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Considerationes circa Analysin Diophanteam

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VII.

Considerationes circa analysin Diophanteam.

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1. Saepe ac multum mecum cogitavi, an non liceret eam analyseos partem, quae Diophantea appellari solet, veluti reliquas matheseos disciplinas, ad certa capita revocare, quibus constitutis universus hujus analyseos complexus perspici, et cuique problemati caput, ad quod referri oporteat, assignari queat, ut hinc principia statim innotescant, ex quibus cujusque problematis solutionem peti conveniat. Verum postquam complures quaestiones huc pertinentes omni studio pertractassem, singulas fere sibi prorsus peculiares methodos et calculi artificia postulare deprehendi, ut propemodum totidem hujus analyseos capita constituenda videantur, quot problemata particularia in hoc genere proponi possunt. Ex quo nullo adhuc modo intelligere licet, quomodo pro hac analyseos parte principia generalia constitui, eamque in certas partes distribui oporteat.

2. Divisio quidem hujus analyseos in duas partes statim se offert, quarum altera ejusmodi problemata in se complectitur, quorum solutiones ita per formulas generales exhiberi queant, ut omnes plane solutiones in iis contineantur, ex iisque derivari queant. Altera autem pars ejusmodi quaestionibus solvendis versatur, quarum solutio generalis neutiquam in certis formulis comprehendi potest, sed ita institui solet, ut ex qualibet solutione jam inventa aliae novae deduci queant, quo in negotio tamen iterum fere infinita varietas pro diversa problematum indole cernitur; et quoniam nunc quidem maxima pars problematum, quae in analysi Diophantea tractari solent, ad hanc alteram partem est referenda, multo minus methodi ad ea solvenda accommodatae ad certas classes revocari posse videntur.

3. Neque vero illa divisio problematum inde petita, quod alia solutionem generalem, in certis formulis analyticis contentam, admittant, alia vero tantum solutiones particulares recipiant, quae tamen continuo ad alias novas perducant, tam certo est stabilita, ut haec duo problematum genera ob suam naturam penitus a se invicem dirimantur, cum utique evenire queat, ut problemata, quae ad posteriorem partem referenda videntur, certis adhibitis artificiis generaliter resolvi queant. Cujusmodi est problema de tribus cubis inveniendis, quorum summa sit cubus, cujus solutiones antehac tantum particulares sunt datae, donec equidem ejus solutionem generalem exhibui, ita ut hoc problema nunc primae parti accensendum videatur.

4. Cum igitur hactenus plura problemata Diophantea sim perscrutatus, unde multitudo ac varietas methodorum, quibus ad ea solvenda uti convenit, maxime elucet, nunc aliud ejus generis problema, quod quidem apud auctores passim occurrit, contemplabor, quod ita se habet:

Invenire tres numeros v, x, y, *quorum binorum productum, summa eorundem auctum, producat numerum quadratum, ita ut hae tres formulae* vx+v+x, vy+v+y, xy+x+y *quadrata reddi debeant.*

Deinde vero eandem quaestionem ad quatuor hujusmodi numeros extendam, quandoquidem tum maximae difficultates occurrunt, dum haec quaestio, uti est proposita, generaliter resolvi potest, ac solutio tantum in numeris integris certa artificia postulat.

5. Ad hoc problema resolvendum, ponamus $v+1=A$, $x+1=B$ et $y+1=C$, ut sequentes tres formulae $AB-1$, $AC-1$ et $BC-1$ quadrata fieri debeant. Statuamus igitur primo $AB=pp+1$, $AC=qq+1$ et $BC=rr+1$, eritque $ABC=\sqrt{(pp+1)(qq+1)(rr+1)}$. Quo jam haec formula facilius rationalis efficiatur, litteras p et q ut datas spectemus, ponamusque $(pp+1)(qq+1)=mm+nn$, ut sit $m=pq\pm 1$, et $n=p\mp q$, fietque

$$ABC=\sqrt{(mm+nn)(rr+1)}=\sqrt{((mr+n)^2+(nr-m)^2)},$$

quae radix statuatur $=mr+n+t(nr-m)$, ut prodeat $nr-m=2mrt+2nt+nrtt-mtt$,

$$\text{hincque}\quad r=\frac{m(tt-1)-2nt}{n(tt-1)+2mt}.$$

6. Erit ergo $rr+1=\frac{(mm+nn)(tt+1)^2}{(n(tt-1)+2mt)^2}$ et $ABC=\frac{(mm+nn)(tt+1)}{n(tt-1)+2mt}$, unde ob $BC=rr+1$, reperitur

$$A=\frac{n(tt-1)+2mt}{tt+1},\quad \text{et ob}\quad mm+nn=(pp+1)(qq+1),$$

$$B=\frac{(pp+1)(tt+1)}{n(tt-1)+2mt}\quad \text{et}$$

$$C=\frac{(qq+1)(tt+1)}{n(tt-1)+2mt},$$

existente $m=pq\pm 1$ et $n=p\mp q$.

7. En ergo solutionem maxime generalem nostri problematis, in qua adeo binos numeros p et q pro lubitu accipere licet, ita ut binae formulae $AB-1$ et $AC-1$ datis quadratis aequentur; et cum littera t etiamnunc arbitrio nostro permittatur, pro tertia formula $BC-1$ infinita quadrata reperiri possunt, unde hoc problema sine ullo dubio ad primam partem, ubi solutiones generales exhibere licet, erit referendum. Verum cum hoc modo terni numeri quaesiti plerumque prodeant fracti, si solutiones in integris desiderentur, alia artificia in hunc finem adhiberi conveniet, quae hic exposuisse juvabit.

Solutio problematis per numeros integros.

8. Quoniam numeros p et q ut datos spectamus, solutio ita facilius obtinetur. Posito

$$AB=pp+1\quad \text{et}\quad AC=qq+1,\quad \text{ut sit}\quad B=\frac{pp+1}{A}\quad \text{et}\quad C=\frac{qq+1}{A},$$

erit statim

$$BC-1=\frac{(pp+1)(qq+1)}{AA}-1=\frac{mm+nn-AA}{AA},$$

quae forma cum esse debeat quadratum, sumatur $A=n=p-q$, seu $p=q+A$, fietque

$$B=A+2q+\frac{qq+1}{A}\quad \text{et}\quad C=\frac{qq+1}{A},$$

unde numeros A et q ita accipi conveniet, ut A sit divisor ipsius $qq+1$. Quare necesse est pro

A capi summam duorum quadratorum, ac tum semper pro q infinitos valores assignare licebit, ut $qq+1$ divisionem per A admittat, veluti ex sequentibus exemplis patebit:

1) Sit $A=1$ et $q=u$, erunt tres numeri quaesiti $A=1$, $B=uu+2u+2$ et $C=uu+1$.

2) Sit $A=2$, sumique oportet $q=2u-1$, unde prodeunt numeri quaesiti:

$$A=2,\quad B=2uu+2u+1,\quad C=2uu-2u+1.$$

3) Sit $A=5$, sumique oportet $q=5u\pm 2$, unde duae resultant solutiones

$$A=5,\quad B=5uu+14u+10,\quad C=5uu+4u+1$$
$$A=5,\quad B=5uu+\ \ 6u+\ \ 2,\quad C=5uu-4u+1;$$

sicque binis A et C duplex valor ipsius B respondet, scilicet

$$A=5,\quad C=5uu+4u+1,\qquad B=5uu+14u+10$$
$$\text{vel}\quad B=5uu-\ \ 6u+\ \ 2.$$

4) Sit $A=ff+gg$ et k minimus numerus, cujus quadratum unitate auctum per A sit divisibile, ut sit $\frac{kk+1}{ff+gg}=h$. Jam ponatur $q=(ff+gg)\,u+k$, eruntque tres numeri quaesiti

$A=ff+gg$, $C=(ff+gg)\,uu+2ku+h$ et $B=(ff+gg)\,uu+2\,(ff+gg+k)\,u+ff+gg+2k+h$,

ubi observo, si ambo numeri k et u capiantur negative, ut valor ipsius C maneat idem, tum pro B alium insuper prodire valorem

$$B=(ff+gg)\,uu-2\,(ff+gg-k)\,u+ff+gg-2k+h.$$

9. Alio autem modo prorsus singulari solutiones in integris facile inveniri possunt, qui ita procedit: Capiantur binae fractiones $\frac{a}{b}$ et $\frac{c}{d}$ tam parum a se invicem discrepantes, ut sit $ad-bc=\pm 1$; inde formetur tertia $\frac{c\pm a}{d\pm b}$, quae ad utramque priorum simili modo erit comparata. Quo facto tres numeri quaesiti ita se habebunt:

$$A=aa+bb,\quad B=cc+dd,\quad C=(c\pm a)^2+(d\pm b)^2,$$

namque ob $ad-bc=\pm 1$, erit $AB=(ac+bd)^2+1$,

$$AC=(ac\pm aa+bd\pm bb)^2+1,$$
$$BC=(cc\pm ac+dd\pm bd)^2+1.$$

10. Hinc simpliciores solutiones obtinentur sequentes:

$\frac{a}{b}$,	$\frac{c}{d}$,	$\frac{a+c}{b+d}$	A	B	C
$\frac{0}{1}$,	$\frac{1}{f}$,	$\frac{1}{f+1}$	1	$ff+1$	$ff+2f+2$
$\frac{1}{1}$,	$\frac{f-1}{f}$,	$\frac{f}{f+1}$	2	$2ff-2f+1$	$2ff+2f+1$
$\frac{1}{2}$,	$\frac{f}{2f-1}$,	$\frac{f+1}{2f+1}$	5	$5ff-4f+1$	$5ff+6f+2$
$\frac{1}{2}$,	$\frac{f}{2f-1}$,	$\frac{f-1}{2f-3}$	5	$5ff-4f+1$	$5ff-14f+10$
$\frac{1}{3}$,	$\frac{f}{3f-1}$,	$\frac{f+1}{3f+2}$	10	$10ff-6f+1$	$10ff+14f+5$
$\frac{1}{3}$,	$\frac{f}{3f-1}$,	$\frac{f-1}{3f-4}$	10	$10ff-6f+1$	$10ff-26f+17$

unde patet has solutiones convenire cum praecedentibus.

11. Datis autem duobus numeris A et B, ut sit $AB-1=\square=pp$, tertius C infinitis modis inveniri potest sequenti modo: Cum tam $AC-1$ quam $BC-1$ quadratum esse debeat, statuatur primo productum $ABCC-(A+B)C+1=(mC+1)^2$, unde reperitur

$$C=\frac{A+B+2m}{AB-mm}, \quad \text{unde fit} \quad AC-1=\frac{(A+m)^2}{AB-mm} \quad \text{et} \quad BC-1=\frac{(B+m)^2}{AB-mm}.$$

Tantum ergo superest ut $AB-mm=pp+1-mm$ reddatur quadratum puta $=nn$, seu ut sit $mm+nn=pp+1$. Hunc in finem sumantur duae fractiones a et α, ut sit $aa+\alpha\alpha=1$, fiatque $m=ap+\alpha$ et $n=\alpha p-a$, ex quo habebitur

$$C=\frac{A+B\pm 2(ap+\alpha)}{(\alpha p-a)^2},$$

ubi sumtis pro lubitu duobus numeris f et g, capi oportet

$$a=\frac{ff-gg}{ff+gg} \quad \text{et} \quad \alpha=\frac{2fg}{ff+gg}.$$

12. Hinc adeo plures valores pro C in integris inveniri possunt; sumto enim $f=1$ et $g=0$, prodit $C=A+B\pm 2p$. Deinde posito $f=2p$ et $g=1$, prodit

$$C=(A+B)(4pp+1)^2\pm 2p(4pp+1)(4pp+3).$$

Tum vero etiam sumendo $f=4pp+1$ et $g=2p$, fit

$$C=(A+B)(16p^4+12pp+1)^2\pm 2p(16p^4+12pp+1)(16p^4+20pp+5).$$

Porro, positio $f=8p^3+4p$ et $g=4pp+1$ dat quoque duos novos valores integros. Ex quo intelligere licet, in genere formam tertii numeri C fore

$$C=(A+B)M^2\pm 2pMN,$$

ubi quantitates M et N has series recurrentes constituunt:

$$M=+1, \quad 4pp+1, \quad 16p^4+12pp+1, \quad 64p^6+80p^4+24pp+1, \text{ etc.},$$
$$N=+1, \quad 4pp+3, \quad 16p^4+20pp+5, \quad 64p^6+112p^4+56pp+7, \text{ etc.},$$

quarum utriusque scala relationis est $4pp+2, -1$, ita ut in genere sit

$$M=\frac{(\sqrt{(1+pp)}+p)^{2\lambda+1}+(\sqrt{(1+pp)}-p)^{2\lambda+1}}{2\sqrt{(1+pp)}} \quad \text{et} \quad N=\frac{(\sqrt{(1+pp)}+p)^{2\lambda+1}-(\sqrt{(1+pp)}-p)^{2\lambda+1}}{2p}.$$

Vel posito $2p=q$, et denotante n numerum parem quemcunque erit

$$M=q^n+\frac{(n-1)}{1}q^{n-2}+\frac{(n-2)(n-3)}{1.2}q^{n-4}+\frac{(n-3)(n-4)(n-5)}{1.2.3}q^{n-6}+\text{etc.}$$

$$N=q^n+\frac{(n+1)}{1}q^{n-2}+\frac{(n+1)(n-2)}{1.2}q^{n-4}+\frac{(n+1)(n-3)(n-4)}{1.2.3}q^{n-6}+\text{etc.}$$

13. **Problema 2.** *Invenire quatuor numeros, ut binorum productum una cum summa eorundem binorum faciat quadratum; seu, quod eodem redit, invenire quatuor numeros* A, B, C, D, *ut binorum producta, unitate minuta, sint quadrata, sicque hae sex formulae*

$$AB-1, \quad AC-1, \quad AD-1, \quad BC-1, \quad BD-1, \quad CD-1$$

fiant quadrata.

Pro solutione hujus problematis spectemus duos numeros A et B tanquam datos, ut sit

$$AB-1=pp, \quad \text{seu} \quad AB=pp+1,$$

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ac sumto $aa + \alpha\alpha = 1$, statuatur tertius numerus $C = \frac{A+B+2(ap+\alpha)}{(\alpha p - a)^2}$. Simili modo sumto $bb + \beta\beta = 1$, ponatur quartus numerus $D = \frac{A+B+2(bp+\beta)}{(\beta p - b)^2}$, sicque jam erit satisfactum his conditionibus

$$AB - 1 = \square, \quad AC - 1 = \square, \quad BC - 1 = \square, \quad AD - 1 = \square, \quad BD - 1 = \square,$$

ita ut tantum restet sexta conditio implenda, qua esse debet $CD - 1 = \square$, quae propterea dat

$$(A+B)^2 + 2(A+B)((a+b)p + \alpha + \beta) + 4(ap+\alpha)(bp+\beta) - (\alpha p - a)^2(\beta p - b)^2 = \square,$$

cui ita satisfieri oportet, ut simul fiat $AB = pp + 1$.

14. Praeter a, α, b, β spectetur etiam p ut datum, et cum sit

$$B = \frac{pp+1}{A} \quad \text{et} \quad A + B = \frac{AA+pp+1}{A},$$

quadratum effici debebit haec forma:

$$\begin{aligned} A^4 + 2A^3(a+b)p + 2A^2(pp+1) \qquad & + 2A(pp+1)(a+b) + (pp+1)^2 \\ + 2A^3(\alpha+\beta) + 4A^2(ap+\alpha)(bp+\beta) & + 2A(pp+1)(\alpha+\beta) \\ - A^2(\alpha p - a)^2(\beta p - b)^2, & \end{aligned}$$

cujus radix statuatur

$$\begin{aligned} AA + A(a+b)p - pp - 1 \\ + A(\alpha+\beta), \end{aligned}$$

unde nascetur haec aequatio

$$\left.\begin{aligned} & AA(a+b)^2pp \\ & + 3AA(a+b)(\alpha+\beta)p \\ & + AA(\alpha+\beta)^2 \\ & - 4AA(pp+1) \\ & - 4AA(ap+\alpha)(bp+\beta) \\ & + AA(\alpha p - a)^2(\beta p - b)^2 \end{aligned}\right\} = 4A(pp+1)((a+b)p + \alpha + \beta),$$

ex qua elicitur

$$A = \frac{4(pp+1)((a+b)p+\alpha+\beta)}{((a+b)p+\alpha+\beta)^2 - 4(pp+1) - 4(ap+\alpha)(bp+\beta) + (\alpha p - a)^2(\beta p - b)^2}.$$

15. Quanquam haec solutio neutiquam est generalis, siquidem ex formula quarti ordinis est derivata, tamen quoniam numeros a, α, b, β cum p pro arbitrio assumere licet, dum sit

$$aa + \alpha\alpha = 1 \quad \text{et} \quad bb + \beta\beta = 1,$$

innumerabiles suppeditat solutiones, circa quas nil aliud desiderari videtur, nisi quod numeri prodeant non solum fracti sed etiam praemagni, ac subinde etiam negativi. Simpliciores autem solutiones ex casu quo $\alpha = 0$, $\beta = 0$ et $a = 1$, $b = -1$ obtinebuntur, ubi fit $C = A + B + 2p$ et $D = A + B - 2p$, existente $AB = pp + 1$; tum igitur erit $CD - 1 = (A+B)^2 - 4pp - 1 = \square$. Quare si statuatur $(A+B)^2 = qq + 4pp + 1$, erit $(A-B)^2 = qq - 3$, ex quo fiat $A - B = q - r$, ut prodeat $q = \frac{rr+3}{2r}$ et $A - B = \frac{3-rr}{2r}$. Jam invento q sit $A + B = 2p + s$, fietque

$$p = \frac{qq+1-ss}{4s} \quad \text{et} \quad A+B = \frac{qq+1+ss}{2s}.$$

Si hic capiatur $r = 1$, erunt numeri quaesiti

$$A = \frac{ss+2s+5}{4s}, \qquad B = \frac{ss-2s+5}{4s}, \qquad C = \frac{5}{s}, \qquad D = s,$$

Posito $r = 2$, ut sit $q = \frac{7}{4}$, habebuntur

$$A = \frac{16ss+8s+65}{64s}, \qquad B = \frac{16ss-8s+65}{64s}, \qquad C = \frac{65}{16s}, \qquad D = s,$$

qui fient omnes unitate majores sumto $s = \frac{7}{2}$:

$$A = \frac{289}{224}, \qquad B = \frac{233}{224}, \qquad C = \frac{65}{56}, \qquad D = \frac{7}{2},$$

et sumto $s = \frac{15}{4}$: $\qquad A = \frac{4}{3}, \qquad B = \frac{13}{12}, \qquad C = \frac{13}{12}, \qquad D = \frac{15}{4}.$

16. Praeterea etiam solutio particularis notari meretur, qua sumtis $b = -a$ et $\beta = -\alpha$ fit

$$C = \frac{A+B+2(ap+\alpha)}{(\alpha p-a)^2} \quad \text{et} \quad D = \frac{A+B-2(ap+\alpha)}{(\alpha p-a)^2}, \quad \text{et ob} \quad B = \frac{pp+1}{A}$$

prodit haec aequatio

$$\left.\begin{array}{l} A^4 + 2AA(pp+1) + (pp+1)^2 \\ -4AA(ap+\alpha)^2 \\ -\ AA(\alpha p-a)^4 \end{array}\right\} = \square,$$

qua reducta ad hanc formam $(AA-pp-1)^2 + AA(\alpha p-a)^2(4-(\alpha p-a)^2) = \square$, evidens est hoc fieri sumendo $\alpha p - a = 2$, seu $p = \frac{2+a}{\alpha}$, hincque

$$\text{unde } B = \frac{pp+1}{A}, \qquad C = \frac{A+B+\frac{2(2a+1)}{\alpha}}{4} = \frac{\alpha(A+B)+4a+2}{4\alpha} \quad \text{et} \quad D = \frac{\alpha(A+B)-4a-2}{4\alpha},$$

ubi adeo A pro lubitu accipi potest.

17. Si hic ponamus $a = \frac{ff-gg}{ff+gg}$ et $\alpha = \frac{2fg}{ff+gg}$, et pro m et n sumamus numeros quoscunque, quatuor numeri quaesiti prodibunt sequenti modo expressi:

$$A = \frac{m(ff+gg)}{2nfg}, \qquad B = \frac{n(9ff+gg)}{2mfg},$$

$$C = \frac{(m+3n)^2 ff + (m-n)^2 gg}{8mnfg}, \qquad D = \frac{(m-3n)^2 ff + (m+n)^2 gg}{8mnfg},$$

quae solutio, etsi est particularis, tamen satis late patet, ob quatuor numeros f, g, m, n arbitrio nostro relictos. Sit, exempli gratia, $f = 1$, $g = 2$, et $m = 5$, $n = 6$, erunt numeri satisfacientes

$$A = \frac{25}{24}, \qquad B = \frac{39}{10}, \qquad C = \frac{533}{480}, \qquad D = \frac{653}{480}, \qquad \text{unde fit}$$

$$AB - 1 = \left(\frac{7}{4}\right)^2, \qquad AC - 1 = \left(\frac{19}{48}\right)^2, \qquad AD - 1 = \left(\frac{31}{48}\right)^2,$$

$$BC - 1 = \left(\frac{73}{40}\right)^2, \qquad BD - 1 = \left(\frac{83}{40}\right)^2, \qquad CD - 1 = \left(\frac{343}{480}\right)^2.$$

Cum autem hae solutiones omnes in numeris fractis consistant praeter simplicissimam, quae est

$$A=1,\quad B=2,\quad C=5,\quad D=1,$$

quaestio oritur satis curiosa, num praeterea non aliae solutiones in numeris integris reperiri queant?

18. **Problema 3.** *Invenire quatuor numeros, ut binorum productum dato numero* n *auctum sit numerus quadratus.*

Sint A, B, C, D quatuor numeri quaesiti, et cum $AB+n$ esse debeat quadratum, ponatur $A=naa-bb$ et $B=ncc-dd$, ut fiat $AB=(nac-bd)^2-n(ad-bc)^2$, quare dum sit $ad-bc=\pm 1$, haec conditio adimpletur. Quare ejusmodi fractiones $\frac{a}{b}$ et $\frac{c}{d}$ investigare oportet, ut sit $ad-bc=\pm 1$, quod cum facile praestetur, idem eveniet in fractionibus $\frac{a+c}{b+d}$ et $\frac{a-c}{b-d}$, cum utraque illarum conjunctis. Statuamus ergo

$$A=naa-bb,\qquad B=ncc-dd,$$
$$C=n(a+c)^2-(b+d)^2,\qquad D=n(a-c)^2-(b-d)^2,$$

nihilque aliud superest nisi ut $CD+n$ reddatur quadratum, hoc est

$$\left.\begin{array}{l} nn(aa-cc)^2-2n(ab-cd)^2+(bb-dd)^2 \\ \quad -2n(ad-bc)^2 \\ \quad +n \end{array}\right\}=\square,$$

seu ob $(ad-bc)^2=1$,

$$\left.\begin{array}{l} nn(aa-cc)^2-2n(ab-cd)^2+(bb-dd)^2 \\ \quad -n \end{array}\right\}=\square,$$

quae autem aequatio tantum solutionem particularem continet.

19. Solutionem autem generalem, ut supra impetrabimus ponendo $AB=pp-n$, tum quia pro C esse debet tam $AC+n=\square$ quam $BC+n=\square$, ponatur productum

$$nn+n(A+B)C+ABCC=nn+2nCx+CCxx,$$

fiet

$$C=\frac{n(A+B-2x)}{xx-AB}\quad\text{et}\quad AC+n=\frac{n(A-x)^2}{xx-AB},$$

unde patet $\frac{xx-AB}{n}$ quadratum esse debere. Statuatur ergo

$$xx-AB=xx-pp+n=nyy,\quad\text{seu}\quad xx-nyy=pp-n.$$

Simili modo ponamus $vv-nzz=pp-n$, ut obtineamus

$$C=\frac{A+B-2x}{yy}\quad\text{et}\quad D=\frac{A+B-2v}{zz},$$

ac superest, ut reddatur quadratum

$$(A+B)^2-2(x+v)(A+B)+nyyzz+4xv.$$

At ob $B=\frac{pp-n}{A}$ et $A+B=\frac{AA+pp-n}{A}$, habebitur

$$A^4 - 2A^3(x+v) + 2AA(pp-n) - 2A(pp-n)(x+v) + (pp-n)^2$$
$$+ nAAyyzz$$
$$+ 4AAxv$$

quadrato aequandum. Statuatur radix $AA - A(x+v) - (pp-n)$, eritque

$$AA(x+v)^2 - 4AA(pp-n) - nAAyyzz - 4AAxv + 4A(x+v)(pp-n) = 0,$$

seu $A = \frac{4(x+v)(pp-n)}{nyyzz + 4(pp-n) - (x+v)^2 + 4xv}$, seu $A = \frac{4(x+v)(pp-n)}{n(yy-1)(zz-1) + 2xv + 2pp - 3n}$, seu

$$A = \frac{4(x+v)(pp-n)}{nyyzz - 2n(yy+zz) + (v+x)^2}.$$

20. Solutio particularis satis concinna hinc obtinetur ut supra sumendo $v = -x$, ut fiat $z = y$ atque $C = \frac{A+B-2x}{yy}$ et $D = \frac{A+B+2x}{yy}$, existente $AB = pp - n = xx - nyy$: cum enim quadratum esse debeat haec forma

$$A^4 + 2AA(pp-n) + nAAy^4 - 4AAxx + (pp-n)^2,$$

seu $$(AA - pp + n)^2 + nAAyy(yy-4),$$

evidens est hoc fieri sumto $y = 2$, ut sit $pp = xx - 3n$. Ponatur $p = x - t$, fiet

$$x = \frac{tt+3n}{2t} \quad \text{et} \quad p = \frac{3n-tt}{2t}, \quad \text{seu} \quad p = \frac{3nuu-tt}{2tu} \quad \text{et} \quad x = \frac{3nuu+tt}{2tu},$$

hinc $$AB = \frac{(nuu-tt)(9nuu-tt)}{4ttuu}.$$

Quocirca habebimus

$$A = \frac{f(nuu-tt)}{2gtu}, \qquad B = \frac{g(9nuu-tt)}{2ftu},$$

$$C = \frac{n(f+3g)^2uu - (f-g)^2tt}{8fgtu}, \qquad D = \frac{n(f-3g)^2uu - (f+g)^2tt}{8fgtu}.$$

Circa hanc solutionem notari convenit esse $C + D = \frac{A+B}{2}$.

21. **Problema 4.** *Invenire quatuor numeros, ut binorum producta singula, summa numerorum aucta, fiant numeri quadrati.*

Inventis, per problema praecedens, quatuor numeris A, B, C, D, quorum binorum producta dato numero n aucta fiunt quadrata, statuantur numeri quatuor quaesiti mA, mB, mC, mD, et cum sit $mm(AB+n)$ quadratum, seu $mmAB + mmn = \square$, efficiendum erit tantum, ut numerus mmn aequalis fiat summae horum quatuor numerorum $m(A+B+C+D)$, unde statim reperitur multiplicator quaesitus

$$m = \frac{A+B+C+D}{n}.$$

Quodsi ergo numeri A, B, C, D ex § praecedente accipiantur, ob $C + D = \frac{A+B}{2}$ habebitur

$$m = \frac{3(A+B)}{2n} = \frac{3n(ff+9gg)uu - 3(ff+gg)tt}{4nfgtu}.$$

Hic igitur non solum quatuor litterae f, g et t, u, sed etiam numerus n pro lubitu accipi possunt, ita ut haec solutio latissime pateat, etiamsi non sit generalis.

22. Quoniam autem hic numerus n arbitrio nostro relinquitur, ex aequatione § praecedentis $pp = xx - 3n$ statim sumamus $n = \frac{xx-pp}{3}$, ut sit $AB = \frac{4pp-xx}{3}$; hinc ponamus

$$A = \frac{f(2p+x)}{3g} \quad \text{et} \quad B = \frac{g(2p-x)}{f}, \quad \text{erit}$$

$$A + B = \frac{2(ff+3gg)p+(ff-3gg)x}{3fg},$$

hinc
$$C = \frac{2(ff+3gg)p+(ff-6fg-3gg)x}{12fg},$$

$$D = \frac{2(ff+3gg)p+(ff+6fg-3gg)x}{12fg}.$$

Nunc igitur ob $A+B+C+D = \frac{2(ff+3gg)p+(ff-3gg)x}{2fg}$, erit multiplicator communis

$$m = \frac{6(ff+3gg)p+3(ff-3gg)x}{2fg(xx-pp)}.$$

23. Possunt hic adeo bini numeri A et B pro lubitu assumi, unde fit

$$2p + x = \frac{3Ag}{f} \quad \text{et} \quad 2p - x = \frac{Bf}{g}, \quad \text{hinc}$$

$$p = \frac{3Agg+Bff}{4fg} \quad \text{et} \quad x = \frac{3Agg-Bff}{2fg}, \quad \text{atque}$$

$$n = \frac{(9Agg-Bff)(Agg-Bff)}{16ffgg}, \quad \text{tum vero}$$

$$C = \frac{A+B}{4} + \frac{3Agg-Bff}{4fg}, \qquad D = \frac{A+B}{4} - \frac{3Agg+Bff}{4fg}$$

ac denique multiplicator $m = \frac{3(A+B)}{2n}$. Si hic ad fractiones tollendas ponamus $A = 4afg$ et $B = 4bfg$, erit $C = (a+b)fg + 3agg - bff$ et $D = (a+b)fg - 3agg + bff$, atque $n = (9agg - bff)(agg - bff)$, tum vero $m = \frac{6(a+b)fg}{(9agg-bff)(agg-bff)}$.

24. Si sumamus $f = 1$ et $g = 1$, erit $A = 4a$, $B = 4b$, $C = 4a$, $D = 2b - 2a$ et multiplicator

$$m = \frac{6(a+b)}{(9a-b)(a-b)} = \frac{6(a+b)}{(b-a)(b-9a)},$$

qui ut fiat positivus, capi debet $b > 9a$; sit ergo

1) $a = 1$, $b = 10$, erit $A = 4$, $B = 40$, $C = 4$, $D = 18$ et $m = \frac{6.11}{9.1} = \frac{22}{3}$.

2) $a = 1$, $b = 11$, erit $A = 4$, $B = 44$, $C = 4$, $D = 20$ et $m = \frac{6.12}{10.2} = \frac{18}{5}$.

3) $a = 1$, $b = 13$, erit $A = 4$, $B = 52$, $C = 4$, $D = 24$ et $m = \frac{6.14}{12.4} = \frac{7}{4}$.

unde quatuor numeri quaesiti erunt integri

$$mA = 7, \quad mB = 91, \quad mC = 7, \quad mD = 42, \quad \text{quorum summa est} = 147.$$

Hic autem desiderari potest, quod duo quaesitorum numerorum sint aequales, quod etiam evenit sumendo $f = 3g$.

25. Ut igitur numeros inaequales nanciscamur, sumamus $f=2$ et $g=1$, fietque

$$A=8a,\quad B=8b,\quad C=5a-2b,\quad D=6b-a\quad \text{et}\quad m=\frac{12(a+b)}{(9a-4b)(a-4b)}=\frac{12(a+b)}{(4b-9a)(4b-a)},$$

unde casus simpliciores erunt

1) $a=5$, $b=1$, hinc $A=40$, $B=8$, $C=23$, $D=1$ et $m=\frac{72}{41}$;

2) $a=11$, $b=2$, hinc $A=88$, $B=16$, $C=51$, $D=1$ et $m=\frac{4}{7}$,

3) $a=3$, $b=7$, hinc $A=24$, $B=56$, $C=1$, $D=39$ et $m=\frac{24}{5}$.

Ponamus etiam $f=3$ et $g=2$, ut consequemur

$$A=24a,\quad B=24b,\quad C=18a-3b,\quad D=15b-6a$$

et

$$m=\frac{36(a+b)}{9(4a-b)(4a-9b)}=\frac{4(a+b)}{(4a-b)(4a-9b)},$$

sicque patet hinc praecedentem solutionem enasci.

26. Verum solutio adeo in integris prodit ponendo $f=5$ et $g=1$, unde fit

$$A=20a,\quad B=20b,\quad C=8a-20b,\quad D=30b+2a\quad \text{et}\quad m=\frac{30(a+b)}{(25b-9a)(25b-a)}.$$

Sumatur jam $a=19$, $b=7$ fietque

$$A=380,\quad B=140,\quad C=12,\quad D=248\quad \text{et}\quad m=\frac{30.26}{4.156}=\frac{5}{4}.$$

Quare quatuor numeri quaesiti erunt

I. 475, II. 175, III. 15, IV. 310,

quorum summa est $975=25.39$. Alii numeri integri sunt

I. 504, II. 96, III. 36, IV. 264,

quorum summa est $=900$.

27. Hinc etiam solvi potest hoc problema,

quo quaeruntur quatuor numeri ejusmodi, ut binorum producta, summa omnium minuta, fiant numeri quadrati.

Solutio enim ex praecedente facile deducitur, dum pro multiplicatore m numerus capitur negativus. Unde in numeris integris sequens solutio obtinetur:

$$\text{I}=80,\quad \text{II}=24,\quad \text{III}=8,\quad \text{IV}=44,$$

quorum summa est 156; quibusque hoc modo quaestioni satisfit

$$80.24-156=1764=42^2,\quad 80.8-156=484=22^2,$$
$$80.44-156=3364=58^2,\quad 24.8-156=36=6^2,$$
$$24.44-156=900=30^2,\quad 8.44-156=196=14^2.$$

Appendix.

28. Adjungam hic problema prorsus singulare, olim mihi propositum, quod vires analyseos Diophanteae omnino transcendere videtur, quandoquidem solutio ad formulam sexti gradus quadrato aequandam perducit, dum adhuc operationes istius analyseos non ultra quartum gradum sunt promotae. Problema autem hoc, cujus tandem unam solutionem sum adeptus, ita se habet:

Invenire duos numeros, quorum productum ita sit comparatum, ut sive addatur, sive subtrahatur tam summa quam differentia eorum, semper prodeant numeri quadrati.

Positis ergo numeris quaesitis $\frac{x}{n}$ et $\frac{y}{n}$, requiritur ut sit

$$\text{tam } xy \pm n(x+y) = \text{quadrato}$$
$$\text{quam } xy \pm n(x-y) = \text{quadrato}.$$

Cum nunc sit $AA + BB \pm 2AB$ quadratum, capiatur xy ita, ut duplici modo in duo quadrata resolvi possit. Hunc in finem posito $xy = (pp+qq)(rr+ss)$, erit duplici modo

$$\text{vel } A = pr + qs \text{ et } B = ps - qr,$$
$$\text{vel } A = ps + qr \text{ et } B = pr - qs;$$

quare statuatur

$$n(x+y) = 2(pr+qs)(ps-qr) \text{ et}$$
$$n(x-y) = 2(ps+qr)(pr-qs), \text{ ut fiat}$$
$$\sqrt{(xy+n(x+y))} = pr+qs+ps-qr$$
$$\sqrt{(xy-n(x+y))} = pr+qs-ps+qr$$
$$\sqrt{(xy+n(x-y))} = ps+qr+pr-qs$$
$$\sqrt{(xy-n(x-y))} = ps+qr-pr+qs$$

29. Jam factae positiones praebent

$$\frac{x+y}{x-y} = \frac{(pr+qs)(ps-qr)}{(ps+qr)(pr-qs)} = \frac{pprs+pqss-pqrr-qqrs}{pprs-pqss+pqrr-qqrs},$$

unde fit $\frac{x}{y} = \frac{rs(pp-qq)}{pq(ss-rr)}$. Ponamus ergo $x = mrs(pp-qq)$ et $y = mpq(ss-rr)$, eritque

$$x+y = m(pr+qs)(ps-qr),$$

ideoque $n = \frac{2}{m}$, ut numeri quaesiti fiant

$$\frac{x}{n} = \frac{1}{2}mmrs(pp-qq) \text{ et}$$
$$\frac{y}{n} = \frac{1}{2}mmpq(ss-rr).$$

Nunc ob $xy = (pp+qq)(rr+ss)$, habebimus hanc aequationem resolvendam

$$mmpqrs(pp-qq)(ss-rr) = (pp+qq)(rr+ss),$$

quae utique ita est comparata, ut per nulla artificia adhuc cognita confici possit.

30. Facile autem perspicitur id effici oportere, ut haec fractio quadratum evadat

$$\frac{pq(pp-qq)(pp+qq)}{rs(ss-rr)(rr+ss)} = \square,$$

tum igitur ob
$$mm = \frac{(pp+qq)(rr+ss)}{pqrs(pp-qq)(ss-rr)},$$
erit
$$\frac{x}{n} = \frac{(pp+qq)(rr+ss)}{2pq(ss-rr)} \quad \text{et} \quad \frac{y}{n} = \frac{(pp+qq)(rr+ss)}{2rs(pp-qq)},$$
dummodo formulae $pq(pp-qq)(pp+qq)$ et $rs(ss-rr)(rr+ss)$ rationem quadratam inter se teneant.

31. Ad hoc praestandum alia non patere videtur via, nisi ut simplicioribus numeris pro A et B assumendis, hujus formulae $AB(AA-BB)(AA+BB)$ plures valores evolvantur, donec duo occurrant rationem quadratam inter se tenentes. Hoc modo reperi istam conditionem impleri sumendo $p=12$, $q=1$, $s=16$ et $r=11$, unde fit
$$pq(pp-qq)(pp+qq) = 4.3.5.11.13.29 \quad \text{et}$$
$$rs(ss-rr)(rr+ss) = 144.3.5.11.13.29.$$
Quamobrem ambo numeri quaesiti sunt
$$\frac{x}{n} = \frac{5.13.29.29}{8.3.5.27} = \frac{13.841}{8.81} = A,$$
$$\frac{y}{n} = \frac{5.13.29.29}{32.11.11.13} = \frac{5.841}{32.121} = B,$$
qui duo numeri si dicantur A et B, fit
$$\sqrt{(AB+A+B)} = \frac{29.329}{16.9.11} = \frac{9541}{1584}; \quad \sqrt{(AB-A-B)} = \frac{29.33}{16.9.11} = \frac{29}{48};$$
$$\sqrt{(AB+A-B)} = \frac{841.11}{16.9.11} = \frac{841}{144}; \quad \sqrt{(AB-A+B)} = \frac{841.3}{16.9.11} = \frac{841}{528}.$$