tt}.$$

Verum hi valores substituti pro $\nu\nu$ dant expressionem inextricabilem. Fieret enim sex dimensionum. Necesse ergo est, ut rem ad formulas simpliciores reducamus.

Casus I. Sumatur $t = x - y$ fietque $z = x - y$ et $S = xx - xy + yy$, hincque porro $\nu\nu = 0$, qui ergo casus prorsus est inutilis.

Casus II. Sumatur $t = y$ fietque $z = \frac{2yy}{x}$ et $S = \frac{xxy + 2y^3}{x}$.

Alius conatus. Ex aequatione $F = 0$ quaeratur valor ipsius xx, qui est

$$xx = yy + zz - \nu\nu \pm 2\sqrt{(yyzz - yy\nu\nu - zz\nu\nu)} = 2S + yy + zz - \nu\nu.$$

Quia hic yy multiplicatur per $zz - \nu\nu$, pro z et ν tales valores sumantur, ut $zz - \nu\nu$ fiat quadratum, quod fit sumendo $z = 5$ et $\nu = 3$; tum erit $S = \sqrt{(16yy - 225)} = 4y - t$, unde $y = \frac{225 + tt}{8t}$ et $S = \frac{225 - tt}{2t}$. Tum igitur erit $xx = yy + 16 \pm 2S$. Sumatur $t = 5$, erit $y = \frac{25}{4}$ et $S = 20$, ergo $xx = \frac{625}{16} + 16 \pm 40 = \frac{39^2}{4^2}$, ergo

$x=\frac{39}{4}$, $y=\frac{25}{4}$, $z=5$ et $v=3$, sive $x=39$, $y=25$, $z=20$, $v=12$. Tentemus etiam casum $t=3$, erit $y=\frac{39}{4}$ et $S=36$, ergo $xx=\frac{625}{16}$, ergo $x=\frac{25}{4}$, unde prodit casus praecedens.

Possemus etiam sumere $z=5$ et $v=4$: erit $S=\sqrt{(9yy-400)}=3y-t$, hinc

$$y=\frac{400+tt}{6t} \quad \text{et} \quad S=\frac{400-tt}{2t}, \quad \text{hinc} \quad xx=yy+9\pm 2S.$$

Sumatur $t=10$, erit $y=\frac{25}{3}$ et $S=15$, hinc $xx=$ incongruo.

Ex priore solutione $vv=2S-xx-yy-zz$ existente $z=\frac{2txy}{xx+yy-tt}$, $S=\frac{xy\,(x^2+y^2+t^2)}{x^2+y^2-t^2}$. Sumatur $x=5$ et $y=4$, fiet $z=\frac{40t}{41-tt}$ et $S=\frac{20\,(41+tt)}{41-tt}$. Porro si $t=\frac{13}{3}$, solutio supra data oritur, ex quo casu derivavi sequentem:

$$t=\frac{185}{153}, \quad \text{eritque} \quad z=\frac{5.185.153}{115693} \quad \text{et} \quad S=\frac{5.496997}{115693}.$$

A. m. T. III. p. 117. 118.

78.

Problème. Trouver trois nombres x, y, z tels, que le carré de chacun avec le produit des deux autres fasse un carré.

Solution. Qu'on pose, pour les deux premières conditions $xx+yz=pp$ et $yy+xz=qq$, et l'on aura $pp-qq=(x-y)(x+y-z)$. Soit donc $p-q=x-y$ et $p+q=x+y-z$, d'où l'on tire $p=x-\frac{1}{2}z$. Cette valeur substituée dans la première équation donne $z=4(x+y)$. Maintenant la troisième équation sera

$$16\,(x+y)^2+xy=\square.$$

Donc la racine sera plus grande que $4(x+y)$. Soit cette racine

$$4x+4y+s, \quad \text{et il y aura} \quad 8sx+8sy-xy=-ss.$$

Ajoutons de part et d'autre $-64ss$, pour avoir

$$(x-8s)(y-8s)=65ss=\frac{5ts}{u}\cdot\frac{13us}{t}.$$

Soit $x-8s=\frac{5ts}{u}$ et $y-8s=\frac{13us}{t}$, et pour ôter les fractions, supposons $s=tu$, et l'on aura $x=8tu+5tt$ et $y=8tu+13uu$. De là $z=4(x+y)=64tu+20tt+52uu$.

Exemple. Soit $t=1$ et $u=1$, et il y aura $x=13$, $y=21$, $z=136$, car alors

$$136^2+13.21=137^2, \quad 21^2+13.136=47^2, \quad 13^2+21.136=55^2.$$

Autre solution. Pour la première formule qu'on prenne $x=\frac{yz-ss}{2s}$, pour avoir $xx+yz=\left(\frac{yz+ss}{2s}\right)^2$. La seconde $yy+xz=\square$ donne $\frac{2syy+yzz-ssz}{2s}=\square=(y+pz)^2$, d'où l'on tire

$$y=\frac{2ppsz+ss}{z-4ps}, \quad \text{et de là} \quad x=\frac{p\,(pzz+2ss)}{z-4ps}.$$

Ces valeurs étant substituées dans la troisième équation $zz+xy=\square$, celle-ci deviendra

$$z^4+(2p^4-8p)\,sz^3+17ppsszz+4p^3s^3z+2ps^4=\square.$$

Pour rendre carré le dernier terme, je prends $p=2$, et la formule sera

$$z^4+16sz^3+68sszz+32s^3z+4s^4=\square.$$

Mais on s'aperçoit d'abord, qu'elle est carrée et que sa racine est $zz+8sz+2ss$; par conséquent z demeure arbitraire. Donc puisque $p=2$, nous aurons $y=\frac{8sz+ss}{z-8s}$ et $x=\frac{4zz+4ss}{z-8s}$, et pour ôter les fractions, mettons dans ces formules t au lieu de z, et ainsi on pourra multiplier tous ces nombres par $t-8s$ et l'on aura

$$x=4(tt+ss),\quad y=s(8t+s)\quad \text{et}\quad z=t(t-8s),$$

d'où il est clair que pour t il faut prendre une valeur $>8s$. En prenant $s=1$ et $t=9$, on aura $x=328$, $y=73$, $z=9$, plus grands que les précédents.

Cependant les deux solutions s'accordent; mais pour avoir le cas le plus simple, il faut prendre $t=13$ et $s=1$; car alors on aura

$$x=680,\quad y=105,\quad z=65,\quad \text{ou bien}\quad x=136,\quad y=21,\quad z=13.$$

A. m. T. III. p. 145.

79.

Ad PROBLEMA, quo quaeruntur tres numeri x, y, z, ut quadratum cujusque una cum producto reliquorum faciat quadratum, cujus solutio specialis facile invenitur haec $x=aa-8ab$, $y=bb+8ab$, $z=4aa+4bb$.

Generaliter statui potest $x=aa+2b$, $y=bb+2a$, $z=ab(ab-4)$, quibus satisfit duabus conditionibus $xx+yz=\square$, $yy+xz=\square$, et ut tertiae quoque $zz+xy=\square$ satisfiat, fieri debet

$$a^4b^4-(8a^3-2)b^3+17a^2b^2+4ab+2a^3=\square.$$

Hinc istos valores inveni: $x=33$, $y=185$, $z=608$, tum vero $x=297$, $y=377$, $z=320$.

A. m. T. III. p. 176.

80.

PROBLEMA. Invenire tria quadrata pp, qq, rr, ut semisumma binorum sit quadratum, scilicet

$$\frac{pp+qq}{2}=zz,\quad \frac{pp+rr}{2}=yy,\quad \frac{qq+rr}{2}=xx.$$

Hinc solutiones simpliciores hujus problematis erunt hae quinque

$$\begin{aligned} p &= 89, && 97, && 119, && 23, && 17\\ q &= 191, && 553, && 833, && 289, && 697\\ r &= 329, && 833, && 1081, && 527, && 1127. \end{aligned}$$

Directa autem hujus problematis solutio ita se habet:

$$p=(ff-2gg)(ss-2tt),\quad q=(ff+2gg+4fg)(2tt+ss+4st)-8fgst$$

Quia hic omnes litterae tam negative quam positive accipi possunt, haec formula plures admittit variationes, quarum una pro q, altera pro r accipi potest. Quo facto necesse est, ut $\frac{qq+rr}{2}$ reddatur quadratum. Praeterea vero notetur, quemlibet numerum formae $aa-2bb$ infinitis modis per similes formas exprimi posse. Ita formula $aa-2bb$ infinitis modis fieri potest $=\pm 1$, scilicet ponendo

$$a=1,\ 3,\ 7,\ 17,\ 41,\ 99,\ \text{etc.}$$
$$b=1,\ 2,\ 5,\ 12,\ 29,\ 70,\ \text{etc.}$$

Cum igitur numeri $ff-2gg$ et $ss-2tt$ infinitis modis per similes formas exprimi queant, formula pro q et r data infinities infinitis modis variari poterit. Si enim fuerit

$$aa-2bb=\pm 1,\quad \text{erit}\quad ff-2gg=(af\pm 2bg)^2-2(ag\pm bf)^2,$$

ita tamen, ut p eundem valorem retineat. At vero hoc modo quaelibet solutio particularis satis difficilem postulat evolutionem; unde praecedens solutio longissime praecellit, cum sumtis numeris c et d pro lubitu valores jam evolutos pro litteris p, q, r suppeditet. Hic etiam notasse juvabit infinitis modis fieri posse

$$aa - 2bb = 2cc - dd.$$

Cum enim fieri debeat $aa + dd = 2(bb + cc)$, hoc fiet sumendo $a = b + c$ et $d = b - c$. Erit ergo

$$c = a - b \quad \text{et} \quad d = 2b - a.$$

A. m. T. III. p. 178. 179.

81.

(Lexell.)

Problema de inveniendis quotcunque numeris p, q, r, s, t, etc., quorum quilibet ductus in summam reliquorum faciat quadratum, facile tentando sine analysi resolvi potest. Sit enim S summa omnium, et cum $p(S-p)$ debeat esse quadratum, ideoque $pS = pp + \square$, evidens est tam p quam S esse debere summam duorum quadratorum, ideoque pro S sumi conveniet talem numerum, qui pluribus modis in bina quadrata se resolvi patiatur, cujusmodi est $S = 130$, qui in binas partes secari debet, quarum productum sit quadratum; quandoquidem si una pars sit p, altera erit $S - p$, tales resolutiones hoc modo exhibemus:

$$S = 130, \qquad \begin{array}{r|rrrrrrrr} p & 2, & 5, & 13, & 26, & 32, & 40, & 49, & 65 \\ S-p & 128, & 125, & 117, & 104, & 98, & 90, & 81, & 65 \end{array}$$

ubi notandum, valores ipsius p ex utraque columna sumi posse, ex his igitur excerpi oportebit vel ternos numeros, vel quaternos, vel quinos, vel senes etc., quorum summa faciat 130, ut sequitur: ternio 32, 49, 49. Sicque quinque habemus numeros 2, 5, 26, 32, 65, quorum quilibet in summam reliquorum ductus, producit quadratum. Alii quini: 2, 13, 26, 40, 49. Alii numeri idonei pro S assumendi, qui plurimas resolutiones admittunt, sunt 2210

$$S = 2210, \qquad \begin{array}{r|llll} p & 1, & 5, & 13 & \ldots \\ S-p & 2209, & 2205, & 2197 & \ldots \end{array}$$

A. m. T. I. p. 112.

82.

Consideratio circa quadrata magica.

I. Quadrata magica facile eo reduci possunt, ut summae per columnas tam horizontales quam verticales evanescant, quod fit admittendo etiam numeros negativos; scilicet si quadratum fuerit impar, numeri inscribendi erunt $0, \pm 1, \pm 2, \pm 3, \pm 4$, etc., sin autem quadratum fuerit par, numeri inscribendi erunt $\pm 1, \pm 3, \pm 5, \pm 7$, etc. Quodsi enim ad singulos addatur numerus quidam impar, et summae per 2 dividantur, orientur numeri naturales; ita si ad singulos illos numeros addatur 9, semisses dabunt hos numeros ordine 4, 5, 3, 6, 2, 7, 8,

II. In omni quadrato magico quaterna loca *connexa* voco, quando duo pro lubitu in columna quadam horizontali accipiuntur, quorum illud primum, alterum secundum voco, duo reliqua vero in alia columna horizontali ita accipiuntur, ut primum et tertium, itemque secundum et quartum in ea columna verticali existant, sive ut lineae 1...2 et 3....4 sint horizontales, rectae vero 1....3 et 2....4 verticales. Jam sumtis hujusmodi quaternis locis, si primo inscribatur numerus quicunque $+a$, secundo $-a$, tertio $-a$ et quarto $+a$, ho modo nec summa horizontalium nec verticalium mutatur; hic enim nondum respicio ad eam conditionem, qua etiam summae per diagonales eaedem, hoc est nostro casu $= 0$ esse debent.

III. Proposito igitur quadrato, in areolas quotcunque more solito diviso; omnes areolae hoc modo litteris inscribantur, ubi nihil impedit, quominus in eandem areolam quandoque duae, vel tres pluresve litterae hoc modo inscribantur: tot scilicet litteras hoc modo inscribi oportet, ut deinceps omnes numeri revera inscribendi obtineri queant, simulque proprietas diagonalium adimpleatur.

IV. Ita si proponatur quadratum novem areolarum, litterae inscribantur, ut in schemate adjecto videre est

$+a$	$+b$	$-a$ $-b$
$+c$	$-b$ $-c$	$+b$
$-c$ $-a$	$+c$	$+a$

pro diagonalibus autem praeterea esse debet $2a-b-c=0$, $2a+2b+2c=0$, unde sequitur $a=0$ et $b+c=0$, seu $c=-b$; unde quadratum ita se habebit

0	$+b$	$-b$
$-b$	0	$+b$
$+b$	$-b$	0

hinc autem numeri propositi obtineri nequeunt.

V. Commode autem in hoc quadrato areola media vacua relinquitur, scilicet cyphra implenda, ac tum inscriptio litterarum ita se habebit

$+a$ $+b$	$-b$	$-a$
$+c$	0	$-c$
$-c$ $-b$ $-a$	$+b$	$+c$ $+a$

ubi diagonales dant $2a+b+c=0$, ideoque $c=-2a-b$, et numeri inscripti erunt

I. $a+b$	II. $-b$	III. $-a$
IV. $-2a-b$	V. 0	VI. $2a+b$
VII. a	VIII. $+b$	IX. $-a-b$

Nunc quaerantur numeri pro a et b sumendi, ut prodeant numeri ± 1, ± 2, ± 3, ± 4. Sit igitur $a=1$ et $b=2$ et habebitur hoc quadratum

3	−2	−1
−4	0	+4
1	2	−3

ac si ad singulos numeros addatur 5, oritur quadratum solitum:

8	3	4
1	5	9
6	7	2

VI. Tentetur quadratum sedecim areolarum. Proprietas diagonalium hic dat

$$2a + 2b + 2e + 2f = 0, \quad 2a + 2b + 2g + 2h = 0, \quad \text{ergo} \quad f = -(a + b + e) \quad \text{et} \quad h = -(a + b + g)$$

et numeri inscripti sunt

I. $a + e$	II. $c - e$	III. $-a - b - c - g$	IV. $b + g$
V. $d - e$	VI. $b + e$	VII. $a + g$	VIII. $-a - b - g - d$
IX. $-d + g$	X. $-b - g$	XI. $-a - e$	XII. $a + b + d + e$
XIII. $-a - g$	XIV. $-c + g$	XV. $a + b + c + e$	XVI. $-b - e$.

Notentur ii, quorum negativa non occurrunt: II. $c - e$, III. $-a - b - c - g$, V. $d - e$, VIII. $-a - b - g - d$, IX. $-d + g$, X. $-b - g$, XII. $a + b + d + e$, XIV. $-c + g$, XV. $a + b + c + e$. Ut horum cuique socius comparetur, statuatur $g = e$, et nunc bini socii junctim repraesententur:

I. $a + e$, II. $c - e$, III. $-a - b - c - e$, IV. $b + e$	XI. $-a - e$, XIV. $-c + e$, XV. $a + b + c + e$, X. $-b - e$
V. $d - e$, VI. $b + e$, VII. $a + e$	IX. $-d + e$, XVI. $-b - e$, XIII. $-a - e$
VIII. $-a - b - d - e$	XII. $a + b + d + e$.

Hic autem quidam bis occurrunt. Verum haec methodus accuratiorem evolutionem postulat.

A. m. T. I. p. 28—30.

83.

N. Fuss I.

Eine LEICHTE REGEL, alle magische Quadrate von ungeraden Zahlen, die sich nicht durch 3 theilen lassen zu verfertigen, in welchen nicht nur alle Horizontal- und Vertikalreihen nebst den beiden Diagonalen, wie gewöhnlich erfordert wird, sondern auch die den Diagonalen parallel gezogenen Zahlenreihen, wenn man sie nämlich durch die gegenüberstehenden, gleich weit entfernten ergänzt, einerlei Summen geben, und wobei man zugleich nach Belieben in jedem Fache anfangen kann.

Dies geschieht vermittelst des bekannten Springerganges im Schachspiel, nach welchem man immer in die folgende Vertikalcolumne abwärts fortgeht, wobei zu merken, dass, wenn man unten an das Ende gekommen, man von da hinaufspringt, und ebenfalls, wenn man auf der rechten Seite ans Ende gekommen, wiederum in die erste Columne linker Hand einschlägt. Wo aber die Stelle schon besetzt ist, prallt man links nach eben dem Gang abwärts zurück, wie aus folgendem Schema zu ersehen:

1	14	22	10	18
25	8	16	4	12
19	2	15	23	6
13	21	9	17	5
7	20	3	11	24

Hievon wird die Ursache deutlicher werden, wenn man eben diese Operation auf eine allgemeine Art mit lateinischen Lettern a, b, c, d, e und griechischen α, β, γ, δ, ε anstellt, und dabei diese Ordnung beobachtet, dass man nach $a\alpha$, $a\beta$, $a\gamma$, $a\delta$, $a\varepsilon$ auf $b\alpha$, $b\beta$, $b\gamma$, u. s. w. fortgeht

$e\delta$	$b\beta$	$d\varepsilon$	$a\gamma$	$c\alpha$
$d\gamma$	$a\alpha$	$c\delta$	$e\beta$	$b\varepsilon$
$c\beta$	$e\varepsilon$	$b\gamma$	$d\alpha$	$a\delta$
$b\alpha$	$d\delta$	$a\beta$	$c\varepsilon$	$e\gamma$
$a\varepsilon$	$c\gamma$	$e\alpha$	$b\delta$	$d\beta$

A. m. T. II. p. 237. 238.

D. Miscellanea.

84.

(*J. A. Euler.*)

Wie blos aus den dreieckigten Zahlen alle vieleckigten Zahlen leicht gefunden werden können.

Wenn die m-eckigte Zahl für die Seite n gefunden werden soll, so suche man die dreieckigte Zahl für eben die Seite n und auch die vorhergehende dreieckigte Zahl, für die Seite $n-1$; diese multiplicire man mit $m-3$ und zum Product addire man jene, so hat man die verlangte vieleckigte Zahl.

Denn für die Seite n ist die dreieckigte Zahl $=\frac{nn+n}{2}$ und für die vorhergehende Seite $n-1$ ist die Dreieckzahl $=\frac{nn-n}{2}$; also diese mit $m-3$ multiplicirt gibt $(m-3)\left(\frac{nn-n}{2}\right)$, hiezu $\frac{nn+n}{2}$ addirt gibt

$$\frac{(m-2)\,nn-(m-4)\,n}{2}.$$

Also wenn die 365-eckigte Zahl von 12 verlangt wird, so ist $m-3=362$, die Dreieckszahl für 12 ist 78, die für 11 ist 66, also die gesuchte Zahl wird sein

$$362.66+78=23970.$$

A. m. T. I. p. 237.

85.

(N. Fuss I.)

THEOREMATA CIRCA PROBLEMA PELLIANUM.

I. Si fuerit $nff-1=gg$, erit $n(2fg)^2+1=(2gg+1)^2$.

DEMONSTRATIO. Cum enim sit $nff=gg+1$, multiplicando per $4gg$ fiet

$$4nffgg=4g^4+4gg$$

et addendo unitatem $$4nffgg+1=4g^4+4gg+1=(2gg+1)^2.$$

II. Si fuerit $nff-4=gg$, erit $nffgg+4=(gg+2)^2$.

DEMONSTRATIO. Cum enim sit $nff=gg+4$ et per gg multiplicando et 4 addendo prodit

$$nffgg+4=g^4+4gg+4=(gg+2)^2. \quad \text{Q. E. D.}$$

III. Si fuerit $nff+4=gg$, erit $nff\left(\frac{gg-1}{2}\right)^2+1=\left(\frac{g^3-3g}{2}\right)^2$.

DEMONSTRATIO. Cum sit $nff=gg-4$, multiplicando per $\left(\frac{gg-1}{2}\right)^2$ et addendo 1 fiet

$$nff\left(\frac{gg-1}{2}\right)^2+1=(gg-4)\left(\frac{gg-1}{2}\right)^2+1=\frac{g^6-6g^4+9gg}{4}=\left(\frac{g^3-3g}{2}\right)^2.$$

Hinc si $n=13$, quia $13.1^2-4=9=3^2$ in theoremate secundo habemus $f=1$ et $g=3$, unde sequitur $13.3^2+4=11^2$. Nunc pro tertio theoremate habemus $f=3$ et $g=11$, hinc $\frac{gg-1}{2}=60$ et $\frac{gg-3}{2}=59$, hinc $\frac{g^3-3g}{2}=649$, ex quo sequitur fore: $13.3^2.60^2+1=649^2=13.180^2+1$.

A. m. T. I. p. 281.

86.

THEOREMA. Si habeantur duo casus hujusmodi $qq-app=k$ et $ss-arr=\pm k$ et capiatur $x=qr\pm ps$ et $y=qs\pm apr$, erit $yy-axx=\pm kk$. At si k sit numerus primus, semper evenit, ut alterutri horum valorum scilicet $x=qr+ps$ et $y=qs+apr$ fiant per k divisibiles, sicque habebitur

$$\frac{yy}{kk}-\frac{axx}{kk}=\pm 1.$$

A. m. T. I. p. 289.

87.

THEOREMA I. Si x fuerit numerus trigonalis, tum etiam $9x+1$ erit numerus trigonalis.

Sit enim $x=\frac{aa+a}{2}$ erit $9x+1=\frac{9aa+9a+2}{2}$. Est vero $9aa+9a+2=(3a+1)(3a+2)$, ideoque $9x+1$ erit numerus trigonalis, cujus radix est $3a+1$.

COROLLARIUM 1. Si ergo x fuerit summa duorum trigonalium, tum etiam $9x+2$ erit summa duorum trigonalium. Sit enim $x=\frac{aa+a}{2}+\frac{bb+b}{2}$, erit $9x+2=\frac{9aa+9a+2}{2}+\frac{9bb+9b+2}{2}$.

COROLLARIUM 2. Simili modo si x fuerit summa trium trigonalium, tum etiam erit $9x+3$ summa trium trigonalium. Si enim sit $x=\Delta+\Delta'+\Delta''$, tum erit $9x+3=9\Delta+1+9\Delta'+1+9\Delta''+1$.

THEOREMA II. Si x fuerit numerus trigonalis, tum etiam $25x+3$ erit numerus trigonalis.

Sit enim $x=\frac{aa+a}{2}$, erit $25x+3=\frac{25aa+25a+6}{2}$. Est vero $25aa+25a+6=(5a+2)(5a+3)$, unde radix trigonalis erit $5a+2$. Hinc si x fuerit summa duorum trigonalium, erit etiam $25x+6$ summa duorum trigonalium; ac si x fuerit summa trium trigonalium, tum etiam erit $25x+9$ summa trium trigonalium.

Theorema III. Si fuerit x numerus trigonalis, erit etiam $49x+6$ numerus trigonalis.

Sit enim $x=\frac{aa+a}{2}$, erit $49x+6=\frac{49aa+49a+12}{2}=\frac{(7a+3)(7a+4)}{2}$ numerus trigonalis, cujus radix est $7a+3$. Hinc si x fuerit summa duorum trigonalium, erit etiam $49x+12$ summa duorum trigonalium; at si fuerit x summa trium trigonalium, erit itidem $49x+18$ summa trium trigonalium.

Theorema IV. Si fuerit x numerus trigonalis, erit etiam $81x+10$ numerus trigonalis.

Sit enim $x=\frac{aa+a}{2}$, erit $81x+10=\frac{81aa+81a+20}{2}=\frac{(9a+4)(9a+5)}{2}$ numerus trigonalis, ejusque radix $=9a+4$.

etc. etc.

Ex his igitur sequitur, si numerus $9x+3$ nullo modo in tres trigonales resolvi queat, tum etiam numerum x in tres trigonales resolvi non posse. Simili modo si numerus $25x+9$ resolutionem in tres trigonales non admittat, etiam numerus x non admittet. Ac si numerus $49x+18$ non admittat resolutionem in tres trigonales, numerus ipse x etiam non admittet.

A. m. T. II. p. 25 26.

88.

Theorema. Si productum $P=2n(2n-1)(2n-2)(2n-3)\ldots(n+1)$ dividatur per potestatem 2^n, quotus erit productum ex omnibus numeris imparibus:

$$1.3.5.7.9\ldots(2n-1).$$

Demonstratio. Cum sit

$$P=\frac{1.2.3.4\ldots.2n}{1.2.3.4\ldots.n}$$

multiplicetur supra et infra per 2^n eritque

$$P=\frac{2^n.1.2.3.4\ldots.2n}{2.4.6.8\ldots\ldots.2n}$$

ac divisione actu facta fiet $P=2^n.1\ 3.5.7\ldots(2n-1)$. Q. E. D.

A. m. T. II. p. 60.

89.

Theorema numericum. Si sumantur quotcunque numeri pro lubitu, veluti quatuor p, q, r, s, hincque formentur bini ordines totidem aliorum, hoc modo

$$a=p,\quad b=p+q,\quad c=p+q+r,\quad d=p+q+r+s$$

similique modo $\alpha=s,\quad \beta=s+r,\quad \gamma=s+r+q,\quad \delta=s+r+q+p.$

tum semper erit $$\frac{1}{abcd}-\frac{1}{abc\alpha}+\frac{1}{ab\alpha\beta}-\frac{1}{a\alpha\beta\gamma}+\frac{1}{\alpha\beta\gamma\delta}=0.$$

Veluti si fuerint numeri dati 1, 2, 3, 4 erit $a=1$, $b=3$, $c=6$, $d=10$, $\alpha=4$, $\beta=7$, $\gamma=9$, $\delta=10$, eritque

$$\frac{1}{1.3.6.10}-\frac{1}{1.3.6.4}+\frac{1}{1.3.4.7}-\frac{1}{1.4.7.9}+\frac{1}{4.7.9.10}=0.$$

A. m. T. II. p. 208.

90.

THEOREMA. Si proposita fuerit haec formula $zz = aa + 2abx + max^2 + 2dex^3 + eex^4$, in qua sit

$$m = bb + dd - ff,$$

tum eadem forma sequentibus modis repraesentari potest:

$$1)\quad zz = (a + bx)^2 + xx(ex + d + f)(ex + d - f)$$

$$2)\quad zz = xx(ex + d)^2 + (a + (b + f)x)(a + (b - f)x),$$

unde sequitur $z = a + bx$ si fuerit vel $x = \frac{-d-f}{e}$, vel $x = \frac{-d+f}{e}$. Ex altera

$$z = x(ex + d) \quad \text{si fuerit vel } x = \frac{-a}{b+f}, \quad \text{vel } x = \frac{-a}{b-f}.$$

Praeter hos quatuor valores operationes vulgares praebent adhuc sequentes sex valores

$$\text{I.}\quad x = \frac{2ae + ff - dd}{2e(a - b)},\qquad \text{II.}\quad x = \frac{2a(b - d)}{2ae + ff - bb},\qquad \text{III.}\quad x = \frac{2ae + ff - dd}{2e(d + b)},\qquad \text{IV.}\quad x = \frac{2a(b + d)}{-2ae + ff - bb},$$

$$\text{V.}\quad x = \frac{3aade + 4ab(ff - dd)}{(ff - dd)^2 - 4aaee},\qquad \text{V.}\quad x = \frac{(ff - bb)^2 - 4aaee}{8abee + 4de(ff - bb)}.$$

A. m. T. III. p. 163.

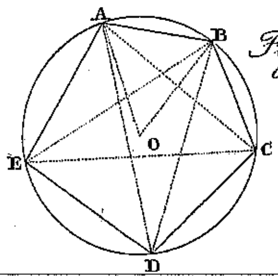


Fig. 1. Pag. 229.

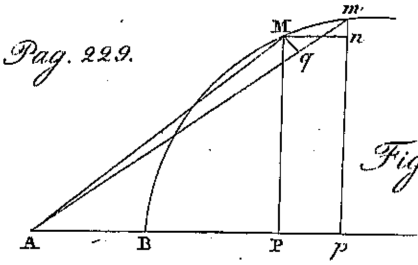


Fig. 2. Pag. 342.

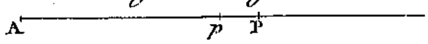


Fig. 3. Pag. 346.

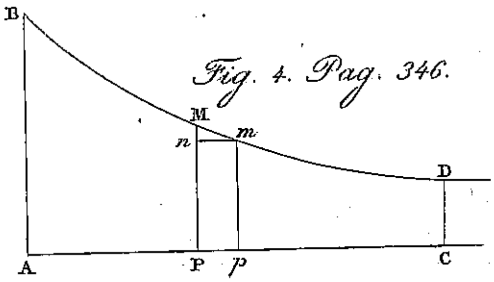


Fig. 4. Pag. 346.

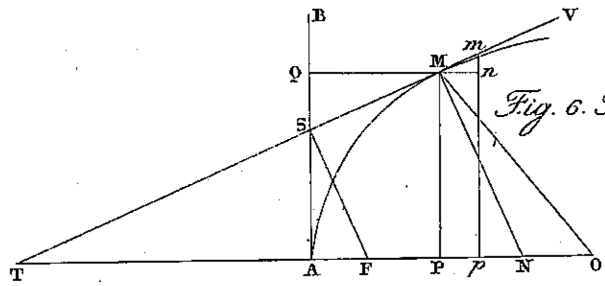


Fig. 6. Pag. 353.

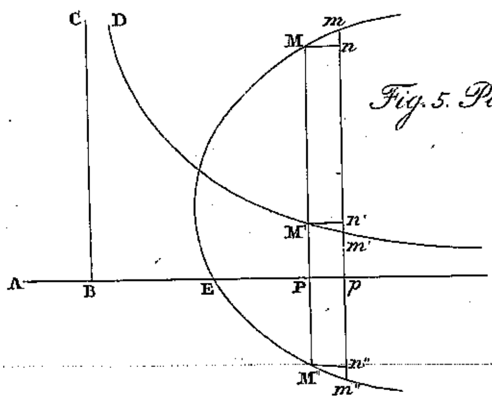
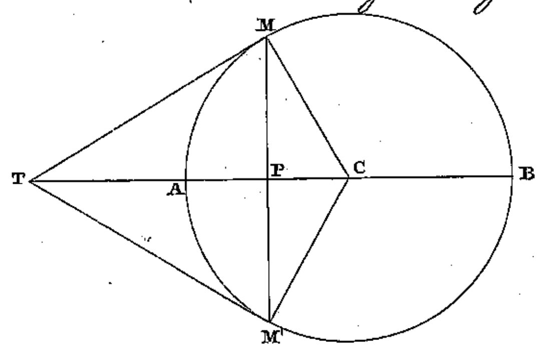


Fig. 5. Pag. 347.

Fig. 7. Pag. 355.



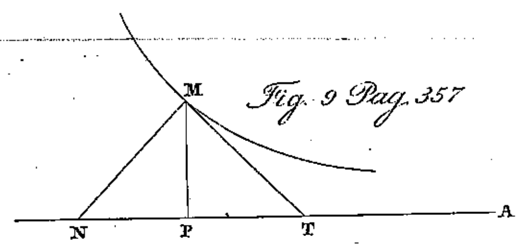


Fig. 9 Pag. 357

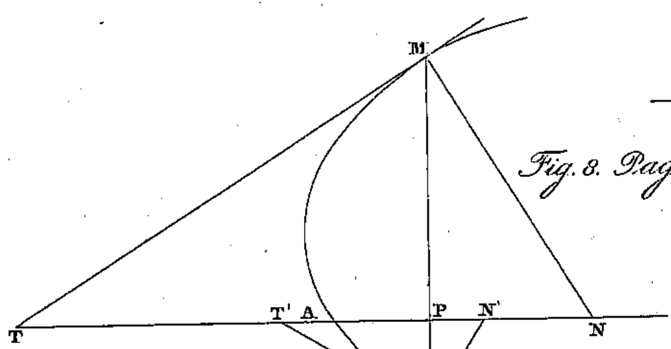


Fig. 8. Pag. 356.

Fig. 10. Pag. 357.